On the equivalence problem for succession rules

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Abstract

The notion of succession rule (system for short) provides a powerful tool for the enumeration of many classes of combinatorial objects. Often, different systems exist for a given class of combinatorial objects, and a number of problems arise naturally. An important one is the equivalence problem between two different systems. In this paper, we show how to solve this problem in the case of systems having a particular form. More precisely, using a bijective proof, we show that the classical system defining the sequence of Catalan numbers is equivalent to a system obtained by linear combinations of labels of the first one.

Keywords: Succession rules; Catalan numbers; ECO method; System equivalence

1. Introduction

The notion of succession rule was introduced by Chung et al. in [8] as a compact notation for generating trees, and flourished later as a powerful tool for the enumeration of combinatorial objects (see, for instance, [1,4,11,20]). More precisely, a succession rule (system) $\Omega = ((b), \mathcal{R})$ consists of an axiom $(b)$ and a set $\mathcal{R}$ of productions or rewriting rules.
and is denoted by

$$\Omega = \begin{cases} (b) \\ (k) \leadsto (e_1(k))(e_2(k)) \cdots (e_k(k)), \quad e_i : \mathbb{N}^+ \to \mathbb{N}^+, \quad b \in \mathbb{N}^+, \quad k \in M \subseteq \mathbb{N}^+, \end{cases}$$

where $M$ is the set of labels. A system $\Omega$ is suitably represented by means of a generating tree where the root is labeled by the axiom $(b)$, and a node labeled $(k)$ produces $k$ sons labeled by $e_1(k), \ldots, e_k(k)$, respectively (consistency condition). Consequently, $\Omega$ defines a non-decreasing sequence of positive integers $\{f_n\}_{n \geq 0}$, the number of nodes at level $n$ (the root is at level 0), and the generating function of $\Omega$ is

$$f_\Omega(x) = \sum_{n \geq 0} f_n x^n.$$

The structure of the productions in a system is closely related to the sequence $\{f_n\}_{n \geq 0}$, and this relationship has been studied in [1] for rational, algebraic and transcendental generating functions.

A well-known system is the one defining the sequence of Schröder numbers [3], 1, 2, 6, 22, 90, 394, ... (sequence A006318 in [19]):

$$\Omega_\mathcal{S} = \begin{cases} (2) \leadsto (3)(3); \\ (k) \leadsto (3)(4) \cdots (k)(k+1)^2, \quad k \geq 3, \end{cases}$$

where the power $(k+1)^2$ stands for the repetition $(k+1)(k+1)$. In Fig. 1 the first levels of the generating tree of (2) are shown. We refer to [4] for more details and examples. A system $\Omega$ is finite if the number of labels in the productions is finite, that is, when $|M| < \infty$. In this particular case, the generating function is rational [1], and sometimes has an interpretation as a regular language or other combinatorial structures [2,11,15].

A classical example of finite system is the one defining the Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13, ... (sequence A000045 in [19]):

$$\Omega_\mathcal{F} = \begin{cases} (1) \leadsto (2); \\ (1) \leadsto (2); \\ (2) \leadsto (1)(2). \end{cases}$$
A succession rule has a factorial form, if a finite modification of the set \( \{1, 2, \ldots, k\} \) is reachable from \( k \). More formally, a factorial succession rule has the form

\[
\Omega = \left\{ \begin{array}{c}
(b) \\
(k) \rightsquigarrow (r_0)(r_0 + 1) \ldots (k - c - 1)(k + d_1)(k + d_2) \ldots (k + d_m),
\end{array} \right. \tag{4}
\]

with \( k \geq r_0 \geq 1, c \geq 0, -c \leq d_1 \leq d_2 \leq \cdots \leq d_m > 0 \), where the consistency condition is satisfied by imposing that \( r_0 + c = m \); the rule in (2) is factorial.

Determining the generating function of a given system is not always an easy task [1]. Therefore, some recent papers focused on the development of some algebraic tools in order to study enumerative properties of succession rules, without computing the corresponding generating functions, by using a linear operator approach [11], or production matrices [9].

