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Asymptotic behavior and zero distribution of Carleman orthogonal polynomials

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Abstract

Let L be an analytic Jordan curve and let $\{p_n(z)\}_{n=0}^{\infty}$ be the sequence of polynomials that are orthonormal with respect to the area measure over the interior of L. A well-known result of Carleman states that

$$\lim_{n\to\infty} \frac{p_n(z)}{\sqrt{(n+1)/\pi}\,[\phi(z)]^n} = \phi'(z) \tag{1}$$

locally uniformly on a certain open neighborhood of the closed exterior of L, where ϕ is the canonical conformal map of the exterior of L onto the exterior of the unit circle. In this paper we extend the validity of (1) to a maximal open set, every boundary point of which is an accumulation point of the zeros of the p_n's. Some consequences on the limiting distribution of the zeros are discussed, and the results are illustrated with two concrete examples and numerical computations.

Keywords: Orthogonal polynomials; Asymptotic behavior; Zeros of polynomials; Conformal maps

1. Introduction

Polynomials of a complex variable that are orthogonal over a bounded domain of the complex plane were first investigated by Carleman [3] in 1922, and considerable progress has been made since then in clarifying questions such as the convergence of Fourier series in these polynomials,

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their completeness in different Banach spaces of analytic functions, their asymptotic behavior and more recently the limiting distribution of their zeros (see, e.g., the monograph by Suetin [15] and the references therein, together with the papers [1,2,5,7,9–12]). Generally speaking, all these questions are dependent of the boundary properties of the orthogonality domain, and in the present paper we specifically consider the case of a domain with analytic boundary, having as the subject of our investigation the asymptotic behavior and zero distribution of the corresponding orthogonal polynomials.

Let L_1 be an analytic Jordan curve in the complex plane $\mathbb{C}$ and let G_1 be its interior domain, that is, the bounded component of $\mathbb{C} \setminus L_1$. By applying the Gram–Schmidt orthonormalization process to the sequence $1, z, z^2, \ldots,$ we can construct a unique sequence of complex analytic polynomials $\{p_n(z)\}_{n=0}^{\infty}$ (each p_n having degree n and positive leading coefficient) that are orthonormal over G_1 with respect to the normalized area measure $\pi^{-1}\mathrm{d}x\mathrm{d}y$, that is, satisfying

$$\frac{1}{\pi}\int_{G_1} p_n(z)\overline{p_m(z)}\mathrm{d}x\mathrm{d}y = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \tag{2}$$

These polynomials were first examined by T. Carleman in his study [3] on the approximation of analytic functions by polynomials over a bounded Jordan domain. In particular, Carleman investigated the behavior of $p_n(z)$ as $n \to \infty$, finding a fundamental result that we state below after setting some needed notation.

For a planar set K and a function f defined on K, $\overline{K}$ and ∂K denote, respectively, the closure and the boundary of K in the extended complex plane $\overline{\mathbb{C}}$, and $f(K) := \{f(z) : z \in K\}$.

Given $r \geq 0$, we set

$$\mathbb{T}_r := \{w : |w| = r\}, \qquad \Delta_r := \{w : r < |w| \leq \infty\}, \qquad \mathbb{D}_r := \{w : |w| < r\}.$$

Let Ω_1 be the unbounded component of $\overline{\mathbb{C}} \setminus L_1$, and let $\psi(w)$ be the unique conformal map of Δ_1 onto Ω_1 satisfying that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. Let $\rho \geq 0$ be the radius of univalency of ψ, that is, the smallest number such that ψ has an analytic and *univalent* continuation to $\{w : \rho < |w| < \infty\}$. Because L_1 is an analytic Jordan curve, $\rho < 1$. For every $\rho \leq r < \infty$, set

$$\Omega_r := \psi(\Delta_r), \qquad L_r := \partial\Omega_r, \qquad G_r := \mathbb{C} \setminus \overline{\Omega_r}, \tag{3}$$

and let

$$\phi(z) : \Omega_\rho \to \Delta_\rho$$

be the inverse of ψ. Observe that for $r > \rho$, L_r is an analytic Jordan curve.

Carleman's fundamental result mentioned above ([3, Satz IV], see also [6, Sec. 2]) states that

$$h_n(z) := \frac{p_n(z)}{\sqrt{n+1}[\phi(z)]^n} - \phi'(z) = o(1) \tag{4}$$

locally uniformly on Ω_ρ as $n \to \infty$. More precisely, Carleman proved that $h_n(z)$ converges uniformly as $n \to \infty$ to zero on each $\overline{\Omega_r}$, $r > \rho$, with the rate

$$h_n(z) = \begin{cases} O\left(\sqrt{n}\rho^n\right), & r \geq 1, \\ O\left(n^{-1/2}(\rho/r)^n\right), & \rho < r < 1. \end{cases} \tag{5}$$

Progress in understanding the behavior of p_n in $\mathbb{C} \setminus \Omega_\rho$ has been recently made in [11], where the following asymptotic integral representation for p_n has been obtained. If $\varphi(z)$ is a conformal map of G_1 onto the unit disk, then φ has an analytic continuation across L_1, so that the composition $\varphi(\psi(w))$ is well-defined and analytic on the unit circle $\mathbb{T}_1$, and we have (see [11, Theorem 2.1.2 and Eq. (14)])

$$p_n(z) = \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_1} \frac{w^n dw}{\varphi(\psi(w)) - \varphi(z)} + \epsilon_n(z), \quad z \in G_1, \ n \geq 0, \tag{6}$$

where the functions $\epsilon_n(z)$ are analytic in $G_{1/\rho}$ and have the following property: if $E \subset G_{1/\rho}$ is such that for some $0 < \tau < 1/\rho$,

$$p_n(z) = O\left(\sqrt{n}\tau^n\right)$$

uniformly on E as $n \to \infty$, then

$$\epsilon_n(z) = O\left(\sqrt{n}(\tau\rho)^n\right)$$

uniformly on E as $n \to \infty$.

This representation is used in [11] to derive finer asymptotics for p_n and its zeros under the assumption that (roughly speaking) the boundary of Ω_ρ is a piecewise analytic curve. As a little bonus, one also obtains from (6) (see [11, Corollary 2.1.3]) that the $\sqrt{n}$ factor occurring in (5) for the case $r \geq 1$ can be dropped.

In the present paper we exploit (6) to extend the validity of Carleman's formula (4) from the band $\Omega_\rho \cap G_1$ toward a maximal open subset Σ_1 of G_1 that is, in general, larger than $\Omega_\rho \cap G_1$. Σ_1 is the largest open subset of G_1 where a strong asymptotic formula like (4) holds true, and every point of $\partial\Sigma_1 \cap G_1$ is an accumulation point of the zeros of the p_n's.

These results are stated in Section 2 as Theorems 1 and 2. Some consequences on the limiting distribution of the normalized counting measures of the zeros of the p_n's are presented as Theorem 3. The definition of Σ_1 and its finding in concrete situations involves the meromorphic continuation of the map $\varphi(\psi(w))$ occurring inside the integrand in (6). We study such a continuation in Propositions 4–6 of Section 2. In Section 3, we discuss two concrete examples to illustrate the main results and the use of the propositions, and the proofs of the results are presented in Sections 4 and 5.

2. Main results

Let φ be a conformal map of G_1 onto $\mathbb{D}_1$. Because L_1 is a Jordan curve, φ can be extended as a continuous and bijective function $\varphi : \overline{G_1} \to \overline{\mathbb{D}}_1$. Moreover, L_1 being analytic, φ has a one-to-one meromorphic continuation to $G_{1/\rho}$, which satisfies

$$\varphi(z) = \frac{1}{\overline{\varphi(z^*)}}, \quad z \in \Omega_\rho \cap G_{1/\rho}, \tag{7}$$

where

$$z^* := \psi\left(1/\overline{\phi(z)}\right) \tag{8}$$

is the Schwarz reflection about L_1 of the point $z \in \Omega_\rho \cap G_{1/\rho}$ (see [4] for details).

The function ψ is analytic and univalent on Δ_ρ, mapping the annulus $\rho < |w| < 1/\rho$ conformally onto the band $\Omega_\rho \cap G_{1/\rho}$, so that

$$\varphi(\psi(w)), \quad \rho < |w| < 1/\rho,$$

is a one-to-one meromorphic function that is analytic on $\rho < |w| \leq 1$.

Definition 1. Let $\mu \geq 0$ be the smallest number such that $\varphi(\psi(w))$ has a *meromorphic* continuation, denoted by $h_\varphi(w)$, to the annulus

$$\{w : \mu < |w| < 1/\rho\}.$$

Let Σ be the set of those points $z \in G_1$ for which the equation

$$h_\varphi(w) = \varphi(z) \tag{9}$$

has at least one solution in $\mu < |w| < 1$. Let $\Sigma_0 := G_1 \setminus \Sigma$.

We say that a solution ω of (9) has multiplicity $\alpha \geq 1$ if

$$h_\varphi^{(\alpha)}(\omega) \neq 0, \qquad h_\varphi^{(j)}(\omega) = 0 \quad 1 \leq j < \alpha.$$

Consider a $z \in \Sigma$. Since $h_\varphi(w)$ is one-to-one on $\rho < |w| < 1$, among the solutions to (9), a finite number, say $\omega_{z,1}, \ldots, \omega_{z,s}$ $(s \geq 1)$, will have largest modulus. Let $\alpha_{z,k}$ denote the multiplicity of $\omega_{z,k}$ $(1 \leq k \leq s)$. We decompose Σ in subsets Σ_p, $p = 1, 2, \ldots,$ defined by the relation

$$z \in \Sigma_p \Leftrightarrow \alpha_{z,1} + \cdots + \alpha_{z,s} = p. \tag{10}$$

Thus, Σ_1 consists of those points $z \in \Sigma$ such that Eq. (9) has exactly one solution in $\mu < |w| < 1$ of largest modulus, and this solution is simple.

