Oscillation and Asymptotic Behavior of Solutions of Nth Order Nonlinear Delay Differential Equations*

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INTRODUCTION

We consider the equation

\[ x^{(n)}(t) + a(t)f(x(q(t))) = 0 \]  

where \( n \geq 2 \), \( a: [0, \infty) \to [0, \infty) \), \( q: [0, \infty) \to (-\infty, \infty) \), and \( f: (-\infty, \infty) \to (-\infty, \infty) \).

We assume \( a(t), q(t) \), and \( f(x) \) are continuous, \( q(t) \leq t \) for all \( t \geq 0 \), \( q(t) \to \infty \) as \( t \to \infty \), and \( xf(x) > 0 \) for \( x \neq 0 \).

Usually, a condition of monotonicity on \( f \) is needed in order to obtain results for Eq. (1) analogous to those of an ordinary differential equation of the same type. Many authors observed that such a monotonicity condition makes it possible to extend many results from an ordinary differential equation of type (1) to delay equations of type

\[ x^{(n)}(t) + a(t)f(x(t), x(q(t))) = 0 \]  

or

\[ x^{(n)}(t) + f(t, x(t), x(q(t))) = 0 \]  

or even more general types. Consequently, many good results such as in [8, 9, 11] have been obtained for delay equations of types (2) and (3) which obviously apply to Eq. (1). Our first observation is that many results concerning Eq. (2) and (3) can be improved if restricted to Eq. (1). The fact that the function \( f \) involved in (1) is a function of one variable presents the possibility that such a function may be written as a product of two monotone functions over intervals not containing zero. In this case it is possible to relax the condition of monotonicity imposed on \( f \) in previous results. If we let \( R = (-\infty, \infty) \) and let \( C_p(R) \) denote the class of functions which can be written as a product of two

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monotone functions over intervals not containing zero, we will show that $C_2(R)$ is precisely the class of functions of bounded variation on finite intervals not containing zero. Our goal in this paper is basically two-fold: first, to extend some of the results known for (1) when $n = 2$ to Eq. (1) when $n \geq 2$; and next, to relax the condition of monotonicity on $f$ by allowing $f$ to belong to the class $C_2(R)$ or to a subclass of $C_2(R)$ when for instance we require that the component functions of $f$ be either bounded or bounded away from zero. Observe that such a condition on the component functions will not reimpose monotonicity on the function under consideration.

**MAIN RESULTS**

As in [3], for any $t_0 \gg 0$ we define $E_{t_0} = \{ s : s = q(t) \leq t_0 \text{ for } t \geq t_0 \} \cup \{ t_0 \}$. The following conditions will be assumed to hold throughout this paper.

(i) $a(t)$ is not eventually identically zero, and
(ii) all solutions of (1) defined on $E_{t_0}$ exist on $[t_0, \infty)$ for every $t_0 \gg 0$.

A solution $x(t)$ of (1) is said to be oscillatory if $x(t)$ has zeros for arbitrarily large $t$. Equation (1) is said to be oscillatory if every solution of (1) is oscillatory.

We begin by proving the following basic lemmas. The first lemma is essentially Kiguradze's lemma [6] applied to Eq. (1) and slightly modified to serve our need. An identical lemma corresponding to $n$ even can be found in [4].

**LEMMA 1.** Suppose (A) holds and $x(t)$ is a solution of (1) which is of constant sign on $[t_0, \infty)$, $t_0 \gg 0$. Then there exists $t^* \geq t_0$ such that on $[t^*, \infty)$ we have

(i) $x^{(k)}(t) x(t) > 0$ whenever $k + n$ is odd and $0 \leq k \leq n - 1$, and

(ii) there exists an integer $l$, $0 \leq l \leq n - 1$, $n + l$ is odd, such that $x^{(k)}(t) x(t) > 0$ for $k = 0$, $1$, $\ldots$, $l$, $(-1)^{n+k-l} x^{(k)}(t) x(t) > 0$ for $k = l + 1$, $l + 2$, $\ldots$, $n - 1$, and $x^{(n)}(t) x(t) \leq 0$.

**Proof.** Assume $x(t) > 0$ for $t \geq t_0$. As $q(t) \to \infty$ where $t \to \infty$, then there exists $t_1 \geq t_0$ such that $q(t) \geq t_0$ for all $t \geq t_1$; hence $x(q(t)) > 0$ for $t \geq t_1$. Thus $x^{(n)}(t) = -a(t) f(x(q(t))) \leq 0$ and hence $x^{(n-1)}(t)$ is nonincreasing for all $t \geq t_1$.

We will show that $x^{(n-1)}(t) > 0$ for $t \geq t_1$. Suppose $x^{(n-1)}(t_2) \leq 0$ for some $t_2 \geq t_1$; then, as $x^{(n)}(t)$ is not eventually identically zero, there exists $t_3 \geq t_2$ so that $x^{(n-1)}(t_3) < 0$ and hence $x^{(n-1)}(t) \leq x^{(n-1)}(t_3)$ for all $t \geq t_3$. By successive integrations we conclude that $x(t) \to -\infty$ as $t \to \infty$, a contradiction. Thus $x^{(n-1)}(t) > 0$ for all $t \geq t_1$ and hence $x^{(n-2)}(t)$ is increasing for all $t \geq t_1$. Now either $x^{(n-2)}(t)$ is eventually positive and so is every lower derivative or $x^{(n-2)}(t) < 0$ for all $t \geq t_1$ and hence $x^{(n-3)}(t) > 0$ for $t \geq t_1$ by the above
argument. Thus derivatives are eventually of constant sign. They must alternate in sign until two consecutive derivatives are eventually positive and hence every lower derivative is eventually positive.

If \( n \) is even, then it follows that odd derivatives are eventually positive. If \( n \) is odd, then even derivatives are eventually positive.

The case \( x(t) < 0 \) is similar and the proof is omitted.

It is already known that if \( x(t) \) is a nonoscillatory solution of (1) such that \( x(t) \to 0 \) as \( t \to \infty \), then \( x^{(k)}(t) \to 0 \) as \( t \to \infty \) for \( k = 0, 1, \ldots, n - 1 \); see [7, 9, 10].

The following lemma will serve to show that if \( x(t) \) is a nonoscillatory solution of (1) such that \( x(t) \to 0 \) as \( t \to \infty \), then a stronger asymptotic behavior of the solution occurs, namely \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \) for \( k = 0, 1, \ldots, n - 1 \). This result is an extension of a result obtained in [5] in connection with a third order linear differential equation.

**Lemma 2.** Suppose (A) holds and \( x(t) \) is a solution of (1) which is of constant sign on \([t_0, \infty)\), \( t_0 \geq 0 \).

Let \( i \) be an integer such that \( 0 \leq i \leq n - 2 \).

If \( x^{(i)}(t) \to 0 \) as \( t \to \infty \), then

(i) there exists \( t_1 \geq t_0 \) so that \( x^{(k)}(t) x^{(k+1)}(t) < 0, k = i, i + 1, \ldots, n - 2 \), and \( x^{(n-1)}(t) x^{(n)}(t) \leq 0 \) for all \( t \geq t_1 \), and

(ii) \( |\int_{t_1}^{t} t^{i-1} x^{(i+j)}(t) \, dt| < \infty \) and \( t x^{(i+j)}(t) \to 0 \) as \( t \to \infty \), \( j = 1, 2, \ldots, n - 1 - i \).

**Proof.** (i) By Lemma 1, there exists \( t_1 \geq t_0 \) such that \( x^{(k)}(t), k = 0, 1, \ldots, n - 2 \), are of constant sign for all \( t \geq t_1 \).

We will show first that \( x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = i, \ldots, n - 1 \). Suppose not; then, for some \( k > i \), there exists \( c > 0 \) such that either \( x^{(k)}(t) \geq c \) for all \( t \geq t_1 \) and hence \( x^{(i)}(t) \to \infty \) as \( t \to \infty \), or \( x^{(k)}(t) \leq -c \) for all \( t \geq t_1 \) and hence \( x^{(i)}(t) \to -\infty \) as \( t \to \infty \), a contradiction. Thus \( x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = i, i + 1, \ldots, n - 1 \).

Next, if, for some \( k \in \{i, i + 1, \ldots, n - 1\}, x^{(k)}(t) > 0 \), then we must have \( x^{(k+1)}(t) < 0 \) for all \( t \geq t_1 \); otherwise, \( x^{(k)}(t) \geq x^{(k)}(t_1) > 0 \) for all \( t \geq t_1 \) and this is a contradiction since \( x^{(k)}(t) \to 0 \) as \( t \to \infty \). Similarly, if \( x^{(k)}(t) < 0 \), then we must have \( x^{(k+1)}(t) > 0 \); otherwise, \( x^{(k)}(t) \leq x^{(k)}(t_1) < 0 \) for all \( t \geq t_1 \) which is also a contradiction. Thus \( x^{(k)}(t) x^{(k+1)}(t) < 0 \) for all \( t \geq t_1, k = i, \ldots, n - 2 \).

(ii) We use induction on \( j \).

First observe from (i) that \( |x^{(k)}(t)| \) for \( k = i, \ldots, n - 1 \) are decreasing for all \( t \geq t_1 \).

We now show that (ii) holds for \( j = 1 \). As

\[
|\int_{t_1}^{t} x^{(i+1)}(s) \, ds| = |x^{(i)}(t) - x^{(i)}(t_1)| = |x^{(i)}(t)| + |x^{(i)}(t_1)| \leq 2 |x^{(i)}(t_2)|
\]
for $t \geq t_1$, then $|\int_{t_1}^{\infty} x^{(i+1)}(s) \, ds| < \infty$ and hence, for any given $\epsilon > 0$, there exists $T \geq t_1$ such that $|\int_{T}^{\infty} x^{(i+1)}(t) \, dt| < \epsilon/2$. As $x^{(k)}(t)$, $k = 0, \ldots, n - 1$, are of constant sign for all $t \geq t_1$, then

$$
\epsilon/2 > \int_{T}^{t} x^{(i+1)}(s) \, ds = \int_{T}^{t} \left| x^{(i+1)}(s) \right| \, ds \geq |x^{(i+1)}(t)| (t - T)
$$

for all $t \geq T$.

Since $x^{(i+1)}(t) \to 0$ as $t \to \infty$, then there exists $T_1 \geq T$ such that $|x^{(i+1)}(t)| T < \epsilon/2$ for all $t \geq T_1$ and hence $t \left| x^{(i+1)}(t) \right| < \epsilon$ for all $t \geq T_1$. Thus $t x^{(i+1)}(t) \to 0$ as $t \to \infty$ and the result for $j = 1$ is proved.