The study of these systems has been systematized by the Italian school [4] in the so-called ECO method, from which we briefly recall some of the basics. Given a class \( \mathcal{O} \) of combinatorial objects, we consider a fixed parameter \( p: \mathcal{O} \rightarrow \mathbb{N} \), such that for all \( n \in \mathbb{N} \), \( p^{-1}(n) \) is finite. If it is possible to define an ECO operator

\[
\vartheta: \mathcal{O}_n \rightarrow 2^{\mathcal{O}_{n+1}},
\]

performing “local expansions” on objects of size \( n \), such that

(i) for each \( O' \in \mathcal{O}_{n+1} \), there exists \( O \in \mathcal{O}_n \) such that \( O' \in \vartheta(O) \),

(ii) for each \( O, O' \in \mathcal{O}_n \) such that \( O \neq O' \), then \( \vartheta(O) \cap \vartheta(O') = \emptyset \),

then the family of sets \( \{ \vartheta(O): O \in \mathcal{O}_n \} \) is a partition of \( \mathcal{O}_{n+1} \).

We refer to [4] for further details, proofs, definitions and examples. The parameter \( p \) being fixed, the recursive construction determined by \( \vartheta \) is described by a generating tree [8], whose vertices are objects of \( \mathcal{O} \). The objects having the same parameter value lie on the same level, and the siblings of an object are the objects produced by \( \vartheta \): if \( |\vartheta(P)| = k \) then the object \( P \) blossoms often according to a system (of the form (1)).

2. The equivalence problem

Two rules \( \Omega_1 \) and \( \Omega_2 \) are said to be equivalent if they define the same number sequence:

\[
\Omega_1 \cong \Omega_2 \iff f_{\Omega_1}(x) = f_{\Omega_2}(x).
\]

For instance, the reader can easily verify that the following rules are equivalent to (2), and define the Schröder numbers [3,6]:

\[
\begin{align*}
\Omega'_S &= \left\{ \begin{array}{c}
(2) \\
(2k) \rightsquigarrow (2)(4)^2 \ldots (2k)^2(2k + 2),
\end{array} \right. \\
\Omega''_S &= \left\{ \begin{array}{c}
(2) \\
(2) \rightsquigarrow (3)(3) \\
(2k - 1) \rightsquigarrow (3)^2(5)^2 \ldots (2k - 1)^2(2k + 1),
\end{array} \right. \\
\Omega''''_S &= \left\{ \begin{array}{c}
(2) \\
(2^k) \rightsquigarrow (2^2)^{2k-1} (4)^{2k-2} (8)^{2k-3} \ldots (2^{k-1})^2(2^k)(2^{k+1}).
\end{array} \right.
\end{align*}
\]
The equivalence problem consists in determining if two different systems are equivalent. In general, as mentioned recently by Robson [16], this problem is not decidable. However, there are classes of systems for which the answer is positive. The easy case of finite systems stems out from formal language theory. Indeed, a PDOL system is a triple (see [18], p. 97) $G = (\Sigma, h, w_0)$ where $\Sigma = \{a_1, \ldots, a_k\}$ is a $k$-letter alphabet, $h$ is an endomorphism defined on the set $\Sigma^+$ of non-empty words, and $w_0 \in \Sigma^+$ is called the axiom. The length of a word $w \in \Sigma$ is denoted by $|w|$. The language of $G$ is defined by

$$L(G) = \{h^i(w_0) : i \geq 0\}.$$ 

The function $f_G(n) = |h^n(w_0)|$, $n \geq 0$, is the growth function of $G$, and the sequence $|h^n(w_0)|$, $n \geq 0$, is its growth sequence. A growth matrix $M_G$ associated with $G$ is defined by

$$M_G[i, j] = |h(a_j)|_{a_i},$$

where $|h(a_j)|_{a_i}$ is the number of occurrences of the letter $a_i$ in $h(a_j)$. The growth sequence is then obtained by the generating function

$$f_G(x) = \frac{[10^{k-1}] \cdot \chi(M) \cdot (I - Mx)^{-1} \cdot [1^k]^t}{x^k \cdot \chi(M)},$$

where $\chi(M)$ is the characteristic polynomial of $M$, $[1^k]$ is the $k$-length vector with all entries 1, and $[10^{k-1}]$ has all entries 0 except the first which is 1 (see [17] for details).