Let the function $\Phi : \Sigma_1 \to \{w : \mu < |w| < 1\}$ be defined as

$$\Phi(z) := \omega_{z,1}, \quad z \in \Sigma_1,$$

and let $r : \mathbb{C} \to [\mu, \infty)$ be defined as

$$r(z) := \begin{cases} |\phi(z)|, & z \in \overline{\Omega}_1, \\ |\omega_{z,1}|, & z \in \Sigma, \\ \mu, & z \in \Sigma_0. \end{cases} \tag{11}$$

It is easy to see (see the first two paragraphs of Section 4) that the number μ, the sets Σ, Σ_p, and the functions $\Phi(z)$, $r(z)$ are, indeed, independent of the choice of the interior map φ. Also (see Corollary 12 in Section 4) Σ and Σ_1 are open, $\Sigma_1 \supset \Omega_\rho \cap G_1$, the map Φ is a *one-to-one* analytic function and $r(z)$ is continuous.

Note that

$$\Phi(z) = \phi(z), \quad z \in \Omega_\rho \cap G_1,$$

and that $r(z) = |\Phi(z)|$ for all $z \in \Sigma_1$. Our main result is the following theorem.

Theorem 1. (a) *For every compact set $E \subset \Sigma_1$, there exists a number $0 < \delta < 1$ such that*

$$\frac{p_n(z)}{\sqrt{n+1}[\Phi(z)]^n} - \Phi'(z) = O(\delta^n)$$

uniformly on E as $n \to \infty$.

(b)

$$\limsup_{n\to\infty} |p_n(z)|^{1/n} = r(z), \quad z \in G_1. \tag{12}$$

This result has several implications on the asymptotic zero distribution of the orthogonal polynomials. Consider the set $\mathcal{Z}$ of accumulation points of the zeros of the p_n's, that is, $\mathcal{Z}$ consists of those points $t \in \overline{\mathbb{C}}$ such that every neighborhood of t contains zeros of infinitely many polynomials p_n. A simple consequence of (4) and Theorem 1(a) is that every closed subset of $\overline{\Omega}_1 \cup \Sigma_1$ may contain zeros of at most finitely many polynomials p_n, and therefore, $\mathcal{Z} \subset G_1 \setminus \Sigma_1$. Moreover, we have

Theorem 2. $\partial \Sigma_1 \cap G_1 \subset \mathcal{Z}$.

Let now $z_{n,1}, \dots, z_{n,n}$ be the zeros of p_n, let δ_z denote the unit point mass at z, and let

$$\nu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_{n,j}}$$

be the so-called normalized counting measure of the zeros of p_n. The sequence $\{\nu_n\}_{n=1}^{\infty}$ is said to converge in the weak*-topology to the measure ν if

$$\lim_{n\to\infty} \int_{\mathbb{C}} f \mathrm{d}\nu_n = \int_{\mathbb{C}} f \mathrm{d}\nu$$

for every continuous function $f : \mathbb{C} \to \mathbb{C}$.

In preparation for the next theorem concerning the weak*-limit points of the sequence $\{\nu_n\}$, we recall that the logarithmic potential of a compactly supported measure ν is the superharmonic function

$$U^{\nu}(z) := -\int_{\mathbb{C}} \log|t - z| \mathrm{d}\nu(t), \quad z \in \overline{\mathbb{C}}.$$

Theorem 3. *There exists a unique measure λ having logarithmic potential*

$$U^{\lambda}(z) = -\log r(z) + \log \phi'(\infty), \quad z \in \mathbb{C}. \tag{13}$$

This λ is a probability measure whose support coincides with $\partial \Sigma_1 \cap G_1$. If the interior of Σ_0 is connected, then some subsequence of $\{\nu_n\}_{n=1}^{\infty}$ converges in the weak-topology to λ, and this is true of the entire sequence $\{\nu_n\}_{n=1}^{\infty}$ if the interior of Σ_0 is empty.*

Remark 1. The recent paper [7] investigates polynomials $q_n(z)$, $n = 0, 1, \dots,$ that are orthonormal with respect to area measure over a set G (briefly called an archipelago) that is a finite union of bounded Jordan domains with mutually disjoint closures. An important role in their study is played by a function $h(z)$ constructed out of the Green function $g_\Omega(z, \infty)$ of $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ with pole at ∞ and the reproducing kernel $K(\xi, z)$ of the Hilbert space of analytic functions in G that are square integrable with respect to area measure. Namely, the function

$$h(z) := \begin{cases} g_\Omega(z, \infty), & z \in \overline{\Omega}, \\ -\log R(z), & z \in G, \end{cases}$$

where $R(z)$ is the supremum of those numbers $R \geq 1$ such that $K(\cdot, z)$ has an analytic continuation to $\mathbb{C} \setminus \{z \in \overline{\Omega} : g_\Omega(z, \infty) \geq \log R\}$.

Theorem 6.1 of [7] establishes the connection between $h(z)$ and the nth root/zero asymptotic behavior of the q_n's. Briefly, that

$$\limsup_{n\to\infty} \frac{1}{n} \log|q_n(z)| = h(z), \quad z \in \mathbb{C}, \tag{14}$$

with equality holding true if lim sup is replaced by lim and z lies outside the convex hull of $\overline{G}$, and that there exists a unique positive probability measure β with support contained in $\overline{G}$, whose logarithmic potential $U^{\beta}(z)$ satisfies

$$U^{\beta}(z) = \log \frac{1}{\text{cap}(\partial G)} - h(z), \quad z \in \mathbb{C},$$

(here $\text{cap}(\partial G)$ is the logarithmic capacity of the compact set ∂G). Moreover, sufficient conditions are given for this measure β to be a weak*-limit point of the normalized counting measures of the q_n's.

In the case treated in the present paper of a single-island archipelago $G = G_1$ with analytic boundary, we have $g_{\Omega}(z, \infty) = \log |\phi(z)|$, $\text{cap}(\partial G) = 1/\phi'(\infty)$, and

$$K(\xi, z) = \frac{\overline{\varphi'(z)}\varphi'(\xi)}{\pi[1 - \overline{\varphi(z)}\varphi(\xi)]^2}.$$

Comparing (14) with (12), we see that Theorem 1 carries a statement about the location of the nearest singularities of $K(\cdot, z)$, namely, that $R(z) = 1/r(z)$ for $z \in G_1$, a fact that can be verified directly with the help of Proposition 4 below. One can also deduce from that proposition that if $z \in \Sigma_p$, $p \geq 1$, then $K(\cdot, z)$ has finitely many singularities on $L_{1/r(z)}$, all of which are poles whose total multiplicity is p, while if $z \in \Sigma_0$, then $K(\cdot, z)$ has a singularity other than a pole on $L_{1/\mu}$.

Also, we see that the measures β and λ are one and the same, thus Theorem 3 provides essential information about the location and structure of the support of β (by identifying it with $\partial \Sigma_1 \cap G_1$) and refines Part (vii) of Theorem 6.1 of [7] for the case of a single-island archipelago with analytic boundary.

Remark 2. Given that, by definition, $h_{\varphi}(w) = \varphi(\psi(w))$ for $\rho < |w| < 1/\rho$, it is easy to verify that

$$\mu = \rho \Leftrightarrow G_1 = \Sigma_1 \cup \Sigma_0 \Leftrightarrow \Sigma_0 = \mathbb{C} \setminus \Omega_{\rho}.$$

Hence if $G_1 = \Sigma_1 \cup \Sigma_0$ and $\Sigma_0 = \{z_0\}$ is a singleton, then $\rho = 0$ and the set G_1 is an open disk centered at z_0, say $G_1 = \{z : |z - z_0| < s\}$ for some $s > 0$. In this case,

$$p_n(z) = \sqrt{n+1} s^{-n-1} (z - z_0)^n, \quad n \geq 0,$$

as can be verified directly from the orthogonality relations (2), so that $\nu_n = \delta_{z_0} = \lambda$, $n \geq 1$.

If $G_1 = \Sigma_1 \cup \Sigma_0$, Σ_0 not a singleton, then $\rho > 0$, $\text{supp}(\lambda) = \partial \Sigma_1 \cap G_1 = \partial \Sigma_0$ and by (13), $U^{\lambda}(z)$ is (a finite) constant on Σ_0. Hence λ is the equilibrium distribution with respect to the logarithmic potential of the compact set Σ_0 (cf. Section III.2 and Theorem III.15 of [16]).

The proofs of Theorems 2 and 3 can be accomplished by using a series of arguments previously developed by Ullman [17] and Kuijlaars and Saff [8] in the context of Faber polynomials. These arguments, which make essential use of several structural properties of the Σ_p sets, are of a very general nature and can be extended to our setting without any essential modification. We shall therefore provide only an outline of these proofs at the end of Section 4.

For concrete instances of a curve L_1, the difficulty of finding the corresponding number μ and set Σ_1 may be reduced with the use of the following three propositions. These establish some properties of the meromorphic continuation of the map $\varphi(\psi(w))$. Their use is illustrated in the examples of the next section.

For a domain $\mathfrak{D} \subset \overline{\mathbb{C}}$, we denote by $\mathfrak{D}^*$ the reflection of $\mathfrak{D}$ about the unit circle, i.e.,

$$\mathfrak{D}^* := \{1/\overline{w} : w \in \mathfrak{D}\}.$$

Proposition 4. *Let φ be a conformal map of G_1 onto $\mathbb{D}_1$ and let us denote by the same letter φ its meromorphic continuation to $G_{1/\rho}$. Let $\mathfrak{D}$ be a domain such that $\{w : \rho < |w| < 1\} \subset \mathfrak{D} \subset \mathbb{D}_1$.*

The function $\varphi(\psi(w))$, originally defined in $\rho < |w| < 1/\rho$, has a meromorphic continuation to $\mathfrak{D}$, if and only if it has a meromorphic continuation to $\mathfrak{D}^*$, if and only if $\varphi(z)$ has a meromorphic continuation to $\overline{G}_1 \cup \psi(\mathfrak{D}^*)$. If $h_\varphi(w)$ denotes the meromorphic continuation of $\varphi(\psi(w))$ to $\mathfrak{D} \cup \mathbb{T}_1 \cup \mathfrak{D}^*$, then

$$h_\varphi(w) = \frac{1}{\overline{h_\varphi(1/\overline{w})}} \tag{15}$$

for all $w \in \mathfrak{D} \cup \mathbb{T}_1 \cup \mathfrak{D}^*$. In particular, for μ as in Definition 1, we have

$$\begin{aligned}
&\sup\{r \geq 1/\rho : \varphi(\psi(w)) \text{ has a meromorphic continuation to } \rho < |w| < r\} \\
&\quad = \sup\{r \geq 1/\rho : \varphi \text{ has a meromorphic continuation to } G_r\} \\
&\quad = 1/\mu.
\end{aligned}$$

The next proposition tells us that if $\mathfrak{D} \subset \mathbb{D}_1$ is a domain that can be exhausted by continuously expanding domains $\mathfrak{D}_t$, each satisfying that $\psi(\overline{\mathfrak{D}}_t) \subset \overline{G}_1 \cup \psi(\mathfrak{D}_t^*)$, then $\varphi(\psi(w))$ has a meromorphic continuation to $\mathfrak{D}$. The precise formulation is as follows.

