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Graceful 2-regular graphs and Skolem sequences

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Abstract

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The purpose of the paper is to study relations between graceful numbering of certain 2-regular graphs and certain Skolem sequences.

In this paper, all graphs will be finite, without loops or multiple edges. For any graph G, the symbols $V(G)$ and $E(G)$ will denote its vertex set and its edge set, respectively. A graceful numbering of a graph G with m vertices and n edges is a one-to-one mapping ψ of the set $V(G)$ into the set $\{0, 1, \ldots, n\}$ which has the property that the values of the edges form the set $\{1, 2, \ldots, n\}$ if the value $\bar{\psi}(e)$ of the edge e with the end vertices u, v is defined by $\bar{\psi}(e) = |\psi(u) - \psi(v)|$. A graph is called graceful if it has a graceful numbering. An α-valuation ψ of a graph G is a graceful numbering of G which satisfies the following additional condition: There exists a number r $(0 \leq r \leq |E(G)|)$ such that for any edge $e = (v, w)$,

$$\min(\psi(v), \psi(w)) \leq r < \max(\psi(v), \psi(w)).$$

The concepts of a graceful numbering and of an α-valuation were introduced by Rosa [6]; in his paper as well as in [3] and [4], the term 'β-valuation' was used for graceful numbering.

It should be noted that a graph with an α-valuation is always bipartite.

In this paper, similarly as in some of the papers quoted, a graph G will be called Eulerian if $|E(G)| > 0$ and if every vertex of G is of even degree. Rosa [6] proved that any graceful Eulerian graph G satisfies the condition $|E(G)| \equiv 0$ or $3 \pmod 4$. This implies (see Kotzig [3, Theorem 2]) that any Eulerian graceful bipartite graph satisfies the condition $|E(G)| \equiv 0 \pmod 4$; in particular, any Eulerian graph with an α-valuation satisfies this condition.

Graceful valuations and α-valuations of 2-regular graphs have been studied by Rosa [6] and Kotzig [4–5]; [4] deals mainly with the case when all components are isomorphic. A $Q(r, s)$-graph is a graph with r components, each of which is an s-cycle.

Here are some of the results published in the references:

(1) A $Q(1, s)$-graph (i.e. an s-cycle) is graceful if and only if $s \equiv 0$ or $3 \pmod 4$, Rosa [6]. It has an α-valuation if and only if $s \equiv 0 \pmod 4$, Kotzig [2], Rosa [6].

(2) A $Q(2, s)$-graph has an α-valuation if and only if s is even and $s > 2$, Kotzig [4].

(3) A $Q(3, 4k)$-graph has an α-valuation for each $k > 1$. The $Q(3, 4)$-graph does not have an α-valuation but it is graceful, Kotzig [4].

(4) A $Q(r, 3)$-graph is graceful if and only if $r = 1$. A $Q(r, 5)$-graph is not graceful for any r, Kotzig [4].

(5) A $Q(r, 4)$-graph has an α-valuation at least for $1 \leq r \leq 10$, $r \neq 3$, Kotzig [4].

(6) If G is a graceful 2-regular graph which has ω components of odd length then $|V(G)| \geq \omega(\omega + 2)$. This implies, e.g. that the 2-regular graph G consisting of two triangles and a pentagon is not graceful although $|E(G)| \equiv 3 \pmod 4$, Kotzig [5].

(7) If G is a two-regular graph with a graceful numbering ψ there is exactly one number $x \in \{1, 2, \ldots, |E(G)| - 1\}$ such that $\psi(v) \neq x$ for all $v \in V(G)$. If ψ is an α-valuation and $|E(G)| = 4k$ then either $x = k$ or $x = 3k$. (Kotzig [4, Theorem 2].)

