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On L^p-Multipliers of Mixed-Norm Type

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Given a sequence (g_n) of Fourier multipliers for $L^p(\mathbb{R})$, $1 < p < \infty$, let $g := \sum_{-\infty}^{\infty} g_n \chi_n$, where χ_n denotes the characteristic function of the interval $[2^n, 2^{n+1}]$ in $\mathbb{R}$. Assuming $(g_n) \in \ell^s(M(p))$ for some s with $0 < s \leq \infty$, we determine the values of s for which g is, or is not, a multiplier of $L^p(\mathbb{R})$. Our results sharpen a result of Littman *et al.* who, in 1968, considered the case when $s = \infty$. The same problem is also considered for multipliers in L^p-spaces defined on a locally compact Vilenkin group.

1. MULTIPLIERS FOR $L^p(\mathbb{R})$

We first introduce some notation. For $f \in L^1(\mathbb{R})$ we denote its Fourier transform by $\hat{f}$:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

The inverse Fourier transform of f will be denoted by $f^{\vee}$. A function $g \in L^{\infty}(\mathbb{R})$ is a (Fourier) multiplier of $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, if the operator T_g, defined initially for $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ by $T_g f = (g\hat{f})^{\vee}$, can be extended as a bounded operator to $L^p(\mathbb{R})$; we write $g \in M_p(\mathbb{R})$. The norm of T_g: $L^p(\mathbb{R})$

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$\to L^p(\mathbb{R})$ will be denoted by $\|g\|_{M(p)}$. For $1 \le p \le \infty$ and $0 < q \le \infty$, the mixed-norm space $\ell^q(M(p))$ consists of the set of all sequences of $L^p(\mathbb{R})$-multipliers $(g_n)_{-\infty}^{\infty}$ for which

$$\|g_n\|_{\ell^q(M(p))} := \left(\sum_{n=-\infty}^{\infty} (\|g_n\|_{M(p)})^q \right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$. In this section only, we shall use the notation χ_n to denote the characteristic function of the interval $[2^n, 2^{n+1}]$ in $\mathbb{R}$, $n \in \mathbb{Z}$. As usual, C will denote a generic constant.

In their 1968 paper [2] on $L^p(\mathbb{R})$-multipliers Littman *et al.* gave an example of a sequence of functions $(g_n)_{-\infty}^{\infty}$ such that

(i) $\operatorname{supp}(g_n) \subset [2^n, 2^{n+1}]$ for each $n \in \mathbb{Z}$,

(ii) $(g_n)_{-\infty}^{\infty} \in \ell^{\infty}(M(p))$, where $1 < p < \infty$ and $p \ne 2$,

(iii) $g := \sum_{n=-\infty}^{\infty} g_n \chi_n \notin M_p(\mathbb{R})$.

The example in [2] raises the question stated in the Abstract: given functions g_n such that $(g_n) \in \ell^s(M(p))$ for some $s > 0$, determine the values of s for which g as defined in (iii) belongs, or does not belong, to $M_p(\mathbb{R})$.

In [1] Cowling *et al.* proved that if (g_n) satisfies (ii) and if the function $\tilde{g}$ is defined by $\tilde{g} = \sum_{-\infty}^{\infty} \alpha_n g_n \chi_n$ for some sequence $(\alpha_n) \in \ell^s(\mathbb{Z})$ with $1/s = |1/p - 1/2|$ and $1 < p < \infty$, then $\tilde{g} \in M_p(\mathbb{R})$. Thus, expressing g as

$$g(\xi) = \sum_{n=-\infty}^{\infty} \|g_n\|_{M(p)} (g_n(\xi)/\|g_n\|_{M(p)}) \chi_n(\xi),$$

we obtain immediately the following.

THEOREM 1.1. *Let $1 < p < \infty$ and let $s = |2p/(2-p)|$ (with $s = \infty$ in case $p = 2$). For each $n \in \mathbb{Z}$ let $g_n \in M_p(\mathbb{R})$ and assume $(g_n)_{-\infty}^{\infty} \in \ell^s(M(p))$. Define g by $g(\xi) = \sum_{-\infty}^{\infty} g_n(\xi)\chi_n(\xi)$. If $g \in L^{\infty}(\mathbb{R})$ then $g \in M_p(\mathbb{R})$.*

We now prove the sharpness of Theorem 1.1, thereby significantly extending the result of Littman *et al.* The example we construct in the proof of Theorem 1.2 is obtained by means of some modifications in the example used by Triebel to prove that, for $1 \le p \le \infty$ and $p \ne 2$, the set of multipliers for the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $0 < p \le \infty$ and $s \in \mathbb{R}$, does not coincide with the set of multipliers for $L^p(\mathbb{R}^n)$. Our proof of Theorem 1.2 will be brief; for additional details we refer to [7]. We mention here in passing that the example we shall give also shows that the choice of $1/s = |1/p - 1/2|$ in the aforementioned theorem of Cowling *et al.* is the best possible.