Proposition 5. *Let $\{\mathfrak{D}_t : a \leq t < b\}$ be a collection of domains such that for every $a \leq t_0 < t_1 < b$,*

$$\{w : \rho < |w| < 1\} \subset \mathfrak{D}_{t_0} \subset \mathfrak{D}_{t_1} \subset \mathbb{D}_1, \qquad \bigcap_{t > t_0} \mathfrak{D}_t = \overline{\mathfrak{D}}_{t_0} \setminus \mathbb{T}_1. \tag{16}$$

Let $\mathfrak{D} := \cup_{a \leq t < b} \mathfrak{D}_t$ and suppose that ψ is meromorphic in $\mathfrak{D}$ and satisfies

$$\psi(\mathfrak{D}_a) \subset G_1, \qquad \psi(\overline{\mathfrak{D}}_t) \subset \overline{G}_1 \cup \psi(\mathfrak{D}_t^*), \quad a < t < b. \tag{17}$$

Then, $\varphi(\psi(w))$ admits a meromorphic continuation to $\mathfrak{D}$.

Remark 3. Note that for a domain $\mathfrak{D}$ as in Proposition 5, the meromorphic continuation of $\varphi(\psi(w))$ to $\mathfrak{D} \cup \mathbb{T}_1 \cup \mathfrak{D}^*$ is likewise the composition of two meromorphic functions, since by Proposition 4, φ is meromorphic in $\overline{G}_1 \cup \psi(\mathfrak{D}^*)$ and obviously $\psi(\mathfrak{D} \cup \mathbb{T}_1 \cup \mathfrak{D}^*) \subset \overline{G}_1 \cup \psi(\mathfrak{D}^*)$.

Let $\rho_a \geq 0$ be the smallest number such that ψ has an analytic continuation to $\rho_a < |w| < \infty$, and let $\bar{\mu} \in [\rho_a, 1)$ be a number that has been fixed. Suppose $z \in G_1$ and that the equation $\psi(w) = z$ has no solutions in $\bar{\mu} < |w| < 1$. In this case we assign $z \in C_0^{\bar{\mu}}$. Otherwise, the equation $\psi(w) = z$ has finitely many solutions of largest modulus, say $v_{z,1}, \ldots, v_{z,s}$ $(s \geq 1)$, in $\bar{\mu} < |w| < 1$. Let $\beta_{z,k}$ be the multiplicity of ψ at $v_{z,k}$.

For every integer $p \geq 1$, we define the subset $C_p^{\bar{\mu}}$ of G_1 by the relation

$$z \in C_p^{\bar{\mu}} \Leftrightarrow \beta_{z,1} + \cdots + \beta_{z,s} = p.$$

Finally, for $r \in [\rho_a, \infty)$, we define

$$L_r := \{z = \psi(w) : |w| = r\}.$$

Note that for $r > \rho$, this definition of L_r is equivalent to that given in (3).

Proposition 6. *Suppose $\bar{\mu}$ is a number satisfying that $\rho_a \leq \bar{\mu} < \rho$ and having the property that $L_r \subset G_{1/r}$ for all $\bar{\mu} < r < 1$. Then, $\mu \leq \bar{\mu}$, $\Sigma_p \supset C_p^{\bar{\mu}}$ for all $p \geq 1$, and*

$$\Phi(z) = v_{z,1}, \quad z \in C_1^{\bar{\mu}}.$$

Moreover, if $\mu = \bar{\mu}$, then $\Sigma_p = C_p^{\bar{\mu}}$ for every integer $p \geq 0$.

Suppose, in addition, that there is a sequence $\{w_n\}_{n\geq 1}$, $\bar{\mu} < |w_n| < 1$, such that

$$\psi(w_{n+1}) = \psi(1/\overline{w_n}), \quad n \geq 1. \tag{18}$$

Then, $|w_n| > |w_{n+1}|$ and $\mu = \bar{\mu} = \lim_{n\to\infty} |w_n|$.

3. Examples

Two well-known sequences of polynomials orthogonal over the interior of an analytic Jordan curve are those corresponding to L_1 a circle and L_1 an ellipse. In both instances, the orthogonal polynomials can be computed explicitly. The case of L_1 a circle is quite trivial and has been already discussed in Remark 2 above. When L_1 is an ellipse with foci at -1 and 1, p_n is, up to a multiplicative constant, the nth Tchebichef polynomial of the second kind (see, e.g., [13, pp. 258–259]).

These examples are, however, of little relevance to us because in both of them $\Sigma_1 = \Omega_\rho \cap G_1$, so that Theorem 1(a) reduces to the original result (4) of Carleman. We now provide two examples in which Σ_1 is actually larger than $\Omega_\rho \cap G_1$. In particular, we shall see that the inequalities $\rho_a < \mu < \rho$ and $\mu < \rho_a$ are both possible.

3.1. Cassini ovals

Let $0 < R < 1$ be a number that has been fixed. The set $|z^2 - 1| = R$ consists of two disjoint analytic Jordan curves known as Cassini ovals. One surrounds 1, the other -1. Of these two, let us denote by L_1 the one encircling the point 1.

Observe that the function

$$\varphi(z) := (z^2 - 1)/R$$

conformally maps G_1 (the interior of L_1) onto the unit disk. Given that φ is an entire function, we have in view of Proposition 4 that

$$\mu = 0.$$

Proposition 7. *Let a be the unique solution that the equation*

$$27x^4 - 18x^2 - 4\left(R + R^{-1}\right)x - 1 = 0$$

has in the interval $(-1/3, 0)$. Then

$$h_\varphi(w) = \frac{(1 - aw)w^2}{w - a}, \quad w \in \overline{\mathbb{C}}, \tag{19}$$

$$\psi(w) = \sqrt{1 + \frac{R(1 - aw)w^2}{w - a}}, \quad |w| > 1, \tag{20}$$

where the branch of the square root in (20) is chosen so that $\psi(1/a) = -1$.

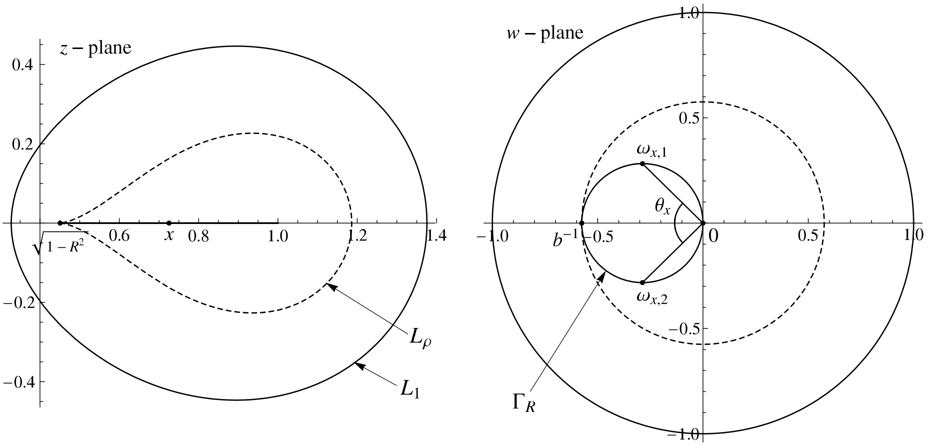

Fig. 1. Curve Γ_R for $a = -.26$ ($R \approx 0.8926$, $\sqrt{1-R^2} \approx 0.4506$).

We shall see during the proof of this proposition that

$$w - a + R(1 - aw)w^2 = -aR(w - b)^2(w - c), \tag{21}$$

with

$$b = \frac{\frac{3a^2+1}{2a} - \sqrt{\left(\frac{3a^2+1}{2a}\right)^2 - 4}}{2}, \qquad c = \frac{1-3a^2}{2a} + \sqrt{\left(\frac{3a^2+1}{2a}\right)^2 - 4}, \tag{22}$$

$$-1 < 1/b < c < a < 0, \tag{23}$$

so that ψ admits an analytic continuation to $\mathbb{C} \setminus [c, a]$ given by

$$\psi(w) = \sqrt{-aR}(w - b)\sqrt{\frac{w-c}{w-a}}, \quad z \in \mathbb{C} \setminus [c, a].$$

Moreover, $\rho = |b|^{-1}$, and so we have

$$0 = \mu < \rho_a = |c| < \rho = |b|^{-1}. \tag{24}$$

It is not difficult to verify that the set

$$\Gamma_R := \left\{ w \in \mathbb{D}_1 : -R \le h_\varphi(w) \le 0 \right\}$$

is an analytic Jordan curve symmetric with respect to the real axis, intersecting it at $1/b$ and 0. The function $h_\varphi(w)$ maps $\Gamma_R \cap \{z : \Im(z) > 0\}$ onto $(-R, 0)$ in an injective way. Hence for every $x \in (\sqrt{1-R^2}, 1)$, the equation

$$h_\varphi(w) = \varphi(x)$$

has exactly two solutions in Γ_R, say $\omega_{x,1}$, $\omega_{x,2}$. These are distinct, and $\omega_{x,1} = \overline{\omega_{x,2}}$.

Let $0 < \theta_x < \pi$ be the angle formed by the two rays emanating from 0 and passing through $\omega_{x,1}$, $\omega_{x,2}$ (see Fig. 1). Recall that ν_n denotes the normalized counting measure of the zeros of p_n.