To show that (ii) holds for an arbitrary $j \in \{1, \ldots, n - 1 - i\}$ we assume that

$$
\int_{T_1}^{t} x^{(i+j)}(t) \, dt < \infty
$$

and $t x^{(i+j)}(t) \to 0$ as $t \to \infty$ for some $j \in \{1, 2, \ldots, n - 2 - i\}$ and show that $|\int_{T_1}^{t} t x^{(i+j+1)}(t) \, dt| < \infty$ and $t^{i+1} x^{(i+j+1)}(t) \to 0$ as $t \to \infty$.

Integration by parts yields

$$
\int_{T_1}^{t} s^{j} x^{(i+j+1)}(s) \, ds = \left| t^{i} x^{(i+j)}(t) - t_1^{i} x^{(i+j)}(t_1) - j \int_{T_1}^{t} s^{j-i} x^{(i+j)}(s) \, ds \right|
$$

$$
\leq \left| t^{i} x^{(i+j)}(t) \right| + \left| t_1^{i} x^{(i+j)}(t_1) \right| + j \left| \int_{T_1}^{t} s^{j-i} x^{(i+j)}(s) \, ds \right| < \infty
$$

since $t x^{(i+j)}(t) \to 0$ as $t \to \infty$. Hence, for any given $\epsilon > 0$, there exists $T \geq t_1$ such that $|\int_{T}^{\infty} t x^{(i+j+1)}(t) \, dt| < \epsilon/[2(j + 1)]$. But

$$
\epsilon/[2(j + 1)] > \int_{T}^{t} s^{j} x^{(i+j+1)}(s) \, ds = \int_{T}^{t} s^{j} |x^{(i+j+1)}(s)| \, ds
$$

$$
\geq |x^{(i+j+1)}(t)| \int_{T}^{t} s^{j} \, ds.
$$

Thus $|x^{(i+j+1)}(t)| (t^{j+1} - T^{j+1})/(j + 1) < \epsilon/[2(j + 1)]$. As $x^{(i+j+1)}(t) \to 0$ when $t \to \infty$, then there exists $T_1 \geq T$ such that $|x^{(i+j+1)}(t)| T^{j+1} < \epsilon/2$ for all $t \geq T_1$ and hence, $t^{j+1} |x^{(i+j+1)}(t)| < \epsilon$ for all $t \geq T_1$. Thus

$$
t^{j+1} x^{(i+j+1)}(t) \to 0
$$

as $t \to \infty$ and the proof now is complete.

The next lemma is intended to reduce computation in the proofs of the following theorems.
LEMMA 3. If \( x(t) \) is a solution of (1) and \( i \) is an integer, \( 0 \leq i \leq n - 1 \), then for any \( t_0 \geq 0 \) and for all \( t \geq t_0 \) we have

\[
\sum_{i=0}^{i} (-1)^{i+k}k\chi(n-i-1+k)(t)/k! + \int_{t_0}^{t} s^i a(s) f(x(q(s))) \, ds/i! - C_i(t_0) = 0 \tag{4}
\]

where \( C_i(t) = \sum_{k=0}^{n-i-1} (-1)^{i+k}k\chi(n-i-1+k)(t)/k! \).

\[
x(t) = D_{n-1}(t_0) + \sum_{k=1}^{n} (-1)^{k+1}k\chi(n)(t)/k!
\]

\[
+ (-1)^{n} \int_{t_0}^{t} (s - t_0)^{n-1}a(s) f(x(q(s))) \, ds/(n - 1)!
\tag{5}
\]

where \( D_i(t) = (-1)^{i}C_i(t) \).

\[
x(t) = x(t_0) + \sum_{k=1}^{n-1} (-1)^{k+1}(t - t_0)^{k}\chi(n)(t)/k!
\]

\[
+ (-1)^{n} \int_{t_0}^{t} (s - t_0)^{n-1}a(s) f(x(q(s))) \, ds/(n - 1)!. \tag{6}
\]

**Proof.** Multiply both sides of Eq. (1) by \( t^i \) and integrate from \( t_0 \) to \( t \) to obtain

\[
\int_{t_0}^{t} s^i x^{(n)}(s) \, ds + \int_{t_0}^{t} s^i a(s) f(x(q(s))) \, ds = 0.
\]

By successive integrations by parts of \( \int_{t_0}^{t} s^i x^{(n)}(s) \, ds \) we may write

\[
\int_{t_0}^{t} s^i x^{(n)}(s) \, ds = s^i x^{(n-1)}(s)\big|_{t_0}^{t} - \int_{t_0}^{t} sdx^{(n-1)}(s) \, ds
\]

\[
= s^i x^{(n-1)}(s) - sdx^{(n-2)}(s)\big|_{t_0}^{t} + \int_{t_0}^{t} sdx^{(n-2)}(s) \, ds
\]

\[
= s^i x^{(n-1)}(s) - sdx^{(n-2)}(s) + \cdots + (-1)^{i}i! x^{(n-i-1)}(s)\big|_{t_0}^{t} + \cdots + (-1)^{i}i! x^{(n-i-1)}(s)\big|_{t_0}^{t}
\]

Divide by \( i! \) to get

\[
\int_{t_0}^{t} s^i x^{(n)}(s) \, ds/i! = s^i x^{(n-1)}(s)/i!
\]

\[
- sdx^{(n-2)}(s)/(i - 1)! + \cdots + (-1)^{i}x^{(n-i-1)}(s)\big|_{t_0}^{t}
\]

\[
= \sum_{k=0}^{i} (-1)^{i+k}k\chi(n-i-1+k)(s)\big|_{t_0}^{t}/k! - C_i(t) - C_i(t_0).
\]
Hence \( C_i(t) - C_i(t_0) + \int_{t_0}^{t} s^i a(s) f(x(q(s))) \, ds/i! = 0 \) and thus (4) is proved.

To prove (5) we replace \( i \) by \( n - 1 \) in (4) and solve for \( x(t) \) to get

\[
(-1)^n x(t) = \sum_{k=1}^{n-1} (-1)^{n-1-k} \frac{k!}{k!} + \int_{t_0}^{t} s^{n-1} a(s) f(x(q(s))) \, ds/(n - 1)! - C_{n-1}(t_0).
\]

Multiply through by \((-1)^n\) to obtain (5).

To obtain (6) we multiply Eq. (1) by \((t - t_0)^{n-1}\) and integrate by parts as above.

**Remark 1.** It is clear from the proof of Lemmas 1, 2, and 3 that these lemmas are also true for Eqs. (2) and (3) provided the corresponding functions \( f \) satisfy respectively the property that if \( x \) and \( y \) have the same sign, then \( f(x, y) \) and \( f(t, x, y) \) have respectively that sign.

**Remark 2.** In connection with the study of solutions of Eq. (1) we consider solutions of the equation

\[
\phi(t) + a(t) f^*(\phi(t)) = 0
\]

(7)

where \( f^* \) is defined as follows:

\[
f^*(x) = f(x) \quad \text{if} \quad x \leq 0
\]

\[
= -f(-x) \quad \text{if} \quad x > 0.
\]

It is easy to see that \( f^* \) is an odd function and \( xf^*(x) > 0 \) for \( x \neq 0 \). Also, if \( \phi(t) \) is a solution of (7), then \( \phi(t) \) is also a solution of (7). Moreover, \( y(t) < 0 \) is a solution of (7) if and only if \( y(t) \) is a solution of (1).

Waltman [11] obtained an oscillation result for Eq. (2) when \( n = 2 \). This result when restricted to Eq. (1) yields:

If \( f \) is nondecreasing and \( \int_{0}^{\infty} a(t) \, dt = \infty \), then (1) is oscillatory when \( n = 2 \).

Our first objective is to generalize this result to Eq. (1) when \( n \geq 2 \) and at the same time allow \( f \) to belong to a larger class of functions, namely, the class of continuous functions which are bounded away from zero in a neighborhood of infinity.

**Theorem 1.** Suppose (A) holds and \( \lim_{x \to \pm \infty} \inf \|f(x)\| > 0 \).

Let \( i \) be an integer such that \( 0 \leq i \leq n - 1 \).

If \( \int_{0}^{\infty} t^i a(t) \, dt = \infty \), then, for \( n \) even, every solution of (1) with bounded \((n - i - 1)\)th derivative oscillates, while, for \( n \) odd, every solution \( x(t) \) of (1) with bounded \((n - i - 1)\)th derivative either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).
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Let $x(t)$ be a solution of (1). If $x(t)$ does not oscillate, then there exists $t_0 \geq 0$ such that $x(t) \neq 0$ for all $t \geq t_0$.

Assume $x(t) > 0$ for all $t \geq t_0$; then, by Lemma 1, there exists $t_1 \geq t_0$ such that $x^{(k)}(t)$, $k = 0, 1, \ldots, n - 1$, are of constant sign for all $t \geq t_1$. In particular, $x^{(n-1)}(t) > 0$ and $x^{(n)}(t) \leq 0$ for $t \geq t_1$. Thus $x(t)$ is monotone for $t \geq t_1$ and hence $x(t) \to l$ as $t \to \infty$, where $0 \leq l \leq \infty$. As $q(t) \to \infty$ when $t \to \infty$, then $x(q(t)) \to l$ as $t \to \infty$. We will show that, under the hypotheses of the theorem, if $x(t)$ does not oscillate, then $l = 0$. Suppose not; then, if $0 < l < \infty$, it follows, from the continuity of $f$, that $f(x(q(t))) \to f(l) > 0$ as $t \to \infty$ and hence there exists $t^* \geq t_1$ such that $f(x(q(t))) > f(l)/2$ for all $t \geq t^*$. If $l = \infty$, and we let $\alpha = \min(1, \lim_{x \to \infty} \inf f(x))$, there exists $x_1 > 0$ such that $f(x) > \alpha/2$ for all $x > x_1$. Choose $T \geq t_1$ so that $x(q(t)) > x_1$ and hence $f(x(q(t))) > \alpha/2$ for all $t \geq T$. Thus if $l \neq 0$, then there exists $t_2 \geq t_1$ and $r > 0$ such that $f(x(q(t))) > r$ for all $t \geq t_2$.

Observe that, for $i = 0$, $x^{(n-i-1)}(t)$ is bounded since $x^{(n)}(t) \leq 0$ for all $t \geq t_1$. As $x^{(n-1)}(t) - x^{(n-1)}(t_2) + \int_{t_2}^{t} a(s) f(x(q(s))) ds = 0$, then

$$r \int_{t_2}^{t} a(s) ds \leq x^{(n-1)}(t_2) - x^{(n-1)}(t) < \infty$$

for all $t \geq t_2$. This is a contradiction since, for $i = 0$, $\int_{0}^{\infty} a(t) dt = \infty$. Thus for $i = 0$, either $x(t)$ oscillates or $x(t) \to 0$ as $t \to \infty$.