Remark that any finite system $\Omega$ can be viewed as a particular PDOL system where the alphabet $\Sigma$ is the set of labels of $\Omega$, and $h$ is defined by the productions of $\Omega$, and $w_0 \in \Sigma$. For instance, rule (3) defines a PDOL system $F$, where $\Sigma = \{1, 2\}$, $w_0 = 1$, and

$$h(1) = 2, \quad h(2) = 12, \quad M_F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

The words in the language of $F$ are $1, 2, 12, 212, 12212, 21212212, \ldots$, and its growth sequence is obtained from $M_F$ by the generating function (5)

$$f_G(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + 34x^8 + \mathcal{O}(x^9).$$

Now, two PDOL systems are growth equivalent if they have the same generating function, which amounts to checking if two polynomials are equal, and, consequently, the equivalence problem is decidable for the class of finite systems. However, the computation of the generating functions can be avoided, by checking the equality of the first few terms of the two sequences as stated below.

**Theorem 1.** The equivalence problem is decidable for the class of finite systems.

**Proof.** Let $\Omega_1$ and $\Omega_2$ be two finite succession rules having $k_1$ and $k_2$ labels, respectively. In view of Theorem 3.3 [18] it is necessary and sufficient to check if the first $k_1 + k_2$ terms of the two sequences defined by $\Omega_1$ and $\Omega_2$ coincide. \qed
For example, both the finite rules

$$\Omega_1 = \begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(3) \end{cases} \quad \text{and} \quad \Omega_2 = \begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(4) \\ (4) \rightsquigarrow (1)(2)(4)(4) \end{cases}$$

(6)

define the sequence of odd index Fibonacci numbers, 1, 2, 5, 13, 34, 89, ... (sequence A001519 in [19]). Their equivalence can be verified by comparing the first five terms of the defined sequences.

In [1] the authors formalize and then apply the so-called kernel method in order to find a solution to the functional equation arising from a factorial system (4). The main result states that a factorial system has an algebraic generating function.

**Theorem 2.** The equivalence problem is decidable for the rules having a factorial form.

**Proof.** A classical result on the equality of algebraic generating functions in several commutative variables shows that the equality is decidable (see [18, Theorem IV.5.1]). □

3. An infinite set of rules for the ballot numbers

For $k, n \in \mathbb{N}$, let $a_{n,k}$ be the set of ballot numbers, defined by the recurrence

$$a_{1,1} = 1,$$

$$a_{n+1,1} = \sum_{j \geq 1} a_{n,j},$$

$$a_{n+1,k} = \sum_{j \geq k-1} a_{n,j}, \quad k \geq 2.$$  

They can be conveniently displayed in a triangular array, called Catalan triangle, and shown in Table 1 (see [7,12] where it appears in a slightly different form; see also [19], sequence A033184).

For any positive integer $h$, a rule defining the sequence in the $h$th column is given by (see [11])

$$\Omega^h = \begin{cases} (h) \\ (k) \rightsquigarrow (2)(3) \ldots (k)(k+1) \end{cases}$$

(7)

Remark that for $h = 1$, we have the rule defining the Catalan numbers.
Table 1

The Catalan triangle

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
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<tbody>
<tr>
<td>1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>1</td>
<td>1</td>
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<td></td>
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<tr>
<td>3</td>
<td>2</td>
<td>2</td>
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<td>4</td>
<td>5</td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>14</td>
<td>9</td>
<td>4</td>
<td>1</td>
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<tr>
<td>6</td>
<td>42</td>
<td>42</td>
<td>28</td>
<td>14</td>
<td>5</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>132</td>
<td>132</td>
<td>90</td>
<td>48</td>
<td>20</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Let \( h, \alpha \in \mathbb{N}^+ \), and \( \beta \in \mathbb{N} \). We first define the following rule:

\[
\Omega_{\alpha, \beta}^h = \begin{cases} 
(h) \\
(1) \rightsquigarrow (2) \\
(2) \rightsquigarrow (2)(3) \\
\vdots \\
(\alpha + \beta - 1) \rightsquigarrow (2)(3) \ldots (\alpha + \beta) \\
(\alpha k + \beta) \rightsquigarrow (1)^k \ldots (\alpha - 1)^k (\alpha + 1) \ldots (\alpha + \beta)(2\alpha + \beta) \ldots \\
\ldots((k+1)\alpha + \beta), \ k \geq 1.
\end{cases}
\]  