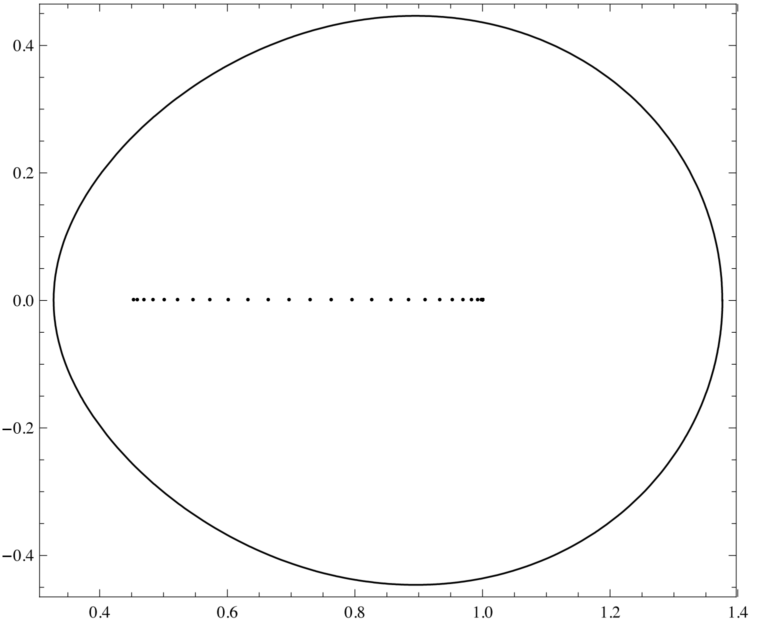

Fig. 2. Zeros of $p_{50}(z)$ for $a = -.26$ ($R \approx 0.8926$, $\sqrt{1-R^2} \approx 0.4506$).

Theorem 8.

$$G_1 = \Sigma_1 \cup \Sigma_2, \quad \Sigma_2 = \left[\sqrt{1-R^2}, 1\right],$$

and $\{\nu_n\}_{n=1}^{\infty}$ *converges in the weak*-topology to* $\sigma + \delta_1/2$*, where* σ *is the measure supported on* $[\sqrt{1-R^2}, 1]$ *with distribution function*

$$\sigma\left(\left[\sqrt{1-R^2}, x\right]\right) = \frac{\theta_x}{2\pi}, \quad \sqrt{1-R^2} \le x \le 1, \tag{25}$$

and δ_1 *is the unit point mass at 1.*

Thus, in this example $\Sigma_0 = \Sigma_p = \emptyset$ for all $p > 2$. The asymptotic formula of Theorem 1(a) holds with $\Phi(z)$ the algebraic function analytic in $\mathbb{C} \setminus [\sqrt{1-R^2}, 1]$ that is the solution of the equation

$$R(1-aw)w^2 - (z^2-1)(w-a) = 0, \qquad \Phi(-1) = 1/a.$$

In Fig. 2, we have plotted the zeros of the polynomial p_{50}. They all seem to lie in the segment $[\sqrt{1-R^2}, 1]$, and only 26 of them show up. This is corroborated by the following theorem, which we derive directly from the orthogonality property of the p_n's.

Theorem 9. *For every integer* $n \ge 0$*,*

$$p_n(z) = (z-1)^{\lfloor n/2 \rfloor} q_n(z),$$

where $q_n(z)$ *is a polynomial with* $n - \lfloor n/2 \rfloor$ *simple roots, all lying in* $(\sqrt{1-R^2}, 1)$.

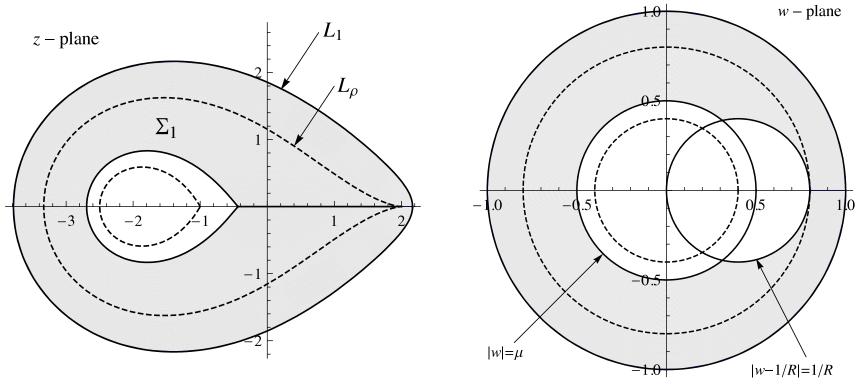

Fig. 3. Sets Σ_1, Σ_2 and Σ_0 for a curve L_1 defined as in (26) for $R = 2.5$.

3.2. Level curves of the inverse of a shifted Joukowsky transformation

Let $R > 2$ be fixed, and set

$$L_1 := \left\{ w - 1 + (w-1)^{-1} : |w| = R \right\}. \tag{26}$$

From very well-known properties of the Joukowsky transformation $u \mapsto u + 1/u$, it follows that L_1 is an analytic Jordan curve, with

$$\psi(w) = Rw - 1 + \frac{1}{Rw - 1}, \qquad z \in \overline{\mathbb{C}}, \tag{27}$$

mapping Δ_1 conformally onto the exterior Ω_1 of L_1. Moreover, ψ maps both $\{w : |w - 1/R| > 1/R\}$ and $\{w : |w - 1/R| < 1/R\}$ conformally onto $\overline{\mathbb{C}} \setminus [-2, 2]$, and for every $z \in \mathbb{C}$, the two solutions of the equation $z = \psi(w)$ are

$$v_{z,1} = \frac{z + 2 + \sqrt{z^2 - 4}}{2R}, \qquad v_{z,2} = \frac{z + 2 - \sqrt{z^2 - 4}}{2R}. \tag{28}$$

Note that $v_{z,1}$ and $\overline{v_{z,2}}$ are reflections of each other about the circle $|w - 1/R| = 1/R$, and that if we choose the branch of the square root in (28) that is positive along $(2, \infty)$, then $|v_{z,1}| > |v_{z,2}|$ for every $z \in \mathbb{C} \setminus [-2, 2]$, with $v_{z,1}$ and $v_{z,2}$ lying, respectively, outside and inside the circle $|w - 1/R| = 1/R$.

We appeal to Proposition 6 and find

Theorem 10.

$$\mu = \frac{R - \sqrt{R^2 - 4}}{2}$$

and $G_1 = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, *where* Σ_1 *is the image by* ψ *of the set of points in* $\mathbb{D}_1$ *that lie exterior to both circles* $|w| = \mu$ *and* $|w - 1/R| = 1/R$, *and* $\Sigma_2 = (R^2\mu^2 - 2, 2]$ *(see Fig. 3).*

Given that Σ_1 is connected, the function Φ is nothing but the analytic continuation of $\phi(z) = (z+2+\sqrt{z^2-4})/(2R)$, and so it follows from Carleman's formula (4) and Theorem 1(a)

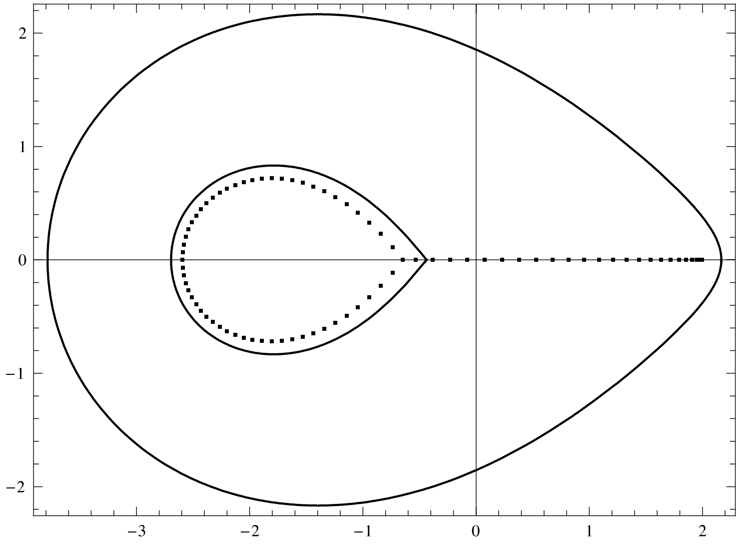

Fig. 4. Zeros of $p_{80}(z)$ for L_1 as in (26) for $R = 2.5$.

that

$$\lim_{n\to\infty} \frac{(2R)^{n+1} p_n(z)}{\sqrt{n+1}\left[z+2+\sqrt{z^2-4}\right]^n} = \frac{z+\sqrt{z^2-4}}{\sqrt{z^2-4}}$$

locally uniformly in $\Sigma_1 \cup \overline{\Omega}_1$. Theorem 2 gives us that every point of $\partial \Sigma_1 \cap G_1$ attracts zeros of the p_n's, a fact illustrated in Fig. 4. Also, being the interior of Σ_0 connected, Theorem 3 guarantees that a subsequent of the normalized counting measures ν_n converges in the weak*-topology to the measure λ.

Observe that, unlike the previous example in which we had $\mu < \rho_a$ (see (24)), we now have

$$\frac{1}{R} = \rho_a < \mu < \rho = \frac{2}{R}.$$

4. Proofs of the results of Section 2

We begin by recalling that if φ_1 and φ are conformal maps of G_1 onto $\mathbb{D}_1$, then they are related through a Möbius transformation, that is, for all $z \in G_1$,

$$\varphi_1(z) = e^{i\theta} \frac{\varphi(z) - \varphi(z_0)}{1 - \overline{\varphi(z_0)}\varphi(z)}, \qquad \varphi(z) = \frac{\varphi_1(z) + e^{i\theta}\varphi(z_0)}{e^{i\theta} + \overline{\varphi(z_0)}\varphi_1(z)},$$

where $z_0 \in G_1$ is such that $\varphi_1(z_0) = 0$, and $\theta = \arg\ (\varphi_1'(z_0)/\varphi'(z_0))$.

Hence $\varphi(\psi(w))$ has a meromorphic continuation to the annulus $\mu < |w| < 1$ if and only if so does $\varphi_1(\psi(w))$. Moreover, the meromorphic continuations $h_\varphi(w)$ and $h_{\varphi_1}(w)$ satisfy

$$h_{\varphi_1}(w) - \varphi_1(z) = \frac{e^{i\theta}\left(1 - |\varphi(z_0)|^2\right)}{1 - \overline{\varphi(z_0)}\varphi(z)} \cdot \frac{h_\varphi(w) - \varphi(z)}{1 - \overline{\varphi(z_0)}h_\varphi(w)},$$

so that for every $z \in G_1$, the equations (in the unknown w) $h_{\varphi_1}(w) = \varphi_1(z)$ and $h_\varphi(w) = \varphi(z)$ share the same solutions, multiplicities included. Therefore, the number μ, the sets Σ, Σ_p, and the functions $\Phi(z)$, $r(z)$ as defined in Section 2, are independent of the choice of the interior conformal map φ.