Now, if $i > 0$ and $x^{(n-i-1)}(t)$ is bounded, then it follows that $x^{(n-i)}(t) \to 0$ as $t \to \infty$; otherwise, as $x^{(n-i)}(t)$ is monotone and of constant sign, there exists $c > 0$ such that either $x^{(n-i)}(t) \geq c$ for all $t \geq t_2$ and hence $x^{(n-i-1)}(t) \to \infty$ as $t \to \infty$, or $x^{(n-i)}(t) \leq -c$ for all $t \geq t_2$ and hence $x^{(n-i-1)}(t) \to -\infty$ as $t \to \infty$. In either case we have a contradiction since $x^{(n-i-1)}(t)$ is bounded. Thus $x^{(n-i)}(t) \to 0$ as $t \to \infty$ and hence, by Lemma 2, $x^{(n-i)}(t), x^{(n-i+1)}(t), \ldots, x^{(n-1)}(t)$ must alternate in sign; i.e., $(-1)^i k x^{(n-i-1+k)}(t)/k! \geq 0$ for all $t \geq t_2$, and hence, by (4), we have

$$(-1)^i x^{(n-i-1)}(t) + \int_{t_2}^{t} s^i a(s) f(x(q(s))) ds/i! - C_i(t_2) \leq 0$$

for all $t \geq t_2$.

Thus

$$r \int_{t_2}^{t} s^i a(s) ds/i! \leq C_i(t_2) + (-1)^i x^{(n-i-1)}(t) < \infty$$

for all $t \geq t_2$.

This contradicts the condition $\int_{0}^{\infty} t^i a(t) dt = \infty$. Thus either $x(t)$ oscillates or $x(t) \to 0$ as $t \to \infty$ and hence, by Lemma 2, either $x(t)$ oscillates or $t^k x^{(k)}(t) \to 0$ as $t \to \infty$, $k = 0, 1, \ldots, n - 1$.

If $n$ is even, then $x'(t) > 0$ for all $t \geq t_2$ and hence $x(t) > x(t_2)$ for all $t \geq t_2$. Thus $x(t)$ must oscillate.
If we now assume \( x(t) < 0 \) for all \( t \geq t_0 \) and we let \( y(t) = -x(t) \), then \( y(t) \) is a solution of (7). As \( y^{(n-i-1)}(t) \) is bounded and \( \lim_{x \to \pm \infty} \inf |f^*(x)| > 0 \), then, by the above proof, the conclusion of the theorem holds for \( y(t) \) and hence for \( x(t) \). This completes the proof of Theorem 1.

**Corollary 1.** Suppose (A) holds and

\[
\lim_{x \to \pm \infty} \inf |f(x)| > 0.
\]

If \( \int^\infty a(t) \, dt = \infty \), then, for \( n \) even, (1) is oscillatory, while, for \( n \) odd, every solution \( x(t) \) of (1) either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).

**Corollary 2.** Suppose (A) holds.

If \( \int^\infty t^{n-1}a(t) \, dt = \infty \), then for \( n \) even, every bounded solution of (1) oscillates, while, for \( n \) odd, every bounded solution \( x(t) \) of (1) either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).

**Proof.** Let \( x(t) \) be as in the proof of Theorem 1 and assume \( x(t) \to l \), \( l \neq 0 \), as \( t \to \infty \). As \( x(t) \) is bounded, then there exists \( r > 0 \) and \( t^* \) sufficiently large so that \( f(x(q(t))) > r \) for all \( t \geq t^* \). Hence, it follows from (4) for \( i = n - 1 \) that \( \int^{t^*} s^{n-1}a(s) \, ds < \infty \) for all \( t \geq t^* \), a contradiction.

**Remark 3.** It is clear that the technique of the proof of Theorem 1 can also be used to obtain an analogous result for Eq. (2) and hence an extension of Waltman's result [11] to a larger class of functions and also an extension of \([8, \text{Theorem 4.1}]\) as the above corollary shows. Furthermore, Theorem 1 yields some information about the nonoscillatory solutions of Eq. (1).

The first question one now would like to ask is: How far can one enlarge the class of functions and still obtain the same result? For instance, is it possible to include the class of continuous functions or its subclass \( C_p(R) \)? The answer is no as the following example shows. The equation

\[
x''(t) + (1 + t)[4t^2]^{-1}x(t)[1 + x^2(t)]^{-1} = 0
\]

has \( x(t) = t^{1/2} \) as a solution on \((0, \infty)\) which neither oscillates nor tends to zero as \( t \to \infty \). Thus the integral condition alone is not enough to obtain the result of Corollary 1 even for the class \( C_p(R) \). Later we will give a sufficient integral condition to yield the above result when the class of functions under consideration is the class \( C_p(R) \).

The next question one then would like to ask is: How good is the integral condition in Corollary 1 for a specific class of functions? Burton and Grimmer [1] answered this question when \( n = 2 \) and showed that for the class of increasing functions this integral condition is the best possible if Eq. (1) is to oscillate for every choice of \( q(t) \). In this paper we will show as we go along that the above
integral condition remains the best possible for \( n \geq 2 \) and for the class \( C_p(R) \) if the result in Corollary 1 is to hold for every choice of \( q(t) \).

The following notations will be used throughout this paper.

\[
R_\alpha = (-\infty, -\alpha] \cup [\alpha, +\infty), \; \alpha > 0,
\]

\[
C(R) = \{ f: R \to R \mid f \text{ is continuous and } xf(x) > 0 \text{ if } x \neq 0 \}.
\]

\[
C'(R_\alpha) = \{ f \in C(R) \mid f \text{ is continuously differentiable in } R_\alpha \}, \text{ and}
\]

\[
C_p(R_\alpha) = \{ f \in C(R) \mid f \text{ is of bounded variation on every } [a, b] \subset R_\alpha \}.
\]

**Lemma 4.** Suppose \( \alpha > 0 \) and \( f \in C(R) \). Then \( f \in C_p(R_\alpha) \) if and only if \( f(x) = g(x) h(x) \) for all \( x \in R_\alpha \), where \( g: R_\alpha \to (0, \infty) \), nondecreasing on \( (-\infty, -\alpha] \) and nonincreasing on \( [\alpha, \infty) \), and \( h: R_\alpha \to R \) and nondecreasing in \( R_\alpha \).

**Proof.** 1. Suppose \( f \in C_p(R_\alpha) \), then clearly \( \log |f| \) is of bounded variation on every \( [a, b] \subset R_\alpha \). If \( x \geq \alpha \), then \( \log f(x) = \log g(x) + \log h(x) \) for some nondecreasing functions \( r \) and \( s \). Hence \( f(x) = e^{r(x)} \cdot e^{-s(x)} \) for all \( x \geq \alpha \). Let \( h(x) = e^{r(x)} \) and \( g(x) = e^{-s(x)} \). If \( x \leq \alpha \), then \( \log(-f(x)) = \log r(x) + \log s(x) \) for some nondecreasing functions \( r \) and \( s \). Hence \( -f(x) = e^{r(x)} \cdot e^{-s(x)} \). Choose \( h(x) = -e^{-s(x)} \) and \( g(x) = e^{r(x)} \). Thus \( f(x) = g(x) h(x) \) for all \( x \geq \alpha \) where \( g \) and \( h \) are as in Lemma 4.

2. Suppose \( f(x) = g(x) h(x) \) for all \( |x| \geq \alpha \) where \( g \) and \( h \) are as in Lemma 4. As \( \log |f(x)| = \log g(x) + \log |h(x)| \), then, for \( x \geq \alpha \), we have \( \log f(x) = \log h(x) - \log [1/g(x)] \). As \( h \) and \( 1/g \) are nondecreasing on \( [\alpha, \infty) \), then \( \log f \) is of bounded variation on every \( [a, b] \subset [\alpha, \infty) \) and so is \( f \). For \( x \leq -\alpha \) we write \( \log(-f(x)) = \log g(x) + \log(-h(x)) = \log g(x) - \log[1/-h(x)] \). As \( g \) and \( -1/h \) are nondecreasing, then \( \log(-f) \) is of bounded variation on every \( [a, b] \subset (-\infty, -\alpha] \) and so is \( f \). Hence \( f \in C_p(R_\alpha) \).

**Definition.** The \( h \) in Lemma 4 will be called a nondecreasing component of \( f \) while \( g \) will be called a positive component of \( f \).

**Remark 4.** If \( f \in C_p(R_\alpha) \) for some \( \alpha > 0 \), then a pair of components \( h \) and \( g \) of \( f \) can be defined as follows:

For \( x \geq \alpha \), we let \( h(x) = \exp[V_\alpha^*(\log f)] \) and \( g(x) = f(x) \exp[-V_\alpha^*(\log f)] \) where \( V_\alpha^*(f) \) denotes the variation of \( f \) over \( [\alpha, x] \). Obviously, \( h \) is nondecreasing for \( x \geq \alpha \). Also, as \( V_\alpha^*(f) - f(x) \) is nondecreasing, \( g \) is nonincreasing for \( x \geq \alpha \). For \( x \leq -\alpha \), we let \( h(x) = f(x) \exp[-V_{-\alpha}^*(\log |f|)] \) and \( g(x) = \exp V_{-\alpha}^*(\log |f|) \).

**Remark 5.** If \( f \in C'(R_\alpha) \) for some \( \alpha > 0 \), then a pair of components \( h \) and \( g \) of \( f \) can be defined as follows:
For \( x \geq \alpha \), we let
\[
h(x) = f(\alpha) \exp \int_{\alpha}^{x} \left[ \frac{f_+(s)/f(s)}{f_+(s)} \right] ds
\]
and
\[
g(x) = \exp \left( -\int_{\alpha}^{x} \left[ f_-(s)/f(s) \right] ds \right)
\]
where \( f_+(x) = \max(0, f'(x)) \) and \( f_-(x) = \max(0, -f'(x)) \).

For \( x \leq -\alpha \), we let
\[
h(x) = f(\alpha) \exp \int_{-\infty}^{x} \left[ \frac{f_+(s)/f(s)}{f_+(s)} \right] ds
\]
and
\[
g(x) = \exp \left( -\int_{-\infty}^{x} \left[ f_-(s)/f(s) \right] ds \right).
\]

We define
\[
C_+(R_\alpha) = \{ f \in C(R_\alpha) : \text{has a positive component bounded away from zero} \}
\]
and
\[
C_+(R_\alpha) = \{ f \in C(R_\alpha) : f \text{ has a bounded nondecreasing component} \}.
\]

**Example 1.** We illustrate the class \( C_+(R_\alpha) \) by constructing a nonmonotone element having a positive component bounded away from zero.