In what follows we prove that, for \( h \leq \alpha + \beta \), system (8) is equivalent to system (7), so the first can be viewed as a generalization of the second, where the labels have been linearly combined according to the positive coefficients \( \alpha \) and \( \beta \). Moreover, the first \( \alpha + \beta \) levels of the two generating trees coincide. As a consequence, we obtain that (8) defines the numbers \( \{a_{n,h} : n \geq 0\} \), for any \( \alpha \) and \( \beta \) such that \( h \leq \alpha + \beta \). In particular, for \( h = 1 \) we have an infinite set of succession rules defining Catalan numbers:

\[
\Omega_{\alpha, \beta}^1 = \begin{cases} 
(1) \\
(1) \rightsquigarrow (2) \\
(2) \rightsquigarrow (2)(3) \\
\vdots \\
(\alpha + \beta - 1) \rightsquigarrow (2)(3) \ldots (\alpha + \beta) \\
(\alpha k + \beta) \rightsquigarrow (1)^k \ldots (\alpha - 1)^k (\alpha + 1) \ldots (\alpha + \beta)(2\alpha + \beta) \ldots \\
\ldots((k+1)\alpha + \beta), \ k \geq 1.
\end{cases}
\]

Instead of using generating functions as in [1], we provide a bijective proof with the ECO method.

3.1. Dyck paths

We consider lattice paths in the plane \( \mathbb{Z} \times \mathbb{Z} \), starting from the origin \((0, 0)\), and using rise steps \( \mathbf{x} = (1, 1) \) and full steps \( \mathbf{y} = (1, -1) \). The set \( \mathcal{D} \) of Dyck paths is the subset of
\( \Sigma^* = \{x, \overline{x}\}^* \) generated by the grammar \( D : = \varepsilon + xD\overline{x}D \), and we refer to paths as words, in which the notions of prefix, suffix have the usual meaning. The height of a point \( P = (P_x, P_y) \) is defined by \( h(P) = P_y \). A Dyck path is called elevated or primitive if it can be written as \( D = xD'\overline{x} \) with \( D' \in \mathcal{D} \), and we denote the stripping operation by \( D' = \text{Top}(D) \).

Given two points \( P' \), \( P \) of a Dyck path \( D \), the factor starting at \( P' \) and ending at \( P \) is denoted by \( D[\overline{P'}x, P] \). By convention, \( D[i, j] = \varepsilon \) if \( i \geq j \). A peak is a factor of the form \( x \). The insertion of a word \( w \) in \( D \) at position \( i \) is defined by

\[
\text{insert}(D, w, i) = D[0, i] \cdot w \cdot D[i, 2n].
\]

The last sequence of fall steps \( \ell_d(D) \), or last descent, of \( D \) satisfies \( \ell_d(D) = \overline{x}^k \) for some \( k \geq 1 \), and \( \mathbf{P}(D) \) is the set of its points. Finally, \( |D| \) denotes the length of the word \( D \). From the grammar above, are easily deduced the properties summarized in the next statement.

**Proposition 3.** Every nonempty Dyck path \( D \) with \( \ell_d(D) = \overline{x}^k \), \( k \geq 1 \), satisfies the conditions

(a) there is a unique decomposition \( D = D_1D_2 \ldots D_m \) where \( D_i \) is primitive for all \( i \leq m \);
(b) \( D = u \overline{x}^k \), \( k \geq 1 \), where the last occurrence of \( \overline{x} \) is the last peak;
(c) for each \( m \leq k \) there exists a longest Dyck path \( c \in \mathcal{D} \) such that \( D = uc \overline{x}^m \) and \( u \overline{x}^m \in \mathcal{D} \).

### 3.2. An ECO operator for \( \mathcal{D} \)

We define now an ECO operator for the generation of Dyck paths according to the rule \( \Omega^1_{\alpha, \beta} \). The operator constructs a partition \( \mathcal{D} = \mathcal{D}^0 \cup \mathcal{D}^1 \cup \mathcal{D}^2 \) where each path belongs to some \( \mathcal{D}^i \) with an extra labeling of those in \( \mathcal{D}^2 \). The ECO operator \( \vartheta : \mathcal{D}_n \to 2^{\mathcal{D}_{n+1}} \) is defined inductively on paths of semi-length \( n \) by setting \( \varepsilon \in \mathcal{D}^0 \), and for each class we have:

\[
[D \in \mathcal{D}_0] \quad \# \text{See Fig. 2 for an example with } \alpha = 3, \beta = 2 \text{ and } |\mathbf{P}(D)| = 4. \]