Hereafter φ is a conformal map of G_1 onto $\mathbb{D}_1$ that has been fixed. We shall employ the notation

$$D_\epsilon(t) := \{z : |z - t| < \epsilon\}, \qquad D^*_\epsilon(t) := D_\epsilon(t) \setminus \{t\}.$$

As in Section 2 above, for $z \in \Sigma$, we denote by $\omega_{z,1}, \ldots, \omega_{z,s}$ the solutions to the equation

$$h_\varphi(w) = \varphi(z) \tag{29}$$

in the annulus $\mu < |w| < 1$ that have largest modulus. The multiplicity of $h_\varphi(w)$ at $\omega_{z,k}$ is denoted by $\alpha_{z,k}$ and Σ_p, $p \geq 1$ is defined by relation (10).

Lemma 11. *Let $z \in \Sigma_p$, $p \geq 1$ and let μ' with $\mu < \mu' < r(z)$ be a number satisfying that $h_\varphi(w)$ has no poles on $\mathbb{T}_{\mu'}$ and that the only solutions to (29) that lie in $\mu' \leq |w| < 1$ are precisely those of largest modulus $\omega_{z,1}, \ldots, \omega_{z,s}$ $(1 \leq s \leq p)$. Let $\delta > 0$ be so small that the closed disks $\overline{D_\delta(\omega_{z,k})}$, $1 \leq k \leq s$, are pairwise disjoint and contained in the annulus $\mu' < |w| < 1$, that $h_\varphi(w)$ has no poles on $\cup_{k=1}^s \overline{D_\delta(\omega_{z,k})}$ and that*

$$h'_\varphi(w) \neq 0, \quad w \in D^*_\delta(\omega_{z,k}),\ 1 \leq k \leq s. \tag{30}$$

There exists $\epsilon > 0$ such that if $0 < |\zeta - z| \leq \epsilon$, then the solutions to the equation

$$h_\varphi(w) = \varphi(\zeta)$$

that lie in $\mu' \leq |w| \leq 1$ are simple and contained in $\cup_{k=1}^s D_\delta(\omega_{z,k})$, and each disk $D_\delta(\omega_{z,k})$ contains exactly $\alpha_{z,k}$ solutions.

Proof. Suppose z, μ' and δ are as in the hypothesis of Lemma 11. Then, for

$$K := \{w : \mu' \leq |w| \leq 1\} \setminus \cup_{k=1}^s D_\delta(\omega_{z,k}), \tag{31}$$

we have that $m := \min_{w \in K} |h_\varphi(w) - \varphi(z)| > 0$. Select $\epsilon > 0$ such that $\overline{D_\epsilon(z)} \subset G_1$ and $|\varphi(\zeta) - \varphi(z)| < m$ for all $\zeta \in \overline{D_\epsilon(z)}$. Then, for this ϵ, the conclusion of Lemma 11 follows from Rouche's theorem and (30). □

Corollary 12. *Both Σ and Σ_1 are open, the function $\Phi : z \mapsto \omega_{z,1}$ is analytic and univalent on Σ_1, and the function $r(z)$ is continuous on G_1.*

Proof. That Σ and Σ_1 are open is a clear consequence of Lemma 11, as well as the fact that $\Phi(z)$ is locally the inverse of $\varphi^{-1}\left(h_\varphi(w)\right)$. Therefore, Φ is analytic in Σ_1, and given that $\varphi(z) = h_\varphi(\Phi(z))$ for all $z \in \Sigma_1$, $\Phi(z)$ is one-to-one in Σ_1 and

$$\Phi'(z) = \frac{\varphi'(z)}{h'_\varphi(\Phi(z))}, \quad z \in \Sigma_1. \tag{32}$$

The function $r(z)$ is by definition constant in Σ_0, and Lemma 11 trivially yields that it is continuous in Σ. We prove now that it is also continuous at every point of $\partial\Sigma_0$. Suppose, on the contrary, that there exists $z \in \partial\Sigma_0$, a sequence $\{z_n\}_{n=1}^\infty \subset \Sigma$ and a number $\mu_1 > \mu$ such that $\lim_{n\to\infty} z_n = z$ and $r(z_n) \geq \mu_1 > \mu$ for all $n \in \mathbb{N}$. For each n, let $\omega_n \in \mathbb{T}_{r(z_n)}$

be such that $h_\varphi(\omega_n) = \varphi(z_n)$. By extracting a subsequence if necessary, we may assume that $\lim_{n\to\infty} \omega_n = \omega$, with $\mu_1 \le |\omega| \le 1$. But then, by the continuity of φ and h_φ, we must have $h_\varphi(\omega) = \varphi(z)$. Given that $z \in G_1$, this is only possible if $|\omega| < 1$, contradicting that $z \in \Sigma_0$. □

Lemma 13. (a) *For every $z \in \Sigma$ and $\delta > 0$, there exist $\epsilon > 0$ and a constant M such that*

$$|p_n(\zeta)| \le M\sqrt{n+1}[r(z)+\delta]^n, \quad \zeta \in D_\epsilon(z),\ n \ge 0.$$

(b) *For every $z \in \Sigma_1$ there exist $\epsilon > 0$ and $0 < v < 1$ such that*

$$p_n(\zeta) = \sqrt{n+1}\,\Phi'(\zeta)[\Phi(\zeta)]^n\left[1 + O(v^n)\right]$$

uniformly in $\zeta \in D_\epsilon(z)$ as $n \to \infty$.

(c) *For every $\sigma \in (\mu, 1)$ and $\delta > 0$, there exists a constant M_1 such that for every ζ with $r(\zeta) \le \sigma$,*

$$|p_n(\zeta)| \le M_1\sqrt{n+1}(\sigma+\delta)^n, \quad n \ge 0.$$

Proof. We first observe that

$$\Sigma = \{z : \mu < r(z) < 1\} = \cup_{k=1}^{\infty}\left\{z : \max\{\mu, \rho^k\} < r(z) < 1\right\},$$

and proceed to prove by mathematical induction on k that if $k \ge 1$, then Parts (a) and (b) of Lemma 13 hold true for every z with $\max\{\mu, \rho^k\} < r(z) < 1$, while Part (c) holds true for every σ with $\max\{\mu, \rho^k\} < \sigma < 1$. That this is true for $k = 1$ clearly follows from (4) and (5), since $\mu \le \rho$ and

$$\{z : \rho < r(z) < 1\} = \Omega_\rho \cap G_1.$$

Then, suppose it is true for some given $k \ge 1$. Let $z \in \Sigma$ be a fixed number such that

$$\max\{\mu, \rho^{k+1}\} < r(z) \le \rho^k.$$

Select $\eta > 0$ so small that

$$\rho(\rho+\eta)^k < r(z) \quad (\Rightarrow \rho + \eta < 1). \tag{33}$$

Let $\omega_{z,1}, \ldots, \omega_{z,s}$ be the solutions to the equation $h_\varphi(w) = \varphi(z)$ in $\mu < |w| < 1$ that have largest modulus, so that $|\omega_{z,k}| = r(z)$, $1 \le k \le s$. Choose μ' and δ satisfying the hypothesis of Lemma 11, with the particularity that

$$\rho(\rho+\eta)^k < \mu', \qquad \cup_{k=1}^{s} D_\delta(\omega_{z,k}) \subset \{w : \mu' < |w| < (\rho+\eta/2)^k\}. \tag{34}$$

Then, by the induction hypothesis that Lemma 13(c) holds whenever

$$\max\{\mu, \rho^k\} < \sigma < 1,$$

there is a constant M_1 such that for every ζ with $r(\zeta) \le (\rho+\eta/2)^k$,

$$|p_n(\zeta)| \le M_1\sqrt{n+1}(\rho+\eta)^{kn}, \quad n \ge 0,$$

and so we obtain from (6) that

$$p_n(\zeta) = \frac{\sqrt{n+1}\,\varphi'(\zeta)}{2\pi \mathrm{i}}\oint_{\mathbb{T}_1}\frac{w^n\,\mathrm{d}w}{h_\varphi(w) - \varphi(\zeta)} + O\left(\sqrt{n}(\rho+\eta)^{kn}\rho^n\right) \tag{35}$$

uniformly on $\{\zeta : r(\zeta) \le (\rho+\eta/2)^k\}$ as $n \to \infty$.

Now, corresponding to the numbers z, μ' and δ above, choose an $\epsilon > 0$ for which the thesis of Lemma 11 holds true, so that (recall (34)) for all $\zeta \in \overline{D_\epsilon(z)}$, $r(\zeta) < (\rho + \eta/2)^k$ and the function $(h_\varphi(w) - \varphi(\zeta))^{-1}$ is analytic on the compact set K defined as in (31). Hence we obtain from (35) that uniformly in $\zeta \in \overline{D_\epsilon(z)}$ as $n \to \infty$,

$$p_n(\zeta) = \frac{\sqrt{n+1}\varphi'(\zeta)}{2\pi \mathrm{i}} \oint_{\mathbb{T}_{\mu'}} \frac{w^n \mathrm{d}w}{h_\varphi(w) - \varphi(\zeta)} + O\left(\sqrt{n}(\rho+\eta)^{kn}\rho^n\right) + \frac{\sqrt{n+1}\varphi'(\zeta)}{2\pi \mathrm{i}} \sum_{k=1}^{s} \oint_{\partial D_\delta(\omega_{z,k})} \frac{w^n \mathrm{d}w}{h_\varphi(w) - \varphi(\zeta)}. \tag{36}$$

Now, the function $(h_\varphi(w) - \varphi(\zeta))^{-1}$ is continuous as a function of (w, ζ) on the compact set $K \times \overline{D_\epsilon(z)}$ and we obtain from (36), (33) and (34) that uniformly in $\zeta \in \overline{D_\epsilon(z)}$ as $n \to \infty$,

$$\begin{aligned} p_n(\zeta) &= O(\sqrt{n}\mu'^n) + O\left(\sqrt{n}[r(z)+\delta]^n\right) + O\left(\sqrt{n}(\rho+\eta)^{kn}\rho^n\right) \\ &= O\left(\sqrt{n+1}[r(z)+\delta]^n\right). \end{aligned}$$

If $z \in \Sigma_1$, i.e., if $s = 1$, then every $\zeta \in D_\epsilon(z)$ belongs to Σ_1 as well, so that $(h_\varphi(w) - \varphi(\zeta))^{-1}$ is analytic on $\overline{D_\delta(\omega_{z,1})}$, except at the point $\Phi(\zeta) := \omega_{\zeta,1}$, where it has a simple pole. Therefore (recall (32)),

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_{\partial D_\delta(\omega_{z,1})} \frac{w^n \mathrm{d}w}{h_\varphi(w) - \varphi(\zeta)} &= (\omega_{\zeta,1})^n \lim_{w \to \omega_{\zeta,1}} \frac{w - \omega_{\zeta,1}}{h_\varphi(w) - h_\varphi(\omega_{\zeta,1})} \\ &= \frac{\Phi'(\zeta)[\Phi(\zeta)]^n}{\varphi'(\zeta)}. \end{aligned}$$

Hence we obtain from (34) and (36) that uniformly in $\zeta \in D_\epsilon(z)$ as $n \to \infty$,

$$p_n(\zeta) = \sqrt{n+1}\,\Phi'(\zeta)[\Phi(\zeta)]^n \left[1 + O\left(v^n\right)\right],$$

with $0 < v = \mu'/(r(z) - \delta) < 1$. Given that δ could have been chosen arbitrarily small, we have proven that Parts (a) and (b) of Lemma 13 hold true if $\max\{\mu, \rho^{k+1}\} < r(z) < 1$.