Let \( h \) be an odd function such that
\[
h(x) = k + \sin^2(x/2) \quad \text{if} \quad 2k\pi < x < (2k + 1)\pi
\]
\[
= k + 1 \quad \text{if} \quad (2k + 1)\pi < x < (2k + 2)\pi, \quad k = 0, 1, 2, \ldots.
\]

Let \( g(x) = (2 + |x|)/(1 + |x|) \) and \( f(x) = g(x) h(x) \). It is easy to see that \( f \) is not monotone and that \( f \in C_+(R_\alpha) \) for any \( \alpha > 0 \) with \( h \) as a nondecreasing component and \( g \) as a positive component.

**Lemma 5.** Suppose \( \alpha > 0 \).

(i) If \( f \in C_+(R_\alpha) \), then there exists \( \beta > 0 \) such that \( f(x_1) \geq \beta f(x_2) \) whenever \( x_1 \geq x_2 \geq \alpha \) and \( f(x_1) \leq \beta f(x_2) \) whenever \( x_1 \leq x_2 \leq -\alpha \). Furthermore, \( \beta \leq \min(\lim_{x \to \alpha} g(x)/g(\alpha), \lim_{x \to -\alpha} g(x)/g(-\alpha)) \) for some positive component \( g \) of \( f \) bounded away from zero.

(ii) If \( f \in C_+(R_\alpha) \), then there exists \( B > 0 \) such that \( f(x_1) \leq B f(x_2) \) whenever \( x_1 \geq x_2 \geq \alpha \) and \( f(x_1) \geq B f(x_2) \) whenever \( x_1 \leq x_2 \leq -\alpha \).
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Proof. (i) As \( f \in C_l(R_\alpha) \), then \( f(x) = g(x) h(x) \) for some positive and non-decreasing components \( g \) and \( h \) respectively with \( \lim_{x \to \pm \infty} g(x) > 0 \), and for all \( |x| \geq \alpha \). Let \( \beta_1 = \min(\lim_{x \to \pm \infty} g(x), \lim_{x \to \pm \infty} g(x)) \).

For \( x_1 \geq x_2 \geq \alpha \) we have \( h(x_1) \geq h(x_2) \geq h(\alpha) \) and \( g(x_1) \geq g(x_2) \geq g(x_1) \geq \beta_1 \).

Hence \( f(x_1) \geq \beta_1 h(x_2) = \beta_1 f(x_2)/g(x_2) \geq \beta_1 f(x_2)/g(\alpha) \geq \beta f(x_2) \).

For \( x_1 \leq x_2 \leq -\alpha \) we have \( h(x_1) \leq h(x_2) \leq h(-\alpha) \) and \( \beta_1 \leq g(x_1) \leq g(x_2) \leq g(-\alpha) \). Hence \( -f(x_1) \geq -\beta_1 h(x_2) \geq -\beta_1 f(x_2)/g(-\alpha) \geq -\beta f(x_2) \) and so \( f(x_1) \leq \beta f(x_2) \).

(ii) As \( f \in C_D(R_\alpha) \), then \( f(x) = g(x) h(x) \) with \( |h(x)| \leq B_1 \) for some \( B_1 > 0 \) and for all \( |x| \geq \alpha \).

For \( x_1 \geq x_2 \geq \alpha \) we have \( B_1 \geq h(x_1) \geq h(x_2) \geq \alpha \) and \( g(x_1) \geq g(x_2) \geq g(x_1) \).

Hence \( f(x_1) \leq B_1 g(x_2) = B_1 f(x_2)/h(x_2) \leq B_1 f(x_2)/h(\alpha) \).

For \( x_1 \leq x_2 \leq -\alpha \) we have \( -B_1 \leq h(x_1) \leq h(x_2) \leq h(-\alpha) \) and \( g(x_1) \leq g(x_2) \leq g(-\alpha) \). Hence \( -f(x_1) \leq B_1 g(x_2) = B_1[-f(x_2)]/[h(-\alpha)] \) and so \( f(x_1) \geq B_1[-f(x_2)]/[h(-\alpha)] \). Choose \( B \geq \max(B_1/h(\alpha), -B_1/h(-\alpha)) \). The proof is now complete.

Remark 6. Let \( \alpha > 0 \). If \( f \in C'(R_\alpha) \) and \( \int_{-\infty}^{\infty} |f'(s)/f(s)| \, ds < \infty \), then \( f \in C_D(R_\alpha) \).

If \( f \in C'(R_\alpha) \) and \( \int_{-\infty}^{\infty} |f'(s)/f(s)| \, ds < \infty \), then \( f \in C_D(R_\alpha) \).

Ladas [8] extended Waltman's result [11] to Eq. (3). This extended result if restricted to Eq. (1) yields:

If \( f \) is nondecreasing and

\[
\int_{-\infty}^{\infty} a(t) f(cq^n-x(t)) \, dt = \pm \infty \quad \text{for every } c \neq 0, \tag{8}
\]

then every solution \( x(t) \) of (1) either oscillates or \( x(n-1)(t) \to 0 \) as \( t \to \infty \).

Observe that (8) does not guarantee oscillation of solutions when \( n \) is even except when \( n = 2 \).

Burton and Grimmer in [1] proved the following result:

If \( f \) is nondecreasing and

\[
\int_{-\infty}^{\infty} a(t) f(\pm cq(t)) \, dt = \pm \infty \quad \text{for every } c > 0, \tag{8}
\]

then for \( n = 2 \) every solution \( x(t) \) of (1) either oscillates or \( x'(t) \to 0 \) as \( t \to \infty \).

In the following theorem we extend Burton and Grimmer's result to (1) when \( n \geq 2 \) and \( f \in C_l(R_\alpha) \) for some \( \alpha > 0 \) and hence obtain an improvement of Ladas' result when restricted to (1) in various directions.

**Theorem 2.** Suppose (A) holds and \( f \in C_l(R_\alpha) \) for some \( \alpha > 0 \). Let \( i \) be an integer with \( 0 \leq i \leq n - 1 \).

If \( \int_{-\infty}^{\infty} t^a(t) f[\pm cq^{n-i-1}](t) \, dt = \pm \infty \) for every \( c > 0 \), then every solution \( x(t) \)
of (1) with bounded \((n - i - 1)\)th derivative either oscillates or \(t^{k}x^{(n-i-1+k)}(t) \to 0\) as \(t \to \infty\), \(k = 0, 1, \ldots, i\).

Proof. For \(i = n - 1\), the result follows from Corollary 2. We may then assume \(i \neq n - 1\).

Let \(x(t)\) be a solution of (1). If \(x(t)\) does not oscillate, then there exists \(t_{0} \geq 0\) such that \(x(t) \neq 0\) for all \(t \geq t_{0}\).

Assume \(x(t) > 0\) for all \(t \geq t_{0}\); then, by Lemma 1, there exists \(t_{1} \geq t_{0}\) such that \(x^{(k)}(t), k = 0, 1, \ldots, n - 1\), are of constant sign for all \(t \geq t_{1}\).

It has been shown, in the proof of Theorem 1, that if \(i > 0\) and \(x^{(n-i-1)}(t)\) is bounded, then \(x^{(n-i)}(t) \to 0\) as \(t \to \infty\) and

\[
(-1)^{i}x^{(n-i-1)}(t) + \int_{t_{1}}^{t} s^{i}a(s)f(x(q(s))) \, ds/i! \leq C_{i}(t_{1}) \quad \text{for all } t \geq t_{1}.
\]

(9)

It is clear from (4) that (9) holds also for \(i = 0\). Hence (9) holds for any \(i \in \{0, 1, \ldots, n - 1\}\) whenever \(x^{(n-i-1)}(t)\) is bounded. We will show that under the conditions of the theorem we have \(x^{(n-i-1)}(t) \to 0\) as \(t \to \infty\).

As \(x^{(n-i-1)}(t)\) is monotone and of constant sign for all \(t \geq t_{1}\), then \(x^{(n-i-1)}(t) \to l\) as \(t \to \infty\), where \(-\infty < l < \infty\). If \(l \neq 0\), then there exists \(c_{1} > 0\) such that either \(x^{(n-i-1)}(t) \geq c_{1}\) or \(x^{(n-i-1)}(t) \leq -c_{1}\) for all \(t \geq t_{1}\). The case \(x^{(n-i-1)}(t) \leq -c_{1}\) implies by successive integrations that \(x(t) \to -\infty\) as \(t \to \infty\) and hence a contradiction. Now assume \(x^{(n-i-1)}(t) \geq c_{1}\) for all \(t \geq t_{1}\); then by successive integrations from \(t_{1}\) to \(t\) we have

\[
x(t) \geq [c_{1}/(n - i - 1)!](t - t_{1})^{n-i-1} + \cdots + x(t_{1}) \quad \text{for all } t \geq t_{1}.
\]

Choose \(t_{2} \geq t_{1}\) and \(c_{2} > 0\) so that \(x(t) \geq c_{2}t^{n-i-1}\) for all \(t \geq t_{2}\). As \(i \neq n - 1\), then there exists \(t_{3} \geq t_{2}\) so that \(x(t) \geq c_{2}t^{n-i-1} \geq \alpha\) for all \(t \geq t_{3}\). Choose \(t_{4} \geq t_{3}\) so that \(q(t) \geq t_{3}\) for all \(t \geq t_{4}\); then \(x(q(t)) \geq c_{2}q^{n-i-1}(t) \geq \alpha\) for all \(t \geq t_{4}\). By Lemma 5 there exists \(\beta > 0\) such that \(f(x(q(t))) \geq \beta f(c_{2}q^{n-i-1}(t))\) for all \(t \geq t_{4}\). Hence, by (9), we have

\[
\beta \int_{t_{4}}^{t} s^{i}a(s)f(c_{2}q^{n-i-1}(s)) \, ds/i! \leq C_{i}(t_{4}) + (-1)^{i+1}x^{(n-i-1)}(t) < \infty
\]

for all \(t \geq t_{4}\).

This is a contradiction. Thus either \(x(t)\) oscillates or \(x^{(n-i-1)}(t) \to 0\) as \(t \to \infty\); consequently, by Lemma 2, either \(x(t)\) oscillates or \(t^{k}x^{(n-i-1+k)}(t) \to 0\) as \(t \to \infty\), \(k = 0, 1, \ldots, i\).

Now assume \(x(t) < 0\) for all \(t \geq t_{0}\) and let \(y(t) = -x(t)\); then \(y(t)\) is a solution of (7). It is clear that \(y(t)\) and \(f^{*}\) satisfy the hypotheses of Theorem 2; hence the conclusion of the theorem holds for \(y(t)\) and consequently for \(x(t)\). The proof now is complete.
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**Corollary 3.** Suppose (A) holds and \( f \in C^1_c(R_+) \) for some \( \alpha > 0 \).