- for each point \( P \in \mathbf{P}(D) \) do
  \[
  D_{h(P)} \leftarrow \text{insert}(D, \overline{x}, P_x);
  \]
  if \( h(P) < \alpha + \beta - 2 \) then \( D_{h(P)} \in \mathcal{D}^0 \) else \( D_{h(P)} \in \mathcal{D}^1 \). \# classifying

Fig. 2. The operator \( \vartheta \) applied to a path in \( \mathcal{D}^0 \).
Remark. In Fig. 2, the class is indicated below each path. Note also that for each path in the class $D_0$ we have $D \in D_0 \Rightarrow h(\ell_d(D)) < \alpha + \beta - 1$.

$[D \in D_1]$ See Fig. 3 for an example with $\alpha = 3$, $\beta = 2$.*/

- for each point $P \in \mathbf{P}(D)$ such that $h(P) > \alpha + \beta - 2$ do
  $D_{h(P)} \leftarrow \text{insert}(D, \mathbf{xX}, P_x); \ D_{h(P)} \in D_1$;
  let $P'$ be the leftmost point of $D$ such that $P'P \in D$;
  /* then $D = uD[P'_x, P_x]v$ with $uv \in D$; the decomposition exists by */
  /* Proposition 3(c) */
  for each point $Q \in \mathbf{P}(D)$ such that $h(Q) \leq \alpha + \beta - 2$ do
  $D_{h(P), h(Q)} \leftarrow \text{insert}(uv, \mathbf{xD}[P'_x, P_x]X, Q_x); \ D_{h(P), h(Q)} \in D_2$;
  Rank($D_{h(P), h(Q)}$) $\leftarrow h(Q)$; /* ranking */
- for each point $P \in \mathbf{P}(D)$ such that $\alpha - 2 < h(P) \leq \alpha + \beta - 2$ do
  $D_{h(P)} \leftarrow \text{insert}(D, \mathbf{xX}, P_x)$;
  if $h(P) = \alpha + \beta - 2$ then $D_{h(P)} \in D_1$ else $D_{h(P)} \in D_0$. 

Fig. 3. The operator $\vartheta$ applied to a path in $D_1$. 

$\vartheta$
Remark. Only the paths in $D_2$ receive a rank equal to the height of the insertion point. 

$[D \in D_2]$ /*See Fig. 4 for an example with $\alpha = 3$, $\beta = 0$. These paths are labeled. */

- for each point $P \in P(D)$ such that $h(P) \leq \text{Rank}(D)$ do 
  
  $D_{h(P)} \leftarrow \text{insert}(D, xx, P_x)$;
  
  if $\text{Rank}(D) < \alpha + \beta - 2$ then $D_{h(P)} \in D^0$ else $D_{h(P)} \in D^1$.

It remains now to prove that the described construction generates all the Dyck paths (i) and that we have a partition (ii). This is achieved for both conditions by induction, according to the inductive definition of $\vartheta$.

(i) For any $D' \in D_{n+1}$ there exists $D \in D_n$, such that $D' \in \vartheta(D)$:

- if $D' \in D^0 \cup D^1$, then $D' = uxx^k$ and the last peak of $D'$ is removed, i.e.
  
  $D = uxx^{k-1} \in D_n$,
  
  such that $D' = \text{insert}(D, xx, |D| - (k - 1))$. See Proposition 3(b).

- if $D' \in D^2$, then for some suffix $v \in \ell(D')$, we have
  
  $\text{Rank}(D') = |v|$ and $D' = uD''v$,

  where $D'' \neq e$. Then, Proposition 3(a) provides the factorization

  $D'' = D''_1 \ldots D''_{m-1} D''_m$,

  where $m \geq 2$ and $D_m$ is primitive. Let $P_x$ be the position of the last point of $D''_{m-1}$. Then,

  $D = \text{insert}(uD''_1, \ldots, D''_{m-1}, \text{Top}(D''_m), P_x)$.