Now, suppose σ is such that $\max\{\mu, \rho^{k+1}\} < \sigma \le \rho^k$, and let $\delta > 0$ be given. By the continuity of the function $r(z)$ and the fact that $r(z)$ approaches 1 as z approaches ∂G_1, we have that the set $\{z : r(z) = \sigma\}$ is compact, and we can find finitely many points $z_1, \ldots, z_m$ in this set, and positive numbers $\epsilon_1, \ldots, \epsilon_m$, M_1, such that $\{z : r(z) = \sigma\} \subset \cup_{j=1}^m D_{\epsilon_j}(z_j)$, and

$$|p_n(\zeta)| \le M_1\sqrt{n+1}(\sigma+\delta)^n \tag{37}$$

for all $\zeta \in \cup_{j=1}^m D_{\epsilon_j}(z_j)$, $n \ge 0$. But the set $\{z : r(z) < \sigma\}$ is a bounded open set whose boundary is precisely $\{z : r(z) = \sigma\}$, so that by the maximum modulus principle for analytic functions, (37) also holds for all ζ with $r(\zeta) \le \sigma$. □

Proof of Theorem 1. Part (a) of Theorem 1 is equivalent to Lemma 13(b). We then pass to prove Part (b).

Let $z \in G_1$. From the definition of $r(z)$, we see that the function (in the variable w) $\left(h_\varphi(w) - \varphi(z)\right)^{-1}$ is analytic in the annulus $r(z) < |w| < 1/\rho$, with a singularity on the circle $\mathbb{T}_{r(z)}$ in case $r(z) > 0$, and therefore, it has a Laurent expansion in said annulus, say

$\sum_{k=-\infty}^{\infty} a_k(z)w^k$, whose coefficients

$$a_{-n}(z) = \frac{1}{2\pi i} \oint_{\mathbb{T}_1} \frac{w^{n-1} dw}{h_\varphi(w) - \varphi(z)}, \quad n \geq 0 \tag{38}$$

satisfy

$$\limsup_{n\to\infty} |a_{-n}(z)|^{1/n} = r(z). \tag{39}$$

Let τ be a number satisfying that $r(z) < \tau < 1$, and in the case $r(z) \neq 0$, that $\tau\rho < r(z)$. By Lemma 13(c), we can find a constant M such that

$$|p_n(z)| \leq M\sqrt{n+1}\tau^n,$$

which combined with (6) and (38) yields

$$p_n(z) = \sqrt{n+1}\varphi'(z)a_{-n-1}(z) + O\left(\sqrt{n}(\tau\rho)^n\right) \quad (n \to \infty).$$

This, in view of (39) and the fact that τ can be taken arbitrarily closed to $r(z)$, forces $\limsup_{n\to\infty} |p_n(z)|^{1/n} = r(z)$. □

Proof of Proposition 4. From (7) and (8) we get that

$$\varphi(\psi(w)) = \frac{1}{\overline{\varphi(\psi(1/\overline{w}))}}, \quad \rho < |w| < 1/\rho. \tag{40}$$

If $\varphi(\psi(w))$ has a meromorphic continuation, denoted by $h_\varphi(w)$, to the domain $\mathfrak{D}$ (resp. to $\mathfrak{D}^*$), then, by virtue of (40), the function $w \mapsto 1/\overline{h_\varphi(1/\overline{w})}$ provides the meromorphic continuation of $\varphi(\psi(w))$ to $\mathfrak{D}^*$ (resp. to $\mathfrak{D}$), and (15) is satisfied.

Suppose now that $\varphi(z)$ is meromorphic in $\overline{G}_1 \cup \psi(\mathfrak{D}^*)$. Then the composition $\varphi(\psi(w))$, originally defined for $\rho < |w| < 1/\rho$, now makes perfect sense for $z \in \mathfrak{D}^*$, and it is obviously meromorphic. Reciprocally, if $\varphi(\psi(w))$ has a meromorphic continuation $h_\varphi(w)$ to $\mathfrak{D}^*$, then $h_\varphi(\phi(z))$ is a meromorphic function in $\psi(\mathfrak{D}^*)$, and

$$h_\varphi(\phi(z)) = \varphi(\psi(\phi(z))) = \varphi(z), \quad z \in \Omega_1 \cap G_{1/\rho}. \quad \square$$

Proof of Proposition 5. By the first inclusion in Eq. (17), the composition $\varphi(\psi(w))$ is well-defined and analytic in $\mathfrak{D}_a$. Hence there exists a largest number $t_0 \in [a, b]$ such that $\varphi(\psi(w))$ has a meromorphic continuation to every $\mathfrak{D}_t$ with $a \leq t < t_0$. Suppose $t_0 < b$. From assumption (17), we see that $\psi(\overline{\mathfrak{D}}_{t_0}) \subset \overline{G}_1 \cup \psi(\mathfrak{D}^*_{t_0})$, which combined with assumption (16) yields the existence of some $t_0 < t_1 < b$ such that $\psi(\mathfrak{D}_{t_1}) \subset \overline{G}_1 \cup \psi(\mathfrak{D}^*_{t_0})$. Since $\varphi(\psi(w))$ is meromorphic in $\mathfrak{D}_{t_0}$, by Proposition 4, $\varphi(z)$ is then meromorphic in $\overline{G}_1 \cup \psi(\mathfrak{D}^*_{t_0})$, so that the composition $\varphi(\psi(w))$ is well-defined and meromorphic in $\mathfrak{D}_{t_1}$, contradicting the definition of t_0. Hence $t_0 = b$ and $\varphi(\psi(w))$ has a meromorphic continuation to $\mathfrak{D} = \cup_{a \leq t < b} \mathfrak{D}_t$. □

Proof of Proposition 6. That $\mu \leq \bar{\mu}$ follows by applying Proposition 5 to the collection of annuli $\mathfrak{D}_t := \{w : \rho + \bar{\mu} - t < |w| < 1\}$, $\bar{\mu} \leq t < \rho$.

To prove that $\Sigma_p \supset C_p^{\bar{\mu}}$, $p \geq 1$, we first make a couple of observations. The first one is that, in view of Proposition 4, φ admits a meromorphic continuation (also denoted by φ) to $G_{1/\mu}$, and since $L_r \subset G_{1/r}$ for $\bar{\mu} < r < 1$, we then have $\psi(\bar{\mu} < |w| < 1/\mu) \subset G_{1/\mu}$ and

$$h_\varphi(w) = \varphi(\psi(w)), \quad \bar{\mu} < |w| < 1/\mu. \tag{41}$$

The second observation is stated as a claim.

Claim 1. *If $z \in G_1$ and w_z with $\bar{\mu} < |w_z| < 1$ are such that $h_\varphi(w_z) = \varphi(z)$, then either $\psi(w_z) = z$ or the equation $h_\varphi(w) = \varphi(z)$ has a solution in $|w_z| < |w| < 1$.*

In effect, suppose first that $\psi(w_z) \in G_1$. Then, by (41) and the fact that $h_\varphi(w_z) = \varphi(z)$, we must have $\psi(w_z) = z$. Next, assume $\psi(w_z) \notin G_1$. Then $\psi(w_z) \notin L_1 = \partial G_1$ either, because φ maps L_1 onto the unit circle and $|h_\varphi(w_z)| = |\varphi(\psi(w_z))| = |\varphi(z)| < 1$. Moreover, since $\psi(w_z) \in L_{|w_z|} \subset G_{1/|w_z|}$ and ψ maps $1 < |w| < 1/|w_z|$ conformally onto $G_{1/|w_z|} \setminus \overline{G}_1$, we see that there is a unique number w'_z with $|w_z| < |w'_z| < 1$ such that $\psi\left(1/\overline{w'_z}\right) = \psi(w_z)$. By (41) and (15), we then have

$$h_\varphi(w'_z) = \frac{1}{\overline{h_\varphi\left(1/\overline{w'_z}\right)}} = \frac{1}{\overline{h_\varphi(w_z)}} = \frac{1}{\overline{\varphi(z)}}.$$

This implies that $\psi(w'_z) \notin \overline{G}_1$, which combined with the fact that $\psi(w'_z) \in L_{|w'_z|} \subset G_{1/|w'_z|}$ yields the existence of a unique w''_z with $|w'_z| < |w''_z| < 1$ such that $\psi\left(1/\overline{w''_z}\right) = \psi(w'_z)$, and so

$$h_\varphi(w''_z) = \frac{1}{\overline{h_\varphi\left(1/\overline{w''_z}\right)}} = \frac{1}{\overline{h_\varphi(w'_z)}} = \varphi(z),$$

which proves the claim.

We now proceed to prove that

$$C_p^{\bar{\mu}} = \left\{z \in \Sigma_p : r(z) > \bar{\mu}\right\}. \tag{42}$$

Suppose $z \in \Sigma_p$ is such that $r(z) > \bar{\mu}$, that is, $z \in G_1$ and there are finitely many numbers $\omega_{z,1}, \ldots, \omega_{z,s}$, with $\bar{\mu} < r(z) = |\omega_{z,1}| = \cdots = |\omega_{z,s}| < 1$, which are the only solutions that the equation $h_\varphi(w) = \varphi(z)$ has in $|\omega_{z,1}| \le |w| < 1$, and moreover $\sum_{k=1}^{s} \alpha_{z,k} = p$, with $\alpha_{z,k}$ being the multiplicity of h_φ at $\omega_{z,k}$.

