JOURNAL OF ALGEBRA **170**, 805–860 (1994)

Three-Dimensional Purely Monomial Group Actions

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Communicated by Richard G. Swan

Received October 21, 1992

1. INTRODUCTION

Let K be any field and $K(x_1, \ldots, x_n)$ the rational function field of n variables over K. A K-automorphism σ of $K(x_1, \ldots, x_n)$ is said to be monomial if

$$\sigma(x_i) = a_i(\sigma) \prod_{j=1}^{n} x_j^{m_{ij}},$$

where $(m_{ij})_{1 \le i, j \le n}$ is an invertible $n \times n$ matrix with integer entries and where $a_i(\sigma) \in K \setminus \{0\}$. If $a_i(\sigma) = 1\ \forall i$, then σ is called purely monomial. It is proved in [6; 8] that if G is any finite group of monomial K-automorphisms of $K(x_1, x_2)$, then its fixed field $K(x_1, x_2)^G$ is rational (= purely transcendental) over K. This does not generalize to the three-variable case since the fixed field of the monomial K-automorphism σ defined on $K(x_1, x_2, x_3)$ by

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto -\frac{1}{x_1 x_2 x_3}$$

is not rational over K when K is the field of rational numbers [6, last paragraph]. Hence we consider the actions of purely monomial groups of $K(x_1, x_2, x_3)$. According to [20; 1] there are 73 conjugacy classes of finite subgroups in $GL(3, Z)$ in total: 34 of them are abelian and 39 of them are not abelian. We have established in [10] that the fixed field of $K(x_1, x_2, x_3)$

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0021-8693/94 $6.00

under any abelian purely monomial group action is rational. We shall study the non-abelian case in this paper. Note that it seems difficult to extend these results to the higher dimensional cases since the fixed field of $K(x_1, x_2, x_3, x_4)$ acted upon by the purely monomial K-automorphism σ defined by

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto \frac{1}{x_1}$$

is not rational over K when K is the field of rational numbers [9, Lemma 3]. What we shall prove in this paper is the following.

MAIN THEOREM. *For any field K, the fixed field of $K(x_1, x_2, x_3)$ under any purely monomial group action is rational over K except for those which are conjugate in $GL(3, Z)$ to the subgroup generated by*

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This "exceptional" conjugacy class is designated as W_{10} in [20, p. 198]. We still do not know whether the fixed subfield under the action of this group is rational or not.

It might be interesting to compare our result with the birational classification of algebraic tori. Let L be a finite Galois extension of K with Galois group G. An algebraic K-torus T which is split by L is a linear algebraic group T defined over K such that $T \otimes_K L$ is isomorphic over L to the direct product of the one-dimensional multiplicative group G_m. Therefore, the function field of an n-dimensional K-torus split by L is simply the fixed field of $L(x_1, x_2, \ldots, x_n)^G$, where some group homomorphism $p : G \to GL(n, Z)$ is determined by the splitting of this torus. It is known that the function field of any two-dimensional algebraic torus is rational over its field of definition [21; 22], and there are exactly 15 conjugacy classes of subgroups in $GL(3, Z)$ giving rise to non-rational function fields of three-dimensional algebraic tori [14]. Note that the subgroup U_8 creating a non-rational algebraic torus in [14, Theorem 1] is conjugate to our "exceptional" case W_{10}. We might mention also that out of the above 15 conjugacy classes of subgroups in $GL(3, Z)$, four of them are abelian and the remaining 11 ones non-abelian.

Besides the applications we have in mind [17; 18, 19; 15; 5], purely monomial group actions are considered in other directions. See, for example, [16; 3; 2] and the references therein.

The proof of the main theorem is contained in Section 2. Please note that results in (2.2), (2.4), (2.7), and (2.14) are of independent interest.

Parts of this work were done while the second named author was visiting Purdue University. We thank our colleagues there, especially Professor T. T. Moh whose encouragement contributed so much to the completion of this paper.

2. Proof of the Main Theorem

Throughout this section, we shall denote by $K(x, y, z)$ the rational function field of three variables over K, where K is any field. Let G be any finite group of purely monomial K-automorphisms acting on $K(x, y, z)$. We may simply regard G as a subgroup of $GL(3, Z)$. Because the case when G is abelian has been completed in [10], it suffices to consider the non-abelian case. Note that conjugation within $GL(3, Z)$ corresponds to a change of the base $\{x, y, z\}$ of $K(x, y, z)$, G is significant only up to the conjugacy class to which it belongs. Tahara has listed 40 conjugacy classes of non-abelian subgroups in $GL(3, Z)$ [20] while the subgroup W_5 in [20, p. 198] should be deleted according to Ascher and Grimmer [1]. Hence, there are exactly 39 non-abelian conjugacy classes. We shall denote by $W_i(j)$ the subgroup W_i which appears on page j of Tahara's paper [20]. We shall describe the subgroup $W_i(j)$ by exhibiting its generators σ, τ (and λ). The rationality of $K(x, y, z)^G$ will be established in a case-by-case manner.

(2.1) The case when $G = W_i(184)$ with $5 \leq i \leq 8$, or $W_i(187)$ with $7 \leq i \leq 10$, or $W_i(191)$ with $2 \leq i \leq 7$, or $W_1(194), W_4(198), W_5(198)$. Specifically these $W_i(j)$ are given by

$$W_5(184) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_6(184) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_7(184) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_8(184) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_7(187) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_8(187) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_9(187) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_{10}(187) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_2(191) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_3(191) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_4(191) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_5(191) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_6(191) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_7(191) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_1(194) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

$$W_4(198) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

$$W_5(198) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

In the above groups, the first vector of the base is always an eigenvector and the subspace generated by the remaining two vectors forms an invariant subspace under the group action. As an illustration, we shall give a proof for $W_4(198)$. All the other cases can be treated similarly.