If \( \int^\infty a(t) f[\pm c q^{n-1}(t)] \, dt = \pm \infty \) for every \( c > 0 \), then every solution \( x(t) \) of (1) either oscillates or \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \).

**Proof.** Let \( x(t) \) be a solution of Eq. (1) and assume \( x(t) \) does not oscillate; then, by Lemma 1, \( |x^{(n-1)}(t)| \) is eventually decreasing and hence \( x^{(n-1)}(t) \) is bounded. Thus, by Theorem 2, \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \).

Marušiak [9] extended Waltman's result to Eq. (3) in a way to insure oscillation of (3) when \( n \) is even. This result if restricted to Eq. (1) yields:

If \( f \) is nondecreasing and \( \int^\infty \frac{t}{a(t)} f[\pm c q^{n-i}(t)] \, dt = \pm \infty \) for every \( c \neq 0 \) and every \( i \in \{2, \ldots, n\} \), then (1) is oscillatory, while, for \( n \) odd, every solution \( x(t) \) of (1) either oscillates or \( x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).

The following theorem will improve this result.

**Theorem 3.** Suppose (A) holds and \( f \in C^1_c(R_+) \) for some \( \alpha > 0 \).

If

\[
\int^\infty t^i a(t) f[\pm c q^{n-i-2}(t)] \, dt = \pm \infty
\]

for every \( c > 0 \) and every \( i \in \{0, 1, \ldots, n - 2\} \), then, for \( n \) even, (1) is oscillatory, while, for \( n \) odd, every solution \( x(t) \) of (1) either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).

**Proof.** Let \( x(t) \) be a solution of Eq. (1) and suppose \( x(t) \) does not oscillate; then, for \( i = 0 \), the integral condition of this theorem implies that of Corollary 3 (Lemma 5) and hence \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \). We propose to show, by induction on \( i \), that \( x(t) \to 0 \) as \( t \to \infty \). Suppose, for some \( i \in \{0, 1, \ldots, n - 2\} \), that \( x^{(n-i-1)}(t) \to 0 \) as \( t \to \infty \); then we show that \( x^{(n-i-2)}(t) \to 0 \) as \( t \to \infty \).

As \( x(t) \) is nonoscillatory, then there exists \( t_0 \geq 0 \) such that \( x(t) \neq 0 \) for all \( t \geq t_0 \). Assume \( x(t) > 0 \) for all \( t \geq t_0 \); then, by Lemma 1, there exists \( t_1 \geq t_0 \) such that \( x^{(k)}(t) \), \( k = 0, 1, \ldots, n - 1 \), are of constant sign for all \( t \geq t_1 \). As \( x^{(n-i-2)}(t) \to 0 \) when \( t \to \infty \), then it follows from Lemma 2 that

\[
\sum_{k=0}^{i} (-1)^{i+k} k x^{(n-i-1+k)}(t) / k! \geq 0 \quad \text{for all} \quad t \geq t_1.
\]

Thus, by (4), we have

\[
\int_{t_1}^{t} s^i a(s) f(x(q(s))) \, ds / i! \leq C(t_1) \quad \text{for all} \quad t \geq t_1.
\]

As \( x^{(n-i-2)}(t) \) is monotone and of constant sign for all \( t \geq t_1 \), then \( x^{(n-i-2)}(t) \to l \) as \( t \to \infty \), where \( 0 \leq l \leq \infty \) since otherwise \( x(t) \) becomes negative. We propose to show that if \( x(t) \) does not oscillate, then \( l = 0 \). Suppose \( i = n - 2 \). If \( 0 < l < \infty \), then \( x(t) \) is bounded and hence, by the integral
condition of the theorem and Corollary 2, we have a contradiction. If \( l = \infty \), then there exists \( T \geq t_1 \) so that \( x(q(t)) \geq \alpha \) for all \( t \geq T \) and hence, by Lemma 5, there exists \( \beta > 0 \) such that \( f(x(q(t))) \geq \beta f(\alpha) \) for all \( t \geq T \). By (10), we have 
\[
\int_{t_1}^{t} s^{n-2}a(s) \, ds < \infty \quad \text{for all} \quad t \geq t_4.
\]
This contradicts the integral conditions of the theorem. If \( i \neq n - 2 \) and \( l \neq 0 \), then, as in the proof of Theorem 2, there exists \( \epsilon_2 > 0 \) and \( t_4 \) sufficiently large so that \( x(q(t)) \geq \epsilon_2 t^{n-i-2}(t) \geq \alpha \) for all \( t \geq t_4 \) and hence, by Lemma 5, there exists \( \beta > 0 \) such that \( f(x(q(t))) \geq \beta f(\epsilon_2 t^{n-i-2}(t)) \) for all \( t \geq t_4 \). Thus, by (10), we obtain
\[
\int_{t_4}^{t} s^{i}a(s) \, f(\epsilon_2 t^{n-i-2}(s)) \, ds < \infty \quad \text{for all} \quad t \geq t_4,
\]
a contradiction. Thus if \( x(t) \) does not oscillate we must have \( l = 0 \) and hence \( x(t) \to 0 \) as \( t \to \infty \). Consequently, by Lemma 2, either \( x(t) \) oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \). If \( n \) is even, then \( x'(t) > 0 \) for all \( t \geq t_2 \) and hence \( x(t) \) cannot tend to zero which implies that \( x(t) \) must oscillate.

If we assume \( x(t) < 0 \) for all \( t \geq t_0 \) and we let \( y(t) = -x(t) \), then \( y(t) > 0 \) is a solution of (7). As \( f^* \) satisfies the hypotheses of Theorem 3, then \( y(t) \) and hence \( x(t) \) satisfy the conclusion of the theorem. The proof is now complete.

**EXAMPLE 2.** Let \( f \) be the function defined in Example 1 and consider the equation

\[
x''(t) + t^{-2}f(x(t/2)) = 0 \quad (\ast)
\]

It is clear that Corollary 1 fails to apply while Theorem 3 implies that every solution \( x(t) \) of (\ast) is either oscillatory or \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, 2 \).

Our next theorem is basic to the proof that the integral condition in Corollary 1 is the best possible under the specified conditions. This theorem also improves [8, Theorem 3.2] when restricted to Eq. (1).

**THEOREM 4.** Suppose \( (\Lambda) \) holds and \( f \in C_{\infty}(R_+^+ \| R_+) \) for some \( \alpha > 0 \).

If, for every solution \( x(t) \) of (1), either \( x(t) \) oscillates or \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \), then
\[
\int_{t_1}^{\infty} a(t) \, h[\pm c q^{n-1}(t)] \, dt = \pm \infty \quad \text{for every} \quad c > 0 \quad \text{and every nondecreasing component} \ h \ \text{of} \ f.
\]

**Proof.** Suppose \( \int_{t_1}^{\infty} a(t) \, h[\pm c q^{n-1}(t)] \, dt < \infty \) for some \( c > 0 \) and some non-decreasing component \( h \). Choose \( t_0 \) so that \( t_0^{-1} \geq \alpha/c \) and choose \( t_1 \) sufficiently large so that \( g(t) \geq t_0 \) for all \( t \geq t_1 \) and \( \int_{t_1}^{\infty} a(t) \, h[\pm c q^{n-1}(t)] \, dt < \alpha_1/[2g(\alpha)] \) where \( \alpha_1 = (n - 1)! c \) and \( g \) is the positive component corresponding to \( h \).

Let \( x(t) \) be a solution of Eq. (1) such that on \( E_{t_1}, x^{(n-k)}(t) = \alpha_1 t^{k-1}/(k - 1)!, k = 1, 2, \ldots, n \).

Observe that \( x(t) = ct^{n-1} \) for all \( t \in E_{t_1} \).

As \( x^{(n-1)}(t_1) = \alpha_1 > 0 \), then there exists \( t_2 > t_1 \) such that \( x^{(n-1)}(t) > 0 \) for
all \( t \in [t_1, t_2] \); hence \( x^{(n-k)}(t) \), \( k = 2, \ldots, n \), are increasing and \( x(q(t)) > 0 \) for all \( t \in [t_1, t_2] \). Thus \( x^{(n)}(t) = -a(t)f(x(q(t))) \leq 0 \) and hence \( x^{(n-1)}(t) \leq x^{(n-1)}(t_1) = \alpha_1 \) for all \( t \in [t_1, t_2] \). By successive integrations from \( t_1 \) to \( t \) we obtain \( x(t) \leq \alpha_1 t^{n-1}/(n-1)! \) or \( x(t) \leq c t^{n-1} \) for all \( t \in [t_1, t_2] \). As \( q(t) \in E_t \cup [t_1, t_2] \) for all \( t \in [t_1, t_2] \), then \( x(q(t)) \leq c q^{n-1}(t) \) for all \( t \in [t_1, t_2] \). On the other hand, for \( t \in [t_1, t_2] \) we have either \( q(t) \in [t_1, t_2] \) and hence \( x(q(t)) > x(t_1) = c t_1^{n-1} > 0 \) or \( q(t) \in E_{t_2} \) and hence \( x(q(t)) = c q^{n-1}(t) > c t_2^{n-1} > \alpha \). Consequently \( \alpha \leq x(q(t)) \leq c q^{n-1}(t) \) for all \( t \in [t_1, t_2] \). By Lemma 4, \( h(x(q(t))) \leq h(c q^{n-1}(t)) \) and \( g(x(q(t))) \leq g(\alpha) h(c q^{n-1}(t)) \) for all \( t \in [t_1, t_2] \). As \( x^{(n)}(t) = -a(t)f(x(q(t))) \geq -g(\alpha) a(t) h(c q^{n-1}(t)) \), then

\[
x^{(n-1)}(t) \geq x^{(n-1)}(t_2) - g(\alpha) \int_{t_1}^{t} a(s) h(c q^{n-1}(s)) \, ds > \alpha_1 - \alpha_1/2 = \alpha_1/2
\]

for all \( t \in [t_1, t_2] \). Thus as long as \( x^{(n-1)}(t) \) remains positive we have \( x^{(n-1)}(t) > \alpha_1/2 \) which implies that \( x^{(n-1)}(t) > \alpha_2/2 \) for all \( t > t_1 \). Hence \( x^{(n-1)}(t) \) does not tend to zero and \( x(t) \) does not oscillate.