The purpose of this paper is to establish a relation between certain graceful numberings of some 2-regular graphs and certain Skolem sequences. A Skolem sequence of order n is a sequence $\{s_1, s_2, \ldots, s_{2n}\}$ of $2n$ terms ($n \geq 1$) such that, for every $k \in \{1, \ldots, n\}$, there exist exactly two subscripts $i(k)$, $j(k)$ for which $s_{i(k)} = s_{j(k)} = k$. These two subscripts satisfy the condition $|i(k) - j(k)| = k$. It is well known (see e.g. Skolem [7]) that a Skolem sequence or order n exists if and only if $n \equiv 0$ or $1 \pmod 4$.

If G is a graceful 2-regular graph on n vertices we will let correspond to a given graceful numbering of G a sequence $S(G) = \{a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n\}$; the terms of this sequence will be determined successively. To simplify the notation we will agree that the edges of G will be numbered $e_1, e_2, \ldots, e_n$ in such a way that the value of e_k in the graceful numbering considered is k, $k = 1, 2, \ldots, n$.

In the first step of the construction of $S(G)$ we select (arbitrarily) one cycle C of G and a direction in which we will move around C. Then we choose an arbitrary edge e_k of C with end vertices having the values, e.g. $i, i + k$. Now we choose (at this moment arbitrarily) one of the pairs (a_i, a_{i+k}), (b_i, b_{i+k}), (a_{i+k}, b_i), (a_i, b_{i+k}) and assign the value k to both its terms. Then we move to the edge adjacent to e_k at its end point (in the direction chosen). At a given (non-initial) stage of the construction of the terms of $S(G)$ corresponding to the cycle C we will say that a term (either a_i or b_j) of $S(G)$ is free if it has not been assigned a value yet. In each stage, exactly one edge of C will be used to assign the values to two free terms in $S(G)$. Further steps in the construction of $S(G)$ are based on the following.

Theorem. *Let e_r be an edge of C not yet used for assigning values to the terms of $S(G)$. Let e_r join the vertices with the values p, $p+r$. Then at least one of the pairs (a_p, a_{p+r}), (b_p, b_{p+r}), (a_{p+r}, b_p), (a_p, b_{p+r}) contains two free terms of $S(G)$.*

Proof. Let us assume that the statement is not correct. Then at least one term of the first pair, e.g. a_p, is not free. (The case when a_{p+r} is not free is similar.) Also, at least one term of the second pair is not free and we will distinguish two cases:

Case (i): *b_p is not free.*

Then there exist two edges e_α, e_β of G, both different from e_r, joining the vertex with the value p to two other vertices. Since G is two-regular, e_r cannot be adjacent to this vertex. We have a contradiction.

Case (ii): *b_{p+r} is not free.*

Then at least one of the terms a_{p+r}, b_p is not free (otherwise the third pair would have two free terms). If b_p is not free we are in Case (i). If a_{p+r} and b_{p+r} are not free we are in a situation similar to Case (i). □

We are now going to complete the description of the construction of $S(G)$. Continuing in the cycle C we consider in each stage the edge adjacent to the edge used in the previous stage at its end vertex (in the direction used). If we consider the edge e_r with the end vertices having the values p, $p+r$, we select one of the pairs enumerated in the above theorem which has two free terms (such a choice is possible according to the theorem) and assign the value r to both terms in this pair; we continue with all edges of the cycle C that have been used. If G has only one cycle, at the end we will be left with one pair (a_x, b_x) containing two free terms; we will put $a_x = b_x = n+1$. If G has more than one cycle we take another cycle of G and repeat the procedure, until we end with only one free pair (a_x, b_x); then we put again $a_x = b_x = n+1$.