Then, by (41), the only possible solutions that the equation $\psi(w) = z$ could have in $|\omega_{z,1}| \le |w| < 1$ are precisely these $\omega_{z,k}$. As a matter of fact, in view of the claim proven above, we have $\psi(\omega_{z,k}) = z$ for all $1 \le k \le s$, and it clearly follows from (41) that the multiplicity of ψ at $\omega_{z,k}$ is $\alpha_{z,k}$. Thus, $z \in C_p^{\bar{\mu}}$.

Assume now that $z \in C_p^{\bar{\mu}}$, that is, $z \in G_1$ and there are finitely many numbers $v_{z,1}, \ldots, v_{z,s}$, with $\bar{\mu} < |v_{z,1}| = \cdots = |v_{z,s}| < 1$, which are the only solutions that the equation $\psi(w) = z$ has in $|v_{z,1}| \le |w| < 1$, and moreover $\sum_{k=1}^{s} \beta_{z,k} = p$, with $\beta_{z,k}$ being the multiplicity of ψ at $v_{z,k}$. These $v_{z,k}$'s are the only possible solutions that the equation $h_\varphi(w) = \varphi(z)$ could have in $|v_{z,1}| \le |w| < 1$, because by the claim proven above, among such solutions those of largest modulus must be mapped by ψ to z. Moreover, by (41), we have that for all $1 \le k \le s$, $h_\varphi(v_{z,k}) = \varphi(z)$ and that $\beta_{z,k}$ is the multiplicity of h_φ at $v_{z,k}$. Hence $z \in \Sigma_p$, and (42) is proven.

Since every element of Σ_p, $p \ge 1$ satisfies $r(z) > \mu$, (42) implies that if $\mu = \bar{\mu}$, then $\Sigma_p = C_p^{\bar{\mu}}$ for all $p \ge 1$, which in turn implies that $\Sigma_0 = C_0^{\bar{\mu}}$.

Finally, suppose that a sequence $\{w_n\}_{n \ge 1}$ satisfying (18) is found. Then, given that $L_r \subset G_{1/r}$ for $\bar{\mu} < r < 1$, we have $\psi\left(1/\overline{w_n}\right) = \psi(w_{n+1}) \in G_{1/|w_{n+1}|}$, so that $|w_n| > |w_{n+1}|$, $n \ge 1$. Moreover, in view of (41) and (15),

$$h_\varphi(w_{n+2}) = h_\varphi\left(1/\overline{w_{n+1}}\right) = \frac{1}{\overline{h_\varphi(w_{n+1})}} = \frac{1}{\overline{h_\varphi\left(1/\overline{w_n}\right)}} = h_\varphi(w_n), \quad n \ge 1.$$

Hence $h_\varphi(w)$ remains constant along an infinite set of points contained in $\bar{\mu} < |w| < 1$. This is only possible if $\lim_{n\to\infty} |w_n| = \bar{\mu} = \mu$. □

Outline of the proof of Theorems 2 and 3. In order to derive Theorems 2 and 3 from Theorem 1, one needs first to establish several structural properties of the sets Σ_p, $p \geq 1$. These properties have been previously established for different, but similarly defined sets. For instance, in [17], Ullman studied the zero distribution of the Faber polynomials $F_n(z)$, $n = 0, 1, 2, \ldots$, associated with a Laurent series about ∞ of the form

$$g(w) = w + b_0 + b_1 w^{-1} + b_2 w^{-2} + \cdots \tag{43}$$

with radius of convergence $\varrho := \limsup_{n\to\infty} |b_n|^{1/n} < \infty$. The function g is locally invertible at ∞, and if $g^{-1}(z)$ denotes its inverse, then $F_n(z)$ is defined as the polynomial part of the Laurent expansion at ∞ of $[g^{-1}(z)]^n$.

For each $p \geq 1$, Ullman introduced the set C_p consisting of those points $z \in \mathbb{C}$ for which the solutions of largest modulus that the equation $g(w) = z$ has in $|w| > \varrho$, say $u_{z,1}, \ldots, u_{z,s}$, have total multiplicity p. Note the similarity of this definition with that of Σ_p given in (10). Setting $C_0 := \mathbb{C} \setminus \cup_{p\geq 1} C_p$, $\Psi(z) := u_{z,1}$ for all $z \in C_1$ and

$$\tilde{\varrho}(z) := \begin{cases} |u_{z,1}|, & z \in \cup_{p\geq 1} C_p, \\ \varrho, & z \in C_0. \end{cases}$$

Ullman proved that (see (3.7), (3.8), (5.1) and (5.4) in [17])

$$\limsup_{n\to\infty} |F_n(z)|^{1/n} = \tilde{\varrho}(z), \quad z \in \mathbb{C}, \tag{44}$$

and more specifically, for points in C_1, that

$$\lim_{n\to\infty} F_n(z)/[\Psi(z)]^n = 1 \tag{45}$$

locally uniformly on C_1. These asymptotic formulas are the analogue of Theorem 1 for the Faber polynomials.

Ullman also proved that the sets C_p have the following properties [17, Lemmas 4.1, 4.2]: *Every $z \in C_p$, $p \geq 1$ has a neighborhood that is fully contained in $\cup_{q=1}^{p} C_q$. Every C_p with $p > 1$ has empty interior. Every neighborhood of a point $z \in C_p$, $p > 1$ contains points that are not in C_1.*

Combining these properties with (44) and (45), Ullman succeeded in proving that [17, Theorem 1(b)] every point of ∂C_1 is an accumulation point of the zeros of the F_n's. Following Ullman's arguments, one can easily see that the properties just stated for the C_p's are word for word valid for the sets Σ_p as well, and that these properties in conjunction with Theorem 1 imply the validity of Theorem 2.

In another paper [18] dealing with the limiting behavior of the eigenvalues of Toeplitz matrices associated with a semi-infinite Laurent series of the form $\sum_{n=-\infty}^{k} c_n w^n$ ($\limsup_{n\to\infty} |c_{-n}|^{1/n} < \infty$), Ullman considered the smallest possible $\tau \geq 0$ for which there exists a meromorphic function $F(w)$ on $|w| > \tau$ having this expansion at ∞. He defined a corresponding set C that for the case $k = 1$ (i.e., a simple pole at ∞) consists of those points $z \in \mathbb{C}$ for which the equation $F(w) = z$ has exactly one solution in $|w| > \tau$ of largest modulus, and this solution is simple (see the definition of the set C in [18, Definition 1]). He proved two important lemmas [18, Lemmas 7 and 8] about the structure of the boundary of the set C, which can be established in a similar way for both the set C_1 (i.e., the C_p corresponding to $p = 1$) and the set Σ_1. Using the extension of these lemmas to C_1 (see [8, Lemmas 2.2 and 2.4]) together with Ullman's asymptotic formulas (44) and (45), Kuijlaars and Saff proved the analogue of Theorem 3 for the Faber polynomials F_n associated with (43) (see [8, Theorems 1.3, 1.4 and 4.1]). Their arguments are based on general facts of logarithmic potential theory and can be used essentially without variation to derive our Theorem 3. □

5. Proofs of the results of Section 3

Proof of Proposition 7. Given that $\varphi(z) = (z^2 - 1)/R$ is an entire function, Proposition 4 implies that $\mu = 0$ and $h_\varphi(w)$ is meromorphic in $\mathbb{C} \setminus \{0\}$. By uniqueness of the meromorphic continuation, we then have

$$h_\varphi(w) = \varphi(\psi(w)) = \frac{[\psi(w)]^2 - 1}{R}, \qquad \rho < |w| < \infty. \tag{46}$$

This and (15) imply that $h_\varphi(w)$ is indeed a meromorphic function in $\overline{\mathbb{C}}$, whose only poles are ∞ and some point a, $0 < |a| < 1$, and whose only zeros are $1/\overline{a}$ and 0. ∞ and 0 are of multiplicity 2, while a and $1/\overline{a}$ are simple. Hence for some complex number β,

$$h_\varphi(w) = \frac{\beta(1 - \overline{a}w)w^2}{w - a}, \qquad w \in \overline{\mathbb{C}}.$$

By symmetry, $\psi(w) = \overline{\psi(\overline{w})}$, which in view of the normalization $\psi'(\infty) > 0$ implies that ψ maps $(-\infty, -1)$ onto $(-\infty, \sqrt{1 - R})$. Hence $-1 < a < 0$. Also, given that $|h_\varphi(w)| = 1$ for $|w| = 1$ and that by (46)

$$\lim_{w \to \infty} h_\varphi(w)/w^2 > 0,$$

we then must have $\beta = 1$, and so (19) is proven.

Equality (20) follows directly from (19) and (46). To find the value of a, first observe that 0 lies outside the curve L_1. Let b be the point in $|w| > 1$ such that $\psi(b) = 0$ (then, $1/a < b < -1$). By (46), b is a double zero of $1 + Rh_\varphi(w)$, so that (21) holds for some c, and the relations

$$\begin{aligned} &aRb^3 - Rb^2 - b + a = 0 \\ &3aRb^2 - 2Rb - 1 = 0 \\ &2ab^2 - \left(3a^2 + 1\right) b + 2a = 0 \end{aligned} \tag{47}$$

are satisfied. From these we get

$$27a^4 - 18a^2 - 4\left(R + R^{-1}\right) a - 1 = 0,$$

and it is easy to see that this equation (in the unknown a) has only two real solutions, one positive, the other contained in $(-1/3, 0)$. This completes the proof of Proposition 7.

The equalities in (22) follow from (21) and (47). Also from (21) and Vieta's formulas we obtain the relations

$$1/a = 1/c + 2/b, \qquad 2b - 2/b = 1/c - c,$$

which, given that $b < -1$, forces the inequalities in (23) to hold true. □

Proof of Theorem 8. We first observe that if $|\xi| < 1$, then the equation

$$\xi = \frac{(1 - aw)w^2}{w - a} \tag{48}$$

has exactly two roots (counting multiplicities) in $|w| < 1$. To see this, suppose w_1, w_2 and w_3 are the roots of (48). Then, not all can be contained in $|w| < 1$, for in such a case $|(1 - aw_j)/(w_j - a)| > 1$, $1 \le j \le 3$, which together with (48) yields

$$|\xi|^{1/2} > |w_j|, \quad 1 \le j \le 3.$$

Since $w_1 w_2 w_3 = \xi$, we would have $|\xi| < |\xi|^{3/2}$, contradicting the assumption that $|\xi| < 1$.