Now, assume \( \int_{t_1}^{t} a(t) h(-c q^{n-1}(t)) \, dt > -\infty \) for some \( c > 0 \) and some nondecreasing component \( h \) of \( f \). Let \( g \) be the positive component of \( f \) corresponding to \( h \). Define \( f^* \) as in Remark 2 and \( h^* \) and \( g^* \) as follows:

\[
h^*(x) = h(x) \quad \text{if } x < -\alpha \\
= -h(-x) \quad \text{if } x \geq \alpha,
\]

\[
g^*(x) = g(x) \quad \text{if } x < -\alpha \\
= g(-x) \quad \text{if } x \geq \alpha.
\]

It is clear that \( f^*(x) = g^*(x) h^*(x) \) for all \( x \geq \alpha \) and

\[
\int_{t_1}^{t} a(t) h^*[c q^{n-1}(t)] \, dt < \infty.
\]

By the above proof there exists a solution \( y(t) > 0 \) of Eq. (7) with the property that \( y(t) \) does not oscillate and \( y^{(n-1)}(t) \) does not tend to zero as \( t \to \infty \). Let \( x(t) = -y(t) \); then \( x(t) \) is a solution of Eq. (1) which does not oscillate and such that \( x^{(n-1)}(t) \) does not tend to zero as \( t \to \infty \). This completes the proof of the theorem.

**Corollary 4.** Suppose (A) holds and \( f \in C_1(R_+) \) for some \( \alpha > 0 \).

For every solution \( x(t) \) of (1), either \( x(t) \) oscillates or \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \) if and only if

\[
\int_{t_1}^{t} a(t) f([-c q^{n-1}(t)]) \, dt = \pm \infty \quad \text{for every } c > 0.
\]
Proof. Suppose (11) holds; then the conclusion follows from Corollary 3.

Now suppose every solution $x(t)$ of (1) either oscillates or $x^{(n-1)}(t) \to 0$ as $t \to \infty$; then, by Theorem 4, \[ \int_{-\infty}^{\infty} a(t) h[\pm cq^{n-1}(t)] \, dt = \pm \infty \] for every $c > 0$ and every nondecreasing component $h$ of $f$. Let $g$ be a positive component of $f$ bounded away from zero and let $h$ be the corresponding nondecreasing component; then there exists $\beta > 0$ such that $\beta \leq g(x) \leq g(\alpha)$ for $x \geq \alpha$ and $\beta \leq g(x) \leq g(-\alpha)$ for $x \leq -\alpha$. As $f(x) = g(x) h(x)$ for all $|x| \geq \alpha$, then for $x \geq \alpha$ we have $f(x) \geq \beta h(x)$ and for $x \leq -\alpha$ we have $f(x) \leq \beta h(x)$. Thus the integral condition of Theorem 4 implies (11) and the proof is complete.

COROLLARY 5. Suppose (A) holds, $f \in C_1(R_\alpha)$ for some $\alpha > 0$ and $f$ is bounded above or below.

For $n$ even, (1) is oscillatory if and only if \[ \int_{-\infty}^{\infty} a(t) f[\pm cq^{n-1}] \, dt = \pm \infty \quad \text{for every } c > 0. \] (12)

For $n$ odd, every solution $x(t)$ of (1) either oscillates, or $t^k x^{(k)}(t) \to 0$ as $t \to \infty$, $k = 0, 1, ..., n - 1$, if and only if (12) holds.

Proof. If for every solution $x(t)$ of (1) either $x(t)$ oscillates or $t^k x^{(k)}(t) \to 0$ as $t \to \infty$, then (12) follows from Corollary 4. Now, assume (12) holds and $f$ is bounded above; then $f(x) \leq M$ for some $M > 0$ and hence $a(t) f(x) \leq Ma(t)$ for all $x$ and all $t \geq 0$. Thus \[ \int_{-\infty}^{\infty} a(t) f[\pm cq^{n-1}] \, dt \leq M \int_{-\infty}^{\infty} a(t) \, dt \] and hence \[ \int_{-\infty}^{\infty} a(t) \, dt = \infty. \] The result then follows from Corollary 1. The case $f$ bounded below is similar and the proof is omitted.

COROLLARY 6. Suppose (A) holds, $f \in C_1(R_\alpha)$ for some $\alpha > 0$, and \[ \lim_{t \to \infty} \inf [q(t)/t] > 0. \]

For every solution $x(t)$ of (1) either $x(t)$ oscillates or $x^{(n-1)}(t) \to 0$ as $t \to \infty$ if and only if \[ \int_{-\infty}^{\infty} a(t) f(\pm ct^{n-1}) \, dt = \pm \infty \quad \text{for every } c > 0. \] (13)

Proof. It follows from Corollary 4 if we show that the integral conditions (11) and (13) are equivalent.

Let $m_1 = \lim_{t \to \infty} \inf [q(t)/t]$; then there exists $t_1 \geq 1$ such that $q(t)/t \geq m_1/2$ for all $t \geq t_1$ and hence $t \geq q(t) \geq m_1 t/2 \geq m_1/2$ for all $t \geq t_1$.

Let $c > 0$ be arbitrary. Choose $t_2 \geq t_1$ so that $ct^{n-1} \geq cq^{n-1} \geq c_1 t^{n-1} \geq \alpha$ for all $t \geq t_2$, where $c_1 = c(m_1/2)^{n-1}$. By Lemma 5 there exists $\beta > 0$ such that \[ f(\alpha) \beta f(cq^{n-1}) \beta^2 f(c_1 t^{n-1}) \leq \beta^2 f(-c_1 t^{n-1}) \quad \text{for all } t \geq t_2 \] and \[ f(0) \geq f(0) \geq f(-cq^{n-1}) \beta f(c_1 t^{n-1}) \geq \beta^2 f(-c_1 t^{n-1}) \quad \text{for all } t \geq t_2 \]. Thus, (11) and (13) are equivalent and the proof is complete.
THEOREM 5. Suppose (A) holds and \( f \in C_\alpha(R_\alpha) \) for some \( \alpha > 0 \).

If, for every choice of \( q(t) \), every solution \( x(t) \) of (1) either oscillates or \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \), then \( \int^\infty a(t) \, dt = \infty \).

Proof. By Theorem 4, \( \int^\infty a(t) \, h[\pm c^n(t)] \, dt = \pm \infty \) for every \( c > 0 \) and every nondecreasing component \( h \) of \( f \). If \( f \) has a nondecreasing component bounded above or below, then \( \int^\infty a(t) \, dt = \infty \) follows at once.

Assume \( f \) has no nondecreasing component which is bounded above or below and let \( h \) and \( g \) be a decomposition of \( f \) on \( R_\alpha \). We may assume \( h \) is increasing for \( x \geq \alpha \) since \( xh \) and \( (1/x)g \) are respectively increasing and decreasing for \( x \geq \alpha \). If \( \int^\infty a(t) \, dt < \infty \), then, as in the proof of [1, Theorem 1], there exists a non-decreasing continuous function \( P_2 : [0, \infty) \to [1, \infty) \) which is onto and such that \( \int^\infty a(t) \, P_2(t) \, dt < \infty \).

As \( h \) is increasing for \( x \geq \alpha \), then \( h^{-1} : [h(\alpha), \infty) \to [\alpha, \infty) \). Choose \( t_1 \) so that \( P_2(t) \geq h(\alpha) \) for all \( t \geq t_1 \). Define \( q(t) \) as follows:

\[
q(t) = \min(t^{-1}, h^{-1}(P_2(t))) \quad \text{for} \quad t \geq t_1
\]

\[
q(t) = \min(t, h^{-1}(P_2(t))) \quad \text{for} \quad 0 \leq t < t_1.
\]

It is clear that \( q(t) \) is continuous, \( q(t) \leq t \) for all \( t \geq 0 \), and \( q(t) \to \infty \) as \( t \to \infty \). Furthermore, \( q^{n-1}(t) \leq h^{-1}(P_2(t)) \) for all \( t \geq t_1 \). Choose \( t_2 \geq t_1 \) so that \( q(t) \geq \alpha^{1/n-1} \) for all \( t \geq t_2 \); then \( \alpha \leq q^{n-1}(t) \leq h^{-1}(P_2(t)) \) and hence \( h(q^{n-1}(t)) \leq P_2(t) \) for all \( t \geq t_2 \). Thus

\[
\int_{t_2}^{t} a(s) \, h(q^{n-1}(s)) \, ds \leq \int_{t_2}^{t} a(s) \, P_2(s) \, ds < \infty
\]

for all \( t \geq t_2 \). This is a contradiction and the theorem is proved.

COROLLARY 6. Suppose (A) holds, \( f \in C_\alpha(R_\alpha) \) for some \( \alpha > 0 \), and \( \lim_{x \to \pm \infty} \inf |f(x)| > 0 \).

For \( n \) even, (1) is oscillatory, for every choice of \( q(t) \), if and only if \( \int^\infty a(t) \, dt = \infty \).

For \( n \) odd, every solution \( x(t) \) of (1) either oscillates of \( x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \), for every choice of \( q(t) \), if and only if \( \int^\infty a(t) \, dt = \infty \).

Proof. It follows from Theorem 5 and Corollary 1.

Remark 7. Theorem 5 is an extension of [1, Theorem 11] which together with Corollary 6 yields a significant improvement of [1, Theorem 11] and its corresponding corollary respectively.

In the following theorem we give a sufficient condition for oscillation and asymptotic behavior of solutions of (1) when \( f \in C_\alpha(R_\alpha) \) for some \( \alpha > 0 \).
THEOREM 6. Suppose (A) holds and \( f \in C_p(R_+) \) for some \( \alpha > 0 \). If
\[
\int_0^{\infty} a(t) g[\pm cq^{n-1}(t)] \, dt = \infty
\]
for every \( c > 0 \) and for some positive component \( g \) of \( f \), then, for \( n \) even, (1) is oscillatory, while, for \( n \) odd, every solution \( x(t) \) of (1) either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).

Proof. Let \( x(t) \) be a solution of (1). If \( x(t) \) does not oscillate, then there exists \( t_0 \geq 0 \) such that \( x(t) \neq 0 \) for all \( t \geq t_0 \). Assume \( x(t) > 0 \) for all \( t \geq t_0 \); then, by Lemma 1, there exists \( t_1 \geq t_0 \) such that \( x^{(k)}(t), k = 0, 1, \ldots, n - 1 \), are of constant sign for all \( t \geq t_1 \). In particular, \( x^{(n-1)}(t) > 0 \) and \( x^{(n)}(t) < 0 \) for all \( t \geq t_1 \) and hence \( x^{(n-1)}(t) \) is nonincreasing for all \( t \geq t_1 \). Thus there exists \( c > 0 \) such that \( x^{(n-1)}(t) < c \) for all \( t \geq t_1 \).