The reader will notice that the sequence $S(G)$ constructed above does not have to be a Skolem sequence, but sometimes it is. To explain this, we will consider two special cases:

Case (I): Let G be a two-regular graceful graph on n vertices, consisting only of cycles of even length. Then $n \equiv 0 \pmod 4$ and we will put $n = 4t$. We will also agree that, in the construction of $S(G)$, we will use only the pairs of the form (a_i, a_{i+j}) or (b_i, b_{i+j}). Let us now see how this influences our construction of $S(G)$. If, at a certain stage of this construction, we consider an edge e_r adjacent to the vertices v, w with $\psi(v) = p$, $\psi(w) = p+r$ it means that we have reached in the preceding step one of these vertices, e.g. v. Then either a_p or b_p is not free. Both cases being similar we will assume that b_p is not free. If a_p were not free either we would just have completed a cycle, which contradicts the above assumption about e_r (only edges not previously used are considered in the procedures). If a_{p+r} were not free the algorithm would have used w before—and both edges incident with w would have been considered; this means that there would be no edge adjacent to w available for consideration in the procedure. This contradiction implies that (a_p, a_{p+r}) must contain two free terms.

It is now easy to see that the sequence $S(G)$ obtained in this way is a Skolem sequence $(s_1, s_2, \ldots, s_{2n+2})$ of order $n+1$ which has the following special properties:

(1) If $1 \leq k \leq n$ and if $s_i = s_{i+k} = k$ then either $i+k \leq n+1$ or $i \geq n+2$. However, if $s_j = s_{j+n+1} = n+1$ then, clearly, $j \leq n+1$, $j+n+1 \geq n+2$.

(2) If the graceful numbering under consideration is an α-valuation then either $x = t$ or $x = 3t$ where $t = n/4$. If $x = t$, and if $k \in \{1, \ldots, n\}$ is such that $s_i = s_{i+k} = k$ with $i+k \leq n+1$ then $i \leq (n/2)+1$, $(n/2)+1 < i+k \leq n+1$. If $x = 3t$ and $s_{i+1} = s_{i+k} = k$ with $i+k \leq n+1$ then $i \leq n/2$, $n/2 < i+k \leq n+1$. If $x = t$ and $s_i = s_{i+k} = k$ with $i \geq n+2$ then $n+2 \leq i \leq (3n/2)+2$ and $(3n/2)+2 < i+k \leq 2n+2$. If $x = 3t$ and $i \geq n+2$ then $n+2 \leq i \leq (3n/2)+1$ and $(3n/2)+1 < i \leq 2n+2$. Conversely, each Skolem sequence with Property (1) generates a graceful numbering of a 2-regular graph G on $n = 4t$ vertices consisting of one or more cycles of even length. Each Skolem sequence with Properties (1) and (2) generates an α-valuation of such a graph.

Case (II): The presence of the components of odd length in a graceful two-regular graph G on n vertices makes it much harder to ensure that $S(G)$ is a Skolem sequence. The reader will realize that, for a cycle of even length, if we start with a pair (a_p, a_{p+r}), we will end with a pair containing b_p (to close the cycle). If we allow only the pairs of the type (a_i, a_{i+j}), (b_i, b_{i+j}), this becomes impossible for cycles of odd length. The reader will easily realize that only some of the pairs (a_i, b_{i+j}) or (b_i, a_{i+j}) can be used if $S(G)$ should be a Skolem sequence. If $a_{p+r} = b_p = r$, r has the positions $p+r+1$ and $p+n+2$ in $S(G)$ and this implies that $p+n+2-(p+r+1) = r$, i.e. $r = (n+1)/2$ if $S(G)$ is a Skolem sequence. We can therefore allow only the case when G has exactly one component of odd length; this component must then contain $e_{(n+1)/2}$. In this case will clearly have $n \equiv 3 \pmod 4$.