In $W_4(198)$, the group G is generated by σ, τ, and λ with

$$\sigma(x) = x, \qquad \sigma(y) = z, \qquad \sigma(z) = z/y,$$
$$\tau(x) = 1/x, \qquad \tau(y) = z, \qquad \tau(z) = y,$$
$$\lambda(x) = \frac{1}{x}, \qquad \lambda(y) = \frac{1}{y}, \qquad \lambda(z) = \frac{1}{z}.$$

Set

$$u := \frac{1}{1+x}.$$

Then

$$K(x,y,z) = K(y,z,u),$$
$$\sigma(u) = u, \qquad \tau(u) = -u + 1, \qquad \lambda(u) = -u + 1.$$

By the following Theorem 2.2, we have

$$K(x,y,z)^G = K(y,z,u)^G = K(y,z)^G(w),$$

where w is some element of $K(x, y, z)$ and $\sigma(w) = \tau(w) = \lambda(w) = w$. By the results of the two-dimensional monomial group actions [8], $K(y, z)^G$ is rational over K. Hence, $K(x, y, z)^G$ is also rational over K.

(2.2) THEOREM. *Let F be any field, let $F(x_1, \ldots, x_n)$ be the rational function field of n variables over F, and let G be a finite group of automorphisms of $F(x_1, \ldots, x_n)$ for each $\sigma \in G$: (i) $\sigma(F) = F$; (ii) $\sigma(x_i) = a_i(\sigma)x_i + b_i(\sigma)$ for $1 \le i \le n$, where $a_i(\sigma) \in F(x_1, x_2, \ldots, x_{i-1}) \setminus \{0\}$ and $b_i(\sigma) \in F(x_1, x_2, \ldots, x_{i-1})$. Then there exist $y_1, \ldots, y_n \in F(x_1, \ldots, x_n)$ such that $\sigma(y_i) = y_i$ for $1 \le i \le n$ for all $\sigma \in G$ and $F(x_1, \ldots, x_n)^G = F^G(y_1, \ldots, y_n)$.*

Remark. In the above theorem, it is not necessary that the automorphisms be F-automorphisms.

Proof. The proof is essentially the same as that given in [10, Theorem 1]. Thus we omit the details.

(2.3) The case when $G = W_i(184)$ with $i = 9, 10$:

$$W_9(184) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$$

$$W_{10}(184) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

We shall establish the rationality of $W_9(184)$. The case of $W_{10}(184)$ is the same. In $W_9(184)$, recall that the group G is generated by σ and τ with

$$\sigma(x) = z, \quad \sigma(y) = x, \quad \sigma(z) = y,$$
$$\tau(x) = \frac{1}{z}, \quad \tau(y) = \frac{1}{y}, \quad \tau(z) = \frac{1}{x}.$$

(i) Suppose that char $K \neq 2$. Let u, v, w be defined by

$$u = \frac{1-x}{1+x}, \quad v = \frac{1-y}{1+y}, \quad w = \frac{1-z}{1+z}.$$

Then we have

$$\sigma(u) = w, \quad \sigma(v) = u, \quad \sigma(w) = v,$$
$$\tau(u) = -w, \quad \tau(v) = -v, \quad \tau(w) = -u.$$

Let r, s, t be defined by

$$r = u + v + w, \qquad s = \frac{u}{v}, \qquad t = \frac{v}{w}.$$

Then the actions on r, s, t are given by

$$\sigma(r) = r, \qquad \sigma(s) = \frac{1}{st}, \qquad \sigma(t) = s,$$

$$\tau(r) = -r, \qquad \tau(s) = \frac{1}{t}, \qquad \tau(t) = \frac{1}{s}.$$

Now

$$\begin{aligned} K(x, y, z)^G &= K(u, v, w)^G \\ &= K(r, s, t)^G \\ &= K(s, t)^G(p) \end{aligned}$$

is rational over K, again by Theorem 2.2 and [8], where p is some element in $K(r, s, t)$ with $\sigma(p) = \tau(p) = p$.

(ii) Suppose that char $K = 2$. Let u, v, w be defined by

$$u = \frac{1}{1+x}, \qquad v = \frac{1}{1+y}, \qquad w = \frac{1}{1+z}.$$

Then we have

$$\sigma(u) = w, \qquad \sigma(v) = u, \qquad \sigma(w) = v,$$

$$\tau(u) = w + 1, \qquad \tau(v) = v + 1, \qquad \tau(w) = u + 1.$$

Let r, s, t be defined by

$$r = u + v, \qquad s = v + w, \qquad t = v.$$

Then we have

$$\sigma(r) = r + s, \qquad \sigma(s) = r, \qquad \sigma(t) = t + r,$$

$$\tau(r) = s, \qquad \tau(s) = r, \qquad \tau(t) = t + 1.$$

Hence, again by Theorem 2.2 we can find $p \in K(r, s, t)$ such that $\sigma(p) = \tau(p) = p$ and

$$K(r, s, t)^G = K(r, s)^G(p).$$

Note that G is the symmetric group, S_3, permuting among r, s, and $r + s$. Hence, $K(r, s)^G$ is generated by the elementary symmetric functions $r + s + (r + s) = 0$, $rs + (r + s)^2$, and $rs(r + s)$. Thus it is rational over K.

Before the proof of $W_{11}(187)$, we state the following theorem whose proof can be found in [11, 4.6 Remark and 6.7 Theorem].

(2.4) THEOREM. *Let F be any field, $E = F(\alpha)$ a separable extension field of degree 2 over F, and σ the non-trivial F-automorphism on E. Let $E(x, y)$ be the rational function field of two variables over E and extend the automorphism σ into $E