Assume now that $|w_1| > 1$, $|w_2| > 1$. Denoting by $\varphi^{-1} : \mathbb{D}_1 \to G_1$ the inverse of $\varphi(z) = (z^2 - 1)/R$, we get from (46) that

$$\psi(w_1) = \psi(w_2) = -\varphi^{-1}(\xi),$$

so that $w_1 = w_2$. Since $h'_\varphi(w)$ only vanishes at b, $1/b$ and 0, we must have $w_1 = w_2 = b$, so that by (46), $\xi = -1/R$, contradicting that $|\xi| < 1$.

Thus, being φ a bijection from G_1 to $\mathbb{D}_1$, we conclude that $G_1 = \Sigma_1 \cup \Sigma_2$. For $|\xi| < 1$, let $w_{\xi,1}$ and $w_{\xi,2}$ denote the two solutions of (48) lying in $|w| < 1$. To prove that $\Sigma_2 = [\sqrt{1 - R^2}, 1]$, we prove the equivalent statement that

$$S := \left\{|\xi| < 1 : |w_{\xi,1}| = |w_{\xi,2}|\right\} = [-R, 0]. \tag{49}$$

Suppose $\xi \in S$. From (48) we obtain that for $j = 1, 2$,

$$\Re(w_{\xi,j}) = \frac{|w_{\xi,j}|^4 + a^2|w_{\xi,j}|^6 - a^2|\xi|^2 - |w_{\xi,j}|^2|\xi|^2}{2a\left(|w_{\xi,j}|^4 - |\xi|^2\right)}.$$

Hence $w_{\xi,1} = \overline{w_{\xi,2}}$, and since $h_\varphi(\overline{w}) = \overline{h_\varphi(w)}$, we deduce that ξ must be real, and consequently, the point $\xi \in (-1, 1)$ belongs to S if and only if Eq. (48) has either a double real root in $(-1, 1)$, or no real roots in $(-1, 1)$.

Since $h'_\varphi(w)$ only vanishes at b, $1/b$ and 0, it follows that Eq. (48) has a double root in $(-1, 1)$ only for $\xi = -R = h_\varphi(1/b)$, $\xi = 0 = h_\varphi(0)$. On the other hand, considering $h_\varphi(x)$ as a function of the real variable x, and analyzing the sign changes of $h'_\varphi(x)$ in $(-1, 1)$, it is easy to see that Eq. (48) has no real roots in $(-1, 1)$ if and only if $\xi \in (-R, 0)$. Thus, (49) is proven.

Since $\Sigma_0 = \emptyset$, Theorem 3 guarantees the convergence of $\{\nu_n\}_{n=1}^\infty$ in the weak*-topology to a measure λ supported on $[\sqrt{1 - R^2}, 1]$ and having logarithmic potential

$$U^\lambda(z) = \Re\left(\log\left[\phi'(\infty)/\Phi(z)\right]\right), \quad z \in G_1 \setminus \left[\sqrt{1 - R^2}, \infty\right), \tag{50}$$

with the convention $0 < \arg\ (\phi'(\infty)/\Phi(z)) < 2\pi$.

We proceed to prove that $\lambda = \sigma + \delta_1/2$, with σ as in (25), for which we use a well-known formula [14, Theorem II.1.4] that allows a measure to be recovered from its logarithmic potential.

By the continuity of the pair of complex conjugate solutions that Eq. (48) has as the parameter ξ varies in the closed interval $[-R, 0]$, it is clear that

$$\Gamma_R := \left\{w \in \mathbb{D}_1 : -R \le h_\varphi(w) \le 0\right\}$$

is a Jordan curve symmetric with respect to the real axis, intersecting it at $1/b$ and 0. It is easy to see that Γ_R is in fact an analytic curve.

The function $\Phi(z)$ maps $G_1 \setminus [\sqrt{1 - R^2}, 1]$ conformally onto the portion of the unit disk that lies exterior to Γ_R. Moreover, if for $x \in (\sqrt{1 - R^2}, 1)$, $\omega_{x,1}, \omega_{x,2} \in \Gamma_R$ are the two complex conjugate solutions that the equation $h_\varphi(w) = \varphi(x)$ has in $\mathbb{D}_1$ (say, with $\Im\omega_{x,1} > 0$), then

$$\lim_{t \to 0+} \Phi(x + it) = \omega_{x,1}, \qquad \lim_{t \to 0-} \Phi(x + it) = \omega_{x,2},$$

and we obtain from these two equalities, (50) and Theorem II.1.4 of [14] that for all $\sqrt{1-R^2} < x < 1$,

$$\lambda\left(\left[\sqrt{1-R^2}, x\right]\right) = \frac{\theta_x}{2\pi},$$

where $0 < \theta_x < \pi$ is the angle formed by the two rays emanating from 0 and passing through $\omega_{x,1}, \omega_{x,2}$. Given that $\lim_{x\to 1-} \theta_x = \pi$, we must have $\lambda(\{1\}) = 1/2$, completing the proof of Theorem 8. □

Proof of Theorem 9. Because G_1 is symmetric about the real axis, each p_n has real coefficients. Let $n \geq 0$ be an integer. Combining the orthogonality property of p_n with Green's formula (see, e.g., [13, p. 241]) and using that $(\overline{z}^2 - 1)(z^2 - 1) = R^2$ for $z \in L_1$, we obtain that for $1 \leq m \leq \lfloor n/2 \rfloor$,

$$\begin{aligned} 0 &= \int_{G_1} p_n(z)\overline{z^{2m-1}}\mathrm{d}x\mathrm{d}y = \frac{1}{4\mathrm{i}m}\int_{L_1} p_n(z)\overline{z^{2m}}\mathrm{d}z \\ &= \frac{1}{2\mathrm{i}(m+1)}\int_{L_1} p_n(z)\frac{(R^2+z^2-1)^m}{(z-1)^m(z+1)^m}\mathrm{d}z. \end{aligned}$$

Hence by the Cauchy integral formula, $p_n^{(j)}(1) = 0$, $0 \leq j \leq \lfloor n/2 \rfloor - 1$. Therefore, $p_n(z) = (z-1)^{\lfloor n/2 \rfloor} q_n(z)$, where $q_n(z)$ is a polynomial of degree $n - \lfloor n/2 \rfloor$.

Similarly, we obtain that for $0 \leq m \leq n - \lfloor n/2 \rfloor - 1$,

$$\begin{aligned} 0 &= \int_{G_1} p_n(z)\overline{z^{2m}}\mathrm{d}x\mathrm{d}y = \frac{1}{2\mathrm{i}(2m+1)}\int_{L_1} p_n(z)\overline{z^{2m+1}}\mathrm{d}z \\ &= \frac{1}{2\mathrm{i}(m+1)}\int_{L_1} p_n(z)\left[\frac{R^2}{z^2-1}+1\right]^m \sqrt{\frac{z+\sqrt{1-R^2}}{1+z}}\sqrt{\frac{z-\sqrt{1-R^2}}{1-z}}\mathrm{d}z. \end{aligned}$$

If we now deform the contour of integration L_1 onto the two-sided segment $[\sqrt{1-R^2}, 1]$ we obtain

$$\int_{\sqrt{1-R^2}}^{1} q_n(x)[f(x)]^m \mathrm{d}\lambda_n(x) = 0, \quad 0 \leq m \leq n - \lfloor n/2 \rfloor - 1, \tag{51}$$

where

$$f(x) = \frac{R^2}{x^2-1} + 1, \qquad \mathrm{d}\lambda_n = (1-x)^{[n/2]}\sqrt{\frac{x^2-(1-R^2)}{1-x^2}}\mathrm{d}x.$$

Let $\alpha_1, \ldots, \alpha_N$ be the roots that the polynomial of real coefficients q_n has in $(\sqrt{1-R^2}, 1)$. Since $f(x)$ is decreasing in $(\sqrt{1-R^2}, 1)$, we have

$$\int_{\sqrt{1-R^2}}^{1} q_n(x)\prod_{k=1}^{N}[f(x)-f(\alpha_k)]\mathrm{d}\lambda_n(x) \neq 0,$$

which in view of (51) forces $N = n - \lfloor n/2 \rfloor$. □

Proof of Theorem 10. The proof is based on applying Proposition 6 to the number $\bar{\mu} := (R - \sqrt{R^2-4})/2$. For this $\bar{\mu}$, we have that $L_r \subset G_{1/r}$ for all $\bar{\mu} < r < 1$. For otherwise, there must exist $r_0 \in (\bar{\mu}, 1)$ for which $L_{r_0} \cap L_{1/r_0} \neq \emptyset$. Hence we can find two numbers w and v such that $|w| = |v| = r_0$ and $\psi(w) = \psi(1/v)$. By (27), this implies that $1 \geq (|Rw| - 1)\,(|R/v| - 1)$, or equivalently, $r_0^2 - Rr_0 + 1 \geq 0$. This last inequality holds if and only if either $r_0 \leq \bar{\mu}$ or $r_0 \geq 1/\bar{\mu}$ (>1), contradicting that $r_0 \in (\bar{\mu}, 1)$.

Consider now the sequence of real numbers $\{w_n\}_{n=1}^{\infty}$ defined recursively as follows: w_1 is any number satisfying $\bar{\mu} < w_1 < 1$, and

$$w_{n+1} = \frac{1}{R - w_n}, \quad n \geq 1.$$

It is easy to prove by induction that $w_n > \bar{\mu}$ for all $n \geq 1$, while straightforward computations yield that $\psi(w_{n+1}) = \psi\,(1/\overline{w_n})$, $n \geq 1$.

We can then invoke Proposition 6 to conclude that $\mu = \bar{\mu}$ and that for each $p \geq 1$, Σ_p consists of those points $z \in G_1$ for which the equation $z = Rw - 1 + (Rw - 1)^{-1}$ has exactly p solutions of largest modulus in $\mu < |w| < 1$ (counting multiplicities), thereby establishing Theorem 10. □

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