To guarantee that we will get a Skolem sequence, the part of the construction of $S(G)$ which uses the cycle of odd length will have to e slightly modified. Let the length of this cycle C_1 of odd length be $2a+1$, and let its vertices, in the order in which we meet them when moving around C_1, be $v_1, v_2, \ldots, v_{2a+1}$. Furthermore, we will agree that the numbering of the vertices of C_1 will be chosen in such a way that $e_{(n+1)/2} = (v_1, v_2)$ with $\psi(v_1) < \psi(v_2)$; we will denote $\psi(v_i) = \alpha(i)$, $i = 1, \ldots, 2a+1$, and $|\alpha(i+1) - \alpha(i)| = r(i)$, $i = 1, \ldots, 2a$, $|\alpha(1) - \alpha(2a+1)| = r(2a+1)$. We will always start with $e_{r(1)} = e_{(n+1)/2}$ and put $b_{\alpha(1)} = a_{\alpha(2)} = (n+1)/2$. Then we put $b_{\alpha(2)} = b_{\alpha(3)} = r(2)$, $a_{\alpha(3)} = a_{\alpha(4)} = r(3)$, $b_{\alpha(4)} = b_{\alpha(5)} = r(4)$ etc. The rest of the construction of $S(G)$ is the same as in Case (I). The Skolem sequence $(s_1, s_2, \ldots, s_{2n+2})$ generated in this way from a 2-regular graceful graph on $n = 4t - 1$ vertices with a single component of odd length containing $e_{(n+1)/2}$ has the following properties: If $1 \leq l \leq n$, $l \neq (n+1)/2$ and if $s_i = s_{i+l} = l$ then either $i \geq n+2$ or $i + l \leq n+1$. If $s_j = s_{j+(n+1)/2} = (n+1)/2$ then $j \leq n+1$, $j + (n+1)/2 \geq n+2$. If $s_k = s_k + n + 1 = n + 1$ then $k \leq n+1$, $k + n + 1 \geq n + 2$. Conversely, each Skolem sequence with this property generates a graceful numbering of a two-regular graph on $n = 4t - 1$ vertices.

The above results generate a correspondence between graceful numberings of certain 2-regular graphs and certain Skolem sequences. It should be noted that this correspondence is not one-to-one: A change in the orientation of a cycle of the graph changes the resulting Skolem sequence. Nevertheless, this correspondence (rephrased as a correspondence between graceful numbering of certain 2-regular graphs and certain equivalence classes of Skolem sequences) might in future help to find estimates for the number of graceful numberings of certain 2-regular graphs, perhaps along the lines used in [1].

To conclude, we will give some examples.

Example 1. For $n = 7$, all Skolem sequences of order $n + 1 = 8$ of the above type are easy to enumerate. They generate twelve graceful numberings of a 7-cycle. If we write the graceful numbering of an n-cycle as $(\psi(v_1), \psi(v_2), \ldots, \psi(v_n))$, where $v_1, v_2, \ldots, v_n$ are the consecutive vertices, we have the following graceful numbers:

$$(0, 7, 1, 6, 5, 2, 4), \quad (0, 7, 1, 6, 3, 5, 4), \quad (0, 7, 1, 6, 3, 2, 4),$$
$$(0, 7, 1, 2, 6, 3, 5), \quad (0, 7, 4, 3, 5, 1, 6), \quad (0, 7, 2, 1, 5, 3, 6)$$

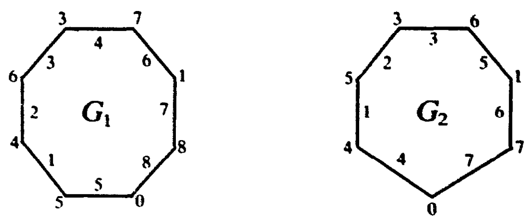
and the six graceful numberings obtained from the given ones by the transformation $\hat{\psi}(v_i) = 7 - \psi(v_i)$, $i = 1, 2, \ldots, 7$. Let us mention here that the 2-regular graph consisting of a 4-cycle and a 3-cycle is graceful (two examples of graceful numberings are $(0, 5, 1, 7)$, $(3, 4, 6)$ and $(0, 6, 2, 7)$, $(1, 3, 4)$) but no graceful numbering of this graph can be obtained from a Skolem sequence. This fact implies that there is no graceful numbering of this graph in which the 3-cycle would contain the edge e_4.