THEOREM 1.2. *Let* $1 < p < \infty$, $p \neq 2$, *and let* $q > s = |2p/(2-p)|$. *Then there exist functions* $g_n \in M_p(\mathbb{R})$, $n \in \mathbb{Z}$, *so that*

(a) $\operatorname{supp}(g_n) \subset [2^n, 2^{n+1}]$ *for each* $n \in \mathbb{Z}$,

(b) $(g_n) \in \ell^q(M(p))$,

(c) *if* $g := \sum_{n=-\infty}^{\infty} g_n$, *then* $g \in L^\infty(\mathbb{R})$ *and* $g \notin M_p(\mathbb{R})$.

Proof. Assume $2 < p < \infty$, so that $1/s = 1/2 - 1/p$, and assume $q < \infty$ (if $q = \infty$ the following proof requires some minor modifications). Choose α so that $1/q < \alpha < 1/s$. Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(\xi) \not\equiv 0$ and $\operatorname{supp}(\psi) \subset \{0 \leq \xi \leq 1\}$. For $k \leq 0$, let $g_k(\xi) = 0$ and for $k \in \mathbb{N}$ define g_k by

$$g_k(\xi) = k^{-\alpha} e^{2\pi i 2^k \xi} \psi(\xi - 2^k).$$

Furthermore, define the function g by $g(\xi) = \sum_1^\infty g_k(\xi)$. It is easy to see that (g_k) satisfies conditions (a) and (b). To prove that $g \notin M_p(\mathbb{R})$, consider for each $N \in \mathbb{N}$ the function f_N defined by

$$f_N(x) = \sum_{k=1}^{N} e^{2\pi i 2^k (x - 2^k)} (\psi)^\vee (x - 2^k).$$

Then

$$(f_N)^\wedge(\xi) = \sum_{k=1}^{N} e^{-2\pi i 2^k \xi} \psi(\xi - 2^k)$$

and on p. 125 (8) of [7] it is shown that $\|f_N\|_p \leq C N^{1/p}$. Also,

$$(g\hat{f}_N)^\vee(x) = \left(\sum_{k=1}^{N} k^{-\alpha} \psi^2(\cdot - 2^k)\right)^\vee (x),$$

so that

$$\begin{aligned}
\|(g\hat{f}_N)^\vee\|_p &\sim \left\| \left(\sum_{k=1}^{N} k^{-\alpha} \psi^2(\cdot - 2^k)\right)^\vee \right\|_{F^0_{p,2}} \\
&= \left\| \left(\sum_{k=1}^{N} |k^{-\alpha} (\psi^2(\cdot - 2^k))^\vee|^2\right)^{1/2} \right\|_p \\
&\geq N^{-\alpha} \left\| \left(\sum_{k=1}^{N} |(\psi^2(\cdot - 2^k))^\vee|^2\right)^{1/2} \right\|_p \\
&\geq C N^{-\alpha} N^{1/2}
\end{aligned}$$

according to [7, p. 125 (9)]. Thus, assuming $g \in M_p(\mathbb{R})$, we have

$$N^{-\alpha+1/2} \leq C\|(g\hat{f}_N)^\vee\|_p \leq C\|f_N\|_p \leq CN^{1/p},$$

that is,

$$N^{-\alpha+1/2-1/p} \leq C \qquad \text{for all } N \in \mathbb{N}.$$

Since $\alpha < 1/2 - 1/p$ we have a contradiction and we may conclude that $g \notin M_p(\mathbb{R})$, that is, (c) holds.

As an application of Theorem 1.1, we give a simple generalization of Theorem 2 in [1].