Remark. At this point the reader has certainly noticed that $D'$ is assumed to belong to some class and has a rank. This can be done by providing a valuation which is independent of the ECO construction. Alas, we did not find a closed formula but rather an inductive algorithm which mimics the inverse of the inductive ECO operator, allowing to compute explicitly from a given Dyck path its ancestors. In fact, the only point we need to prove is that $D'$ has an ancestor; the ECO construction relies on insertions in the last descent, so that removing either the last peak of $D'$ or $D''_m$ yields a path in $D_{\leq n}$, which belongs to some
class by induction hypothesis. Therefore there exists a suffix \( v \) (determining the insertion point and the class) such that \( D' = uD''v \) where \( D'' \) is a primitive Dyck path (perhaps the last peak) such that \( D = uv \).

(ii) Let \( D \) and \( D' \in \mathcal{D}_n \), then \( \vartheta(D) \cap \vartheta(D') = \emptyset \); when \( D \) and \( D' \) are such that \( \vartheta \) performs the insertion of \( \mathbf{x} \mathbf{x} \) in their last descent, the result follows from the fact that, for each \( P \in \mathcal{P}(D) \) and for each \( P' \in \mathcal{P}(D') \), we have

\[
\text{insert}(D, \mathbf{x} \mathbf{x}, P_x) = \text{insert}(D', \mathbf{x} \mathbf{x}, P'_x) \implies D = D'.
\]

When \( \vartheta(D), \vartheta(D') \in \mathcal{D}_2 \), we have two cases:

- \( \text{Rank}(\vartheta(D)) \neq \text{Rank}(\vartheta(D')) \implies \vartheta(D) \neq \vartheta(D') \).
- \( \text{Rank}(\vartheta(D)) = \text{Rank}(\vartheta(D')) \): if \( \vartheta(D) = \vartheta(D') \), construction (i) implies \( D = D' \).

We are now in a position to state our main result.

**Proposition 4.** Let \( \alpha \in \mathbb{N}^+ \) and \( \beta \in \mathbb{N} \), then \( \Omega_{\alpha, \beta}^1 \cong \Omega^1 \).

**Corollary 5.** Let \( h, \alpha \in \mathbb{N}^+ \), \( \beta \in \mathbb{N} \) and \( h \leq \alpha + \beta \). We have \( \Omega_{\alpha, \beta}^h \cong \Omega^h \).

**Proof.** It is a direct consequence of Proposition 4. Indeed, the rules \( \Omega_{\alpha, \beta}^h \) and \( \Omega^h \) both enumerate the class of Dyck paths beginning with \( h \) rise steps. \( \square \)

**Remark.** We then have an infinite set of systems defining ballot numbers. In particular, the following systems define the Catalan numbers:

\[
\Omega_{2,0} = \left\{ \begin{array}{c}
(1) \\
(1) \rightsquigarrow (2) \\
(2k) \rightsquigarrow (1)^k(4)(6) \ldots (2k)(2k+2);
\end{array} \right\} \tag{10}
\]

\[
\Omega_{3,1} = \left\{ \begin{array}{c}
(1) \\
(1) \rightsquigarrow (2) \\
(2) \rightsquigarrow (2)(3) \\
(3) \rightsquigarrow (2)(3)(4) \\
(3k + 1) \rightsquigarrow (1)^k(2)^k(4)(7) \ldots (3k + 1)(3k + 4).
\end{array} \right\} \tag{11}
\]

### 3.3. Factorial rules

The main idea of this paper can be naturally extended by considering the whole set of factorial succession rules. For any given rule of factorial form, say \( \Omega \), it should be possible to determine a class of succession rules equivalent to it. In practice, the productions of each of these succession rules can be distinguished into two different types:

- the set of productions that behave like \( \Omega \) (possibly a finite set, as in the Catalan case);
- the set of productions that are obtained by linearly combining the production of \( \Omega \).
Below we show some interesting examples (stated without proof due to their length) that can be found in the thesis of Duchi ([10, pp. 40–42]).