By successive integrations from \( t_1 \) to \( t \) we conclude that
\[
x(t) < \left[ c/(n-1)! \right] (t - t_1)^{n-1} + \cdots + x(t_1) \quad \text{for all } t \geq t_1.
\]

Choose \( t_2 \geq t_1 \) and \( c_1 > 0 \) so that \( x(t) \leq c_1 t^{n-1} \) for all \( t \geq t_2 \). As \( x(t) \) is monotone for \( t \geq t_2 \), then \( x(t) \to l \) as \( t \to \infty \) where \( 0 < l < \infty \). We propose to show that \( l = 0 \). If \( 0 < l < \infty \), then \( x(t) \) is bounded. Choose \( t_3 \geq t_2 \) so that \( c_1 q^{n-1}(t) > \alpha \) for all \( t \geq t_3 \) and hence \( g[c_1 q^{n-1}(1)] \leq g(\alpha) \). It follows from the integral conditions of the theorem that \( \int_{t_3}^{\infty} a(t) \, dt = \infty \) and hence, by Corollary 2, \( x(t) \) must oscillate or \( x(t) \to 0 \) as \( t \to \infty \), a contradiction. If \( l = \infty \), we choose \( t_3 \geq t_2 \) so that \( \alpha \leq x(q(t)) \leq c_1 q^{n-1}(t) \) for all \( t \geq t_3 \). Let \( h \) be the nondecreasing component of \( f \) corresponding to \( g \); then, by Lemma 4, \( g(\alpha) \geq g(x(q(t))) \geq g(c_1 q^{n-1}(t)) \) and \( h(c_1 q^{n-1}(t)) \geq h(x(q(t))) \geq h(\alpha) \) for all \( t \geq t_3 \). Thus \( f(x(q(t))) \geq h(\alpha) g(c_1 q^{n-1}(t)) \) and hence \( x^{(n)}(t) = -a(t) f(x(q(t))) < -h(\alpha) a(t) g(c_1 q^{n-1}(t)) \) for all \( t \geq t_3 \). Thus \( x^{(n-1)}(t) < x^{(n-1)}(t_3) - h(\alpha) \int_{t_3}^{t} a(s) g(c_1 q^{n-1}(s)) \, ds \) for all \( t \geq t_3 \) and hence by the integral condition of the theorem we conclude that \( x^{(n-1)}(t) \to -\infty \) as \( t \to \infty \), a contradiction. Consequently, either \( x(t) \) oscillates or \( x(t) \to 0 \) as \( t \to \infty \) and hence, by Lemma 2, either \( x(t) \) oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).

If \( n \) is even, then \( x'(t) > 0 \) for all \( t \geq t_1 \) and hence \( x(t) \geq x(t_1) \) for all \( t \geq t_1 \). Thus \( x(t) \) must oscillate.

Assume \( x(t) < 0 \) for all \( t \geq t_0 \) and let \( y(t) = -x(t) \); then \( y(t) \) is a solution of (7). Define \( g^* \) and \( h^* \) as in the proof of Theorem 4; then \( f^* \) and \( g^* \) satisfy the hypotheses of Theorem 6. Thus the conclusion of the theorem follows for \( y(t) \) and hence for \( x(t) \). The proof is now complete.

COROLLARY 7. Suppose (A) holds and \( f \in C_p(R_+) \) for some \( \alpha > 0 \). If
\[
\int_0^{\infty} a(t) f(\pm cq^{n-1}(t)) \, dt = \pm \infty \quad \text{for every } c > 0,
\]
then, for \( n \) even, (1) is oscillatory, while, for \( n \) odd, every solution \( x(t) \) of (1) either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \), \( k = 0, 1, \ldots, n - 1 \).
**Proof.** As \( f \in C_p(R_\alpha) \), then there exists a bounded nondecreasing component \( h \) of \( f \). Let \( g \) be the corresponding positive component; then there exists \( M > 0 \) such that \( |h(x)| \leq M \) for all \( |x| \geq \alpha \) and hence \( |f(x)| \leq Mg(x) \) for all \( |x| \geq \alpha \). Thus the integral condition of the corollary implies the integral condition of Theorem 6 and the proof now is complete.

**Example 3.** The equation \( x''(t) + t^{-\beta}x'(t^2)/(1 + x^2(t^2)) = 0 \) where \( 0 \leq \beta < 1 \), \( 0 < \alpha \leq 1 \), and \( 0 < \beta + \alpha < 1 \) is oscillatory by Theorem 6. Observe that Corollary 1 and Theorem 3 fail to apply.

**Theorem 7.** Suppose \((A)\) holds and \( f \in C_p(R_\alpha) \) for some \( \alpha > 0 \). If, for every solution \( x(t) \) of \((1)\), either \( x(t) \) oscillates or \( x^{(n-1)}(t) \to 0 \) as \( t \to \infty \), then \( \int_{s^\infty}^\infty a(t)f[\pm cq^{n-1}(t)] \, dt = \pm \infty \) for every \( c > 0 \).

**Proof.** Suppose \( \int_{s^\infty}^\infty a(t)f[\pm cq^{n-1}(t)] \, dt < \infty \) for some \( c > 0 \). Choose \( t_0 \) and \( t_1 \geq t_0 \) so that \( ct_0^{n-1} \geq \alpha \) and \( q(t) \geq t_0 \) for all \( t \geq t_1 \). Let

\[
M = \int_{t_1}^{s^\infty} a(t)f[\pm cq^{n-1}(t)] \, dt
\]

and \( \alpha_1 = (n - 1)! \, c \).

Let \( K > \alpha_1 \), \( K \) to be determined, and let \( x(t) \) be a solution of \((1)\) such that on \( E_{\alpha_1} \) we have

\[
x^{(n-k)}(t) = \frac{ht^k}{(k - 1)!}, \quad k = 1, 2, \ldots, n.
\]

As \( x^{(n-1)}(t_1) = K > \alpha_1 \), then there exists \( t_2 > t_1 \) so that \( x^{(n-1)}(t) > \alpha_1 \) for all \( t \in [t_1, t_2] \); hence \( x^{(n-2)}(t) \geq \alpha_1(t - t_1) + x^{(n-2)}(t_1) \geq \alpha_1 t \) for \( t \in [t_1, t_2] \). By successive integrations we conclude that \( x(t) \geq \alpha_1 t^{n-1}((n-1)! \) or \( x(t) \geq ct_0^{n-1} \) for all \( t \in [t_1, t_2] \). As \( q(t) \in E_{\alpha_1} \cup [t_1, t_2] \) for \( t \in [t_1, t_2] \), then \( x(q(t)) \geq cq^{n-1}(t) \geq ct_0^{n-1} \geq \alpha \) for all \( t \in [t_1, t_2] \). By Lemma 5, there exists \( B > 0 \) such that \( f(x(q(t))) \leq Bf[cq^{n-1}(t)] \) and hence \( x^{(n)}(t) = -a(t)f(x(q(t))) \geq -Ba(t)f[cq^{n-1}(t)] \) for \( t \in [t_1, t_2] \). We integrate from \( t_1 \) to \( t \) to get \( x^{(n-1)}(t) \geq x^{(n-1)}(t_1) - B \int_{t_1}^{t} a(s)f[cq^{n-1}(s)] \, ds \geq K - BM \) for all \( t \in [t_1, t_2] \). Choose \( K > 2\alpha_1 + BM \); then \( x^{(n-1)}(t) \geq 2\alpha_1 \) for all \( t \in [t_1, t_2] \). It is clear from the proof that as long as \( x^{(n-1)}(t) > \alpha_1 \) we have \( x^{(n-1)}(t) \geq 2\alpha_1 \) which implies that \( x^{(n-1)}(t) > \alpha_1 \) for all \( t \geq t_1 \). Thus \( x^{(n-1)}(t) \) does not tend to zero as \( t \to \infty \) and \( x(t) \) does not oscillate.

If \( \int_{s^\infty}^\infty a(t)f[-cq^{n-1}(t)] \, dt > -\infty \) for some \( c > 0 \) and if we define \( f^* \) as in Remark 2, then \( \int_{s^\infty}^\infty a(t)f*[cq^{n-1}(t)] \, dt < \infty \). By the above proof, Eq. (7) has a solution \( y(t) > 0 \) which does not oscillate and such that \( y^{(n-1)}(t) \) does not tend to zero as \( t \to \infty \). Let \( x(t) = -y(t) \); then \( x(t) \) is a solution of \((1)\) which has the same property as \( y(t) \). The proof is now complete.
Corollary 8. Suppose (A) holds and \( f \in C_{\alpha}(R_\alpha) \) for some \( \alpha > 0 \).
For \( n \) even, (1) is oscillatory if and only if
\[
\int_{-\infty}^{\infty} a(t) f(\pm c q^{n-1}(t)) \, dt = \pm \infty \quad \text{for every } c > 0.
\] (14)

For \( n \) odd, every solution \( x(t) \) of (1) either oscillates or \( t^k x^{(k)}(t) \to 0 \) as \( t \to \infty \),
\( k = 0, 1, \ldots, n-1 \), if and only if (14) holds.

Proof. It follows from Corollary 7 and Theorem 7.

Remark 8. Corollary 8, together with the theorems from which it is derived,
extends [1, Theorem 6] and improves it.

Example 4. It was mentioned earlier that the equation
\[
x''(t) + [(1 + t)/4t^2] x(t)/[1 + x^\alpha(t)] = 0
\]
has a nonoscillatory solution \( x(t) = t^{1/2} \) and consequently the integral condition
of Corollary 1 is not enough to guarantee oscillation of Eq. (1) when \( n \) is even
unless \( \lim_{x \to \pm \infty} \inf |f(x)| > 0 \). In fact, by Theorem 7, the equation
\[
x''(t) + [(1 + t)/4t^2] x(t)/[1 + x^\alpha(t)] = 0
\]
where \( 0 < \alpha \leq 1 \) has a nonoscillatory solution \( x(t) \) such that \( x'(t) \) does not tend to zero as \( t \to \infty \).

Our next result is a generalization of two results obtained by Burton and
Grimmer in [1, 2]. This result yields a new oscillation criterion for Eq. (1) when
\( n \geq 2 \) is even and \( f \in C_\alpha(R_\alpha) \) for some \( \alpha > 0 \). It also improves the integral
condition of Corollary 1 under these specified conditions.

Theorem 8. Suppose (A) holds and \( f \in C_\alpha(R_\alpha) \) for some \( \alpha > 0 \).
If \( n \) is even and
\[
\int_{-\infty}^{\infty} a(t) f \left( \gamma \int_1^{q(t)} s^{n-1} a(s) f \left( \gamma \int_1^{q(s)} u^{n-1} a(u) \right) ds \right) \, dt = \pm \infty,
\]
then (1) is oscillatory. Here,
\[
\gamma = \min(\lim_{x \to \pm \infty} g(x)/g(\alpha), \lim_{x \to \pm \infty} g(x)/g(-\alpha))/[(n - 1)! 2^{n-1}]
\]
where \( g \) is some positive component of \( f \) bounded away from zero.