THEOREM 1.3. *Let $1 < p < \infty$. Assume that for each $n \in \mathbb{Z}$ we have $g_n \in M_p(\mathbb{R})$ and let $(g_n)_{-\infty}^{\infty} \in \ell^q(M(p))$ for some $q \geq s = |2p/(2-p)|$. Let $\beta = qs/(q-s)$ (with $\beta = s$ if $q = \infty$ and $\beta = \infty$ if $q = s$). Assume $(\alpha_n)_{-\infty}^{\infty} \in \ell^\beta(\mathbb{Z})$ and define $g: \mathbb{R} \to \mathbb{C}$ by $g(\xi) = \sum_{-\infty}^{\infty} \alpha_n g_n(\xi)\chi_n(\xi)$. If $g \in L^\infty(\mathbb{R})$ then $g \in M_p(\mathbb{R})$.*

Proof. We have, by Hölder's inequality,

$$\sum_{n=-\infty}^{\infty} \|\alpha_n g_n\|_{M(p)}^s \leq \left(\sum_{n=-\infty}^{\infty} (\|g_n\|_{M(p)})^q\right)^{s/q} \left(\sum_{n=-\infty}^{\infty} |\alpha_n|^{sq/(q-s)}\right)^{(q-s)/q} < \infty.$$

Thus Theorem 1.1 implies that $g \in M_p(\mathbb{R})$.

2. MULTIPLIERS FOR $L^p(G)$

In this section we prove the analogue of the results of Section 1 for multipliers for Lebesgue spaces defined on a locally compact Vilenkin group G. We start with a brief description of such Vilenkin groups and introduce some additional notation.

DEFINITION 2.1. A locally compact Abelian group G is said to be a locally compact Vilenkin group if there exists a strictly decreasing sequence of compact open subgroups $(G_n)_{-\infty}^{\infty}$ such that $\cup_{-\infty}^{\infty} G_n = G$, $\cap_{-\infty}^{\infty} G_n = \{0\}$ and $\sup\{\text{order } G_n/G_{n+1} : n \in \mathbb{Z}\} < \infty$.

We shall denote the dual group of G by Γ and for each $n \in \mathbb{Z}$, $\Gamma_n := \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}$. We choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $\mu(G_n) = (\lambda(\Gamma_n))^{-1}$ for all $n \in \mathbb{Z}$; we set m_n

$:= \lambda(\Gamma_n)$. The Fourier transform for functions on G and the inverse Fourier transform for functions on Γ will be denoted by $^\wedge$ and $^\vee$, respectively. We define the function $\Delta_n : G \to \mathbb{C}$ by $\Delta_n(x) = m_n \chi_{G_n}(x)$; an easy computation shows that $(\Delta_n)^\wedge = \chi_{\Gamma_n}$. The space of Fourier multipliers for $L_p(G)$, $1 < p < \infty$, is defined as in Section 2 and will be denoted by $\mathcal{M}_p(G)$. If $\phi \in \mathcal{M}_p(G)$ the corresponding operator norm is denoted by $\|\phi\|_{\mathcal{M}(p)}$. The definition of the mixed-norm spaces $\ell^q(\mathcal{M}(p))$, $1 < p < \infty$ and $0 < q \le \infty$, is the same as in Section 1. In this section we shall use the notation χ_n for the characteristic function of the set $\Gamma_{n+1} \backslash \Gamma_n$ in Γ.

Examples of such locally compact Vilenkin groups are the p-adic numbers and, more generally, the additive group of a local field, see [6]. For additional information about the harmonic analysis on such groups, see also [3–5].

We now prove the analogue on G of Theorem 1.1.

THEOREM 2.1. *Let $1 < p < \infty$ and let $s = |2p/(2-p)|$ (with $s = \infty$ if $p = 2$). Let $(\phi_n)_{-\infty}^{\infty}$ be a sequence of functions in $\ell^s(\mathcal{M}(p))$. If $\phi : \Gamma \to \mathbb{C}$ is defined by $\phi(\gamma) = \sum_{-\infty}^{\infty} \phi_n(\gamma)\chi_n(\gamma)$ and if $\phi \in L^\infty(\Gamma)$ then $\phi \in \mathcal{M}_p(G)$.*

Proof. We shall assume that $1 < p < 2$. First we observe that if $f \in L^1(G) \cap L^2(G)$ then, according to Minkowski's inequality,

$$\left\| \sum_{n=-\infty}^{\infty} (\phi_n \chi_n \hat{f})^\vee \right\|_1 \le \sum_{n=-\infty}^{\infty} \|(\phi_n \chi_n \hat{f})^\vee\|_1 .$$