Example 2. For $n = 8$, the enumeration of all Skolem sequences of order 9 of the above described type yields six α-valuations of an 8-cycle, eighteen graceful numberings of an 8-cycle which are not α-valuations, two α-valuations of the 2-regular graph consisting of two 4-cycles (a $Q(2, 4)$-graph), and four graceful numberings of this graph which are not α-valuations. The α-valuations of an 8-cycle are

$$(0, 8, 1, 7, 3, 6, 4, 5), \quad (0, 8, 1, 7, 3, 4, 2, 5), \quad (0, 8, 1, 7, 2, 5, 3, 4),$$
$$(0, 8, 4, 5, 3, 6, 1, 7), \quad (0, 8, 3, 6, 4, 5, 1, 7), \quad (0, 8, 3, 4, 2, 5, 1, 7).$$

An example of a graceful numbering of an 8-cycle which is not an α-valuation is $(0, 8, 1, 7, 2, 6, 5, 3)$. The α-valuations of the $Q(2, 4)$-graph are $(0, 8, 7, 6)$ and $(3, 7, 4, 5)$, $(0, 8, 2, 7)$ and $(1, 5, 3, 4)$. An example of a graceful numbering which is not an α-valuation is $(0, 8, 1, 6)$ and $(2, 5, 7, 3)$. It should be mentioned here as a side remark that the graph consisting of a 5-cycle and a triangle is graceful, too. (An example of its graceful numbering is $(0, 8, 2, 3, 7)$ and $(1, 4, 6)$.)

The Skolem sequences generated by the given graceful numberings of an 8-cycle (graph G_1) and a 7-cycle (graph G_2) are given below.



$S(G_1) = (8, 6, 9, 3, 1, 1, 3, 6, 8, 5, 7, 9, 4, 2, 5, 2, 4, 7)$
$S(G_2) = (7, 5, 8, 2, 4, 2, 5, 7, 4, 6, 8, 3, 1, 1, 3, 6)$

Example 3. For $n = 11$, we can give the following examples of graceful numberings based on a partial enumeration of the corresponding Skolem sequences:

11-cycle: $(0, 11, 1, 6, 4, 3, 10, 2, 8, 5, 9)$,
6-cycle and 5-cycle: $(0, 11, 1, 5, 4, 9)$ and $(2, 10, 3, 6, 8)$,
7-cycle and 4-cycle: $(0, 11, 1, 10, 3, 2, 8, 0)$ and $(4, 7, 5, 9)$,
8-cycle and 3-cycle; $(0, 11, 1, 10, 5, 7, 4, 8, 0)$ and $(2, 3, 9)$,
Two 4-cycles and one 3-cycle: $(0, 11, 1, 9)$ and $(3, 10, 5, 6)$ and $(2, 4, 8)$.

Example 4. For $n = 12$, we can give the following examples of α-valuations based on a partial enumeration of the corresponding Skolem sequences:

12-cycle: $(0, 12, 1, 8, 2, 11, 3, 7, 4, 6, 5, 10)$,
8-cycle and 4-cycle; $(0, 12, 1, 8, 5, 6, 4, 10)$ and $(2, 11, 3, 7)$,
Two 6-cycles: $(0, 12, 1, 8, 4, 10)$ and $(2, 11, 3, 6, 5, 7)$.

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References

[1] J. Abrham, Exponential lower bounds for the number of Skolem and extremal Langford sequences, Ars Combin. 22 (1986) 187–198.

[2] A. Kotzig, Decompositions of a complete graph into 4k-gons (in Russian), Math. časopis 15 (1965) 229–233.

[3] A. Kotzig, On certain vertex valuations of finite graphs, Utilitas Math. 4 (1973) 261–290.

[4] A. Kotzig, β-valuations of quadratic graphs with isomorphic components, Utilitas Math. 7 (1975) 263–279.

[5] A. Kotzig, Recent results and open problems in graceful graphs, Congr. Numer. 44 (1984) 197–219.

[6] A. Rosa, On certain valuations of the vertices of a graph, in: Proc. Théorie des Graphes—Journées Internationales d'Etude, Rome, Juillet 1966 (Dunod, Paris, 1967) 349–355.

[7] Th. Skolem, On certain distributions of integers in pairs with given differences, Math. Scand. 5 (1957) 57–68.