**Example.** An infinite set of rules for the number of m-ary trees. It is common knowledge that for any \( m \geq 2 \), the number of m-ary trees having \( n \) nodes is

\[
\frac{1}{(m-1)n+1} \binom{mn}{n}.
\]

A succession rule defining this sequence is determined in [5]

\[
\Omega^m = \left\{ \begin{array}{l}
(m) \\
(k) \leadsto (m)(m+1) \ldots (k+m-1).
\end{array} \right.
\]

A class of succession rules equivalent to \( \Omega^m \) is then given by

\[
\Omega^m_{\alpha, \beta} = \left\{ \begin{array}{l}
(m) \\
(k) \leadsto (m)(m+1) \ldots (k+m-1), \quad k \neq \alpha + \beta, \ t \in \mathbb{N}, \\
(\alpha k + \beta) \leadsto (1)^k(2)^k \ldots (\alpha - 1)^k(\alpha + m - 1)(\alpha + m) \ldots \\
(\alpha + \beta + m - 2)(\alpha m + \beta)(\alpha(m+1) + \beta) \\
\ldots (\alpha(k + m - 1) + \beta),
\end{array} \right.
\]

with \( \alpha \geq 1, \beta \geq 0 \) and \( \alpha + \beta \geq m \). The specialization \( m = 2 \) again yields the set of rules (8) defining \( \Omega^2_{\alpha, \beta} \).

**Example.** An infinite set of rules defining Motzkin numbers. The sequence of Motzkin numbers, \( 1, 1, 2, 4, 9, 21, 51, 127, \ldots \) (sequence A001006 in [19]), is defined by the rule [4]

\[
\Omega_{\mathcal{M}} = \left\{ \begin{array}{l}
(1) \\
(k) \leadsto (1)(2) \ldots (k-1)(k+1).
\end{array} \right.
\]

An infinite class of succession rules can be derived from \( \Omega_{\mathcal{M}} \), still defining Motzkin numbers:

\[
\left\{ \begin{array}{l}
(1) \\
(1) \leadsto (2) \\
(2) \leadsto (1)(3) \\
(3) \leadsto (1)(2)(4) \\
\vdots \\
(\alpha + \beta) \leadsto (1)(2) \ldots (\alpha + \beta - 1)(\alpha + \beta + 1) \\
(\alpha k + \beta + 1) \leadsto (1)^k(2)^k \ldots (\alpha - 1)^k(\alpha + 1)(\alpha + 2) \ldots \\
(\alpha + \beta)(\alpha + \beta + 1)(2\alpha + \beta + 1) \\
\ldots ((k-1)\alpha + \beta + 1)((k+1)\alpha + \beta + 1).
\end{array} \right.
\]

This rule can be re-written in a simpler way as

\[
\left\{ \begin{array}{l}
(1) \\
(k) \leadsto (1)(2) \ldots (k-1)(k+1), \quad k \leq \alpha + \beta \\
(\alpha k + \beta + 1) \leadsto (1)^k(2)^k \ldots (\alpha - 1)^k(\alpha + 1)(\alpha + 2) \ldots \\
\ldots (\alpha + \beta)(\alpha + \beta + 1)(2\alpha + \beta + 1) \ldots \\
\ldots ((k-1)\alpha + \beta + 1)((k+1)\alpha + \beta + 1).
\end{array} \right.
\]
4. Concluding remarks and open problems

The equivalence relation \( \cong \) partitions a set \( R \subset S \) of systems into equivalence classes, identified by the corresponding number sequence. For instance, if \( R \) is the class of rational systems, those having a rational generating function, we already know that finite systems are in it. On the other hand, there exist rational generating functions that are not the growth sequence of a DOL system [17, Theorem III.4.11]. Therefore, many problems arise naturally concerning

- finiteness of the equivalence classes: Is \( \| \Omega \|_\cong < \infty \) when \( \Omega \) is finite? More generally, for a given rational generating function: Is its class finite?
- the characterization of the rules in a given equivalence class;
- the extension of the decidability of equivalence for finite systems to a larger class, by using the same decision procedure;
- operations on rules (or trees) that provide equivalent systems.

By Theorem 1, the class of finite systems is included in the class of rational systems. On the other hand, Theorem III.4.11 of [17] characterizes the rational functions with integer coefficients that are the generating functions of DOL systems. Therefore, the following problem seems natural.

**Conjecture 6.** Each rational system is equivalent to a finite one.

Actually, a weaker statement is the following.

**Conjecture 7.** A system counting a regular language is equivalent to a finite system.

In some recent discussions with Cyril Banderier, of INRIA, we were led to speculate about algebraic systems.

**Conjecture 8.** A system with algebraic generating function is equivalent to a factorial system.

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