Moreover, since the functions χ_n have mutually disjoint support, Plancherel's equality implies that

$$\left\| \sum_{n=-\infty}^{\infty} (\phi_n \chi_n \hat{f})^\vee \right\|_2^2 = \left\| \sum_{n=-\infty}^{\infty} \phi_n \chi_n \hat{f} \right\|_2^2$$

$$= \sum_{n=-\infty}^{\infty} \|\phi_n \chi_n \hat{f}\|_2^2$$

$$= \sum_{n=-\infty}^{\infty} \|(\phi_n \chi_n \hat{f})^\vee\|_2^2 .$$

Thus, interpolation yields for $1 < p < 2$

$$\left\| \sum_{n=-\infty}^{\infty} (\phi_n \chi_n \hat{f})^\vee \right\|_p^p \le \sum_{n=-\infty}^{\infty} \|(\phi_n \chi_n \hat{f})^\vee\|_p^p .$$

Therefore,

$$\begin{aligned}
\|(\phi\hat{f})^\vee\|_p &= \left\|\sum_{n=-\infty}^{\infty}(\phi_n\chi_n\hat{f})^\vee\right\|_p \\
&\le \left(\sum_{n=-\infty}^{\infty}\|(\phi_n\chi_n\hat{f})^\vee\|_p^p\right)^{1/p} \\
&= \left(\sum_{n=-\infty}^{\infty}\|(\phi_n((\Delta_{n+1}-\Delta_n)*f)^\wedge)^\vee\|_p^p\right)^{1/p} \\
&\le \left(\sum_{n=-\infty}^{\infty}\|\phi_n\|_{\mathcal{M}(p)}^p\|(\Delta_{n+1}-\Delta_n)*f\|_p^p\right)^{1/p} \\
&\le \left(\sum_{n=-\infty}^{\infty}\|\phi_n\|_{\mathcal{M}(p)}^s\right)^{1/s}\left(\sum_{n=-\infty}^{\infty}\|(\Delta_{n+1}-\Delta_n)*f\|_p^2\right)^{1/2} \\
&\le C\left\|\left(\sum_{n=-\infty}^{\infty}|(\Delta_{n+1}-\Delta_n)*f|^2\right)^{1/2}\right\|_p \\
&\le C\|f\|_p.
\end{aligned}$$

The penultimate inequality follows from the assumption of the theorem and from Minkowski's integral inequality, whereas the last inequality is a consequence of the Littlewood–Paley theory for $L^p(G)$-functions. A detailed discussion of this Littlewood–Paley theory in case G is the additive group of a local field can be found in Chapter V of [6]. These results in [6] and their proofs can easily be extended to arbitrary locally compact Vilenkin groups.

Next we prove the sharpness of Theorem 2.1. The example constructed in the proof of Theorem 2.2 is obtained by modifying an example used in the proof of Theorem 3 in [3], where we established the existence of a $\phi \in L^\infty(\Gamma)$ such that $\sup_n\|(\phi\chi_n)^\vee\|_1 < \infty$ and $\phi \notin \mathcal{M}(H^1(G))$.

THEOREM 2.2. *Let $1 < p < \infty, p \ne 2$, and assume $q > s = |2p/(2-p)|$. There exist functions $\phi_n \in \mathcal{M}_p(G)$ such that*

(a) $\text{supp}(\phi_n) \subset \Gamma_{n+1} \setminus \Gamma_n$ *for all* $n \in \mathbb{Z}$,

(b) $(\phi_n)_{-\infty}^{\infty} \in \ell^q(\mathcal{M}(p))$,

(c) *if* $\phi := \sum_{-\infty}^{\infty}\phi_n$ *then* $\phi \in L^\infty(\Gamma)$ *and* $\phi \notin \mathcal{M}_p(G)$.

Proof. We assume that $2 < p < \infty$ so that $s = 2p/(p-2)$. Choose α so that $1/q < \alpha < 1/s$ and for each $n \in \mathbb{N}$ choose an $x_n \in G_{-n}\setminus G_{-n+1}$. Next, define the functions $\phi_n : \Gamma \to \mathbb{C}$ as follows:

$$\phi_n(\gamma) = 0, \qquad \text{if } n \leq 0,$$

$$\phi_n(\gamma) = n^{-\alpha}\overline{\gamma(x_n)}\chi_n(\gamma), \qquad \text{if } n \in \mathbb{N}.$$