Proof. Let \( x(t) \) be a solution of (1). If \( x(t) \) does not oscillate, then there
exists \( t^* > 0 \) such that \( x(t) \neq 0 \) for all \( t \geq t^* \).
Assume \( x(t) > 0 \) for all \( t \geq t^* \); then, by Lemma 1, there exists \( t_0 \geq t^* \) and
an odd integer \( l \), \( 1 \leq l \leq n-1 \), such that \( x^{(k)}(t) > 0 \), \( k = 0, 1, \ldots, l \), and
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\((-1)^{k+1}x^{(k)}(t) > 0, \quad k = l + 1, \ldots, n - 1,\) for all \(t \geq t_0\). Choose \(t_1 \geq t_0\) so that \(q(t) \geq t_0\) for all \(t \geq t_1\). By formula (6), we have

\[
x(t) = x(t_1) + \sum_{k=1}^{n-1} (-1)^{k+1}(t - t_1)^k x^{(k)}(t)/k!
\]

\[+ \int_{t_1}^{t} (s - t_1)^{n-1} a(s) f(x(q(s))) \, ds/(n - 1)! \quad \text{for all } t \geq t_1.
\]

We propose to show that

\[
\sum_{k=1}^{n-1} (-1)^{k+1}(t - t_1)^k x^{(k)}(t)/k! \geq 0 \quad \text{for all } t \geq t_1.
\]

By Lemma 1, \(\sum_{k=1}^{n-1} (-1)^{k+1}(t - t_1)^k x^{(k)}(t)/k! \geq 0\) for all \(t \geq t_1\). Thus we need only show that \(v(t) = \sum_{k=1}^{n-1} (-1)^{k+1}(t - t_1)^k x^{(k)}(t)/k! \geq 0\) for all \(t \geq t_1\).

Differentiate \(v(t)\) to find that

\[
v'(t) = x'(t) - (t - t_1)^{l-1} x^{(l)}(t)/(l-1)!
\]

By Taylor's formula,

\[
x'(t) = x'(t_1) + x''(t_1)(t - t_1) + x'''(t_1)(t - t_1)^2/2! \quad \text{for all } t_1 < t < t.
\]

As \(x^{(i)}(t_1) > 0, \quad i = 1, \ldots, l\), then \(x'(t) \geq x^{(i)}(t_1)(t - t_1)^{l-1}/(l-1)!\). Since \(x^{(i)}(t_1) < 0\) for all \(t \geq t_1\), then \(x^{(i)}(t_1)\) is decreasing and hence \(x^{(i)}(\xi) \geq x^{(i)}(t_1)\).

Thus \(x'(t) \geq x^{(i)}(t_1)(t - t_1)^{l-1}/(l-1)!\) and hence \(v'(t) \geq 0\) for all \(t \geq t_1\). It follows that \(v(t)\) is nondecreasing and hence \(v(t) \geq v(t_1) = 0\) for all \(t \geq t_1\). Thus

\[
x(t) \geq \int_{t_1}^{t} (s - t_1)^{n-1} a(s) f(x(q(s))) \, ds/(n - 1)! \quad \text{for all } t \geq t_1.
\]

Choose \(t_2 \geq 2t_1\) so that \(s - t_1 \geq s/2\) for all \(s \geq t_2\) and hence

\[
x(t) \geq \int_{t_2}^{s} s^{n-1} a(s) f(x(q(s))) \, ds/[(n - 1)! 2^{n-1}] \quad \text{for all } t \geq t_2 \quad (15)
\]

It is clear from the integral condition of the theorem that \(\int_{t_0}^{\infty} t^{n-1} a(t) \, dt = \infty\). As \(x(t)\) is increasing for \(t \geq t_0\), then \(x(t) \to l\) as \(t \to \infty\). If \(l\) is finite, then \(x(t)\) is bounded and hence, by Corollary 2, \(x(t)\) must oscillate, a contradiction. If \(l = \infty\), then \(x(q(t)) \to \infty\) as \(t \to \infty\) and hence there exists \(t_2 \geq t_2\) so that \(x(q(t)) \geq \alpha\) for all \(t \geq t_3\). By Lemma 5, \(f(x(q(t))) \geq \beta f(\alpha)\) for all \(t \geq t_3\), where \(\beta = \min(\lim_{x \to \infty} g(x)/g(\alpha), \lim_{x \to -\infty} g(x)/g(-\alpha))\). Thus, by (15),

\[
x(t) \geq \gamma \int_{t_3}^{t} s^{n-1} a(s) f(\alpha) \, ds \quad \text{for all } t \geq t_3.
\]

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where \( \gamma = \beta / [(n - 1)! \ 2^{n-1}] \). Choose \( t_4 \geq t_3 \) so that \( \gamma \int_{t_3}^{t} s^{n-1} a(s) f(\alpha) \, ds \geq \alpha \) for all \( t \geq t_4 \); then \( x(t) \geq \gamma \int_{t_0}^{t} s^{n-1} a(s) f(\alpha) \, ds \geq \alpha \) for all \( t \geq t_4 \). Choose \( t_5 \geq t_4 \) so that \( q(t) \geq t_4 \) for all \( t \geq t_5 \); then \( x(q(t)) \geq \gamma \int_{t_0}^{q(t)} s^{n-1} a(s) f(\alpha) \, ds \geq \alpha \) for all \( t \geq t_5 \). By Lemma 5,

\[
f(x(q(t))) \geq \beta f \left( \gamma \int_{t_3}^{q(t)} s^{n-1} a(s) f(\alpha) \, ds \right) \geq \beta^2 f(\alpha) \quad \text{for all} \quad t \geq t_5.
\]

By (15),

\[
x(t) \geq \gamma \int_{t_3}^{t} s^{n-1} a(s) f(x(q(s))) \, ds / [(n - 1)! \ 2^{n-1}]
\geq \gamma \int_{t_3}^{t} s^{n-1} a(s) f \left( \gamma \int_{t_3}^{q(s)} u^{n-1} a(u) f(\alpha) \, du \right) \, ds
\geq \beta \gamma \int_{t_3}^{t} s^{n-1} a(s) f(\alpha) \, ds \quad \text{for all} \quad t \geq t_5.
\]

Choose \( t_6 \geq t_5 \) so that \( \beta \gamma \int_{t_3}^{t} s^{n-1} a(s) f(\alpha) \, ds \geq \alpha \) for all \( t \geq t_6 \) and choose \( t_7 \geq t_6 \) so that \( q(t) \geq t_6 \) for all \( t \geq t_7 \); then

\[
x(q(t)) \geq \gamma \int_{t_3}^{q(t)} s^{n-1} a(s) f \left( \gamma \int_{t_3}^{q(s)} u^{n-1} a(u) f(\alpha) \, du \right) \, ds \geq \alpha \quad \text{for all} \quad t \geq t_7,
\]

and hence by Lemma 5,

\[
f(x(q(t))) \geq \beta f \left( \gamma \int_{t_3}^{q(t)} s^{n-1} a(s) f \left( \gamma \int_{t_3}^{q(s)} u^{n-1} a(u) f(\alpha) \, du \right) \, ds \right) \geq \beta^2 f(\alpha)
\quad \text{for all} \quad t \geq t_7.
\]

By (15),

\[
x(t) \geq \gamma \int_{t_3}^{t} s^{n-1} a(s) f \left( \gamma \int_{t_3}^{q(s)} u^{n-1} a(u) f(\alpha) \, du \right) \, ds
\geq \beta \gamma \int_{t_3}^{t} s^{n-1} a(s) f(\alpha) \, ds \geq \alpha \quad \text{for all} \quad t \geq t_7.
\]

Choose \( t_8 \geq t_7 \) so that \( q(t) \geq t_7 \) for all \( t \geq t_8 \); then

\[
x(q(t)) \geq \gamma \int_{t_3}^{q(t)} s^{n-1} a(s) f \left( \gamma \int_{t_3}^{q(s)} u^{n-1} a(u) f(\alpha) \, du \right) \, ds
\geq \alpha \quad \text{for all} \quad t \geq t_8.
\]
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and hence, by Lemma 5,

\[ f(x(q(t))) \geq \beta f \left( \gamma \int_{t_0}^{q(t)} s^{n-1} a(s) \int_{t_0}^{s} u^{n-1} a(u) \right. \\
\left. \cdot f \left( \gamma \int_{t_0}^{q(u)} r^{n-1} a(r) f(\alpha) dr \right) du \right) ds \quad \text{for all } t \geq t_0. \]

In general, there exists \( T \) sufficiently large so that

\[ f \left( \gamma \int_{t_0}^{q(t)} s^{n-1} a(s) \int_{t_0}^{s} u^{n-1} a(u) \right. \\
\left. \cdot f \left( \gamma \int_{t_0}^{q(u)} r^{n-1} a(r) f(\alpha) dr \right) du \right) ds \quad \text{for all } t \geq T. \]

From Eq. (1), we conclude that

\[ x^{(n)}(t) \leq -\beta a(t) f \left( \gamma \int_{t_0}^{q(t)} s^{n-1} a(s) \int_{t_0}^{s} u^{n-1} a(u) \right. \\
\left. \cdot f \left( \gamma \int_{t_0}^{q(u)} r^{n-1} a(r) f(\alpha) dr \right) du \right) ds \quad \text{for all } t \geq T. \]

Thus, by the integral condition of the theorem, \( x^{(n-1)}(t) \to -\infty \) as \( t \to \infty \) which is a contradiction. Thus \( x(t) \) must oscillate.

Assume \( x(t) < 0 \) for all \( t \geq t^* \) and let \( y(t) = -x(t) \); then \( y(t) \) is a solution of (7). As \( f^* \) satisfies the conditions of the theorem, then \( y(t) \) is oscillatory and so is \( x(t) \). The proof is now complete.

**Remark 9.** If \( f \) in Theorem 8 is nondecreasing, then \( \gamma = 1/[(n-1)! 2^{n-1}] \) and \( \alpha \) can be chosen so that \( f(\alpha) = 1 \) or \( f(-\alpha) = -1 \).

It is also clear that Theorem 8 is an improvement of Theorem 3 when \( n = 2 \). Interesting examples in this case can be found in [1, 2]. However, for higher order equations, Theorem 8 shows that the equation \( x''(t) + \frac{t^{-1/2}}{x^{(1/2)}} = 0 \) is oscillatory while Theorem 3 fails to apply.

**REFERENCES**

2. T. Burton and R. Grimmer, Oscillatory solutions of \( x''(t) + a(t) f(x(q(t))) = 0 \), in