Then we have for $n \in \mathbb{N}$, $(\phi_n)^\vee(x) = n^{-\alpha}(\Delta_{n+1} - \Delta_n)(x - x_n)$, so that $\|(\phi_n)^\vee\|_1 \leq 2n^{-\alpha}$. Hence,

$$\|\phi_n\|_{\mathcal{M}(p)} = \sup\{\|(\phi_n\hat{f})^\vee\|_p : \|f\|_p \leq 1\} \leq 2n^{-\alpha},$$

so that the sequence $(\phi_n)_{-\infty}^{\infty}$ satisfies conditions (a) and (b). Next, let $\beta = 1/2 - \alpha$ and for every $n \in \mathbb{N}$ choose a character $\gamma_n \in \Gamma_{n+1}\backslash\Gamma_n$. If we define the functions $h_n : G \to \mathbb{C}$ by $h_n(x) = n^{-\beta}\overline{\gamma_n(x - x_n)}\chi_{G_0}(x - x_n)$, then

$$(h_n)^\wedge(\gamma) = n^{-\beta}\gamma(x_n)\chi_{\Gamma_0}(\gamma - \gamma_n).$$

Let $\phi(\gamma) = \sum_{n=1}^{\infty} \phi_n(\gamma)$ and $h(x) = \sum_{n=1}^{\infty} h_n(x)$. Since $\operatorname{supp}(h_n) \subset x_n + G_0 \subset G_{-n}\backslash G_{-n+1}$, we see immediately that

$$\begin{aligned}\|h\|_p^p &= \int_G \left|\sum_{n=1}^{\infty} h_n(x)\right|^p dx \\ &\leq \int_G \sum_{n=1}^{\infty} |h_n(x)|^p dx \\ &\leq \sum_{n=1}^{\infty} n^{-\beta p} < \infty,\end{aligned}$$

because $\beta p = (1/2 - \alpha)p > 1$. Now we consider $f := (\phi\hat{h})^\vee$. First observe that

$$f(x) = \sum_{n=1}^{\infty} n^{-\alpha-\beta}\overline{\gamma_n(x)}\chi_{G_0}(x).$$

A straightforward computation in which we use the Littlewood–Paley theory for $\|f\|_p$ shows that

$$\begin{aligned}\|f\|_p &\geq C\left\|\left(\sum_{k=-\infty}^{\infty} |(\Delta_{k+1} - \Delta_k) * f(\cdot)|^2\right)^{1/2}\right\|_p \\ &= C\left\|\left(\sum_{k=1}^{\infty} |k^{-\alpha-\beta}\gamma_k(\cdot)\chi_{G_0}(\cdot)|^2\right)^{1/2}\right\|_p \\ &= C\left(\int_{G_0}\left(\sum_{k=1}^{\infty} |k^{-\alpha-\beta}|^2\right)^{p/2} dx\right)^{1/p} = \infty,\end{aligned}$$

because $2(\alpha + \beta) = 1$. Thus $h \in L^p(G)$ and $(\phi\hat{h})^\vee \notin L^p(G)$; consequently $\phi \notin \mathcal{M}_p(G)$. This completes the proof of Theorem 2.2.

Clearly, we can also formulate an analogue of Theorem 1.3 for Vilenkin groups. Such a result extends Theorem 2.4 in [4] from Hardy spaces $H^p(G)$, $0 < p \leq 1$, to the Lebesgue spaces $L^p(G)$, $1 < p < \infty$. We omit the trivial proof of the following theorem.

THEOREM 2.3. *Let $1 < p < \infty$. For each $n \in \mathbb{Z}$ let $\phi_n \in \mathcal{M}_p(G)$ and assume $(\phi_n)_{-\infty}^{\infty} \in \ell^q(\mathcal{M}(p))$ for some $q > s = |2p/(2-p)|$. Let $(\alpha_n) \in \ell^\beta(\mathbb{Z})$ for $\beta = qs/(q-s)$. If $\phi : \Gamma \to \mathbb{C}$ is defined by $\phi(\gamma) = \sum_{n=-\infty}^{\infty} \alpha_n \phi_n(\gamma)\chi_n(\gamma)$, then $\phi \in \mathcal{M}_p(G)$.*

Concluding remark. We have also considered multipliers on Hardy spaces $H^p(\mathbb{R}^n)$ and $H^p(G)$, $0 < p \leq 1$, where the assumption on the multiplier is expressed in terms of a mixed-norm condition. The techniques for proving multiplier theorems for Hardy spaces, however, are very different from the techniques used in this paper. For the spaces $H^p(G)$ our main result in this context is Corollary 2.3 in [4]. The results we have obtained for such multipliers on $H^p(\mathbb{R}^n)$ are presented in [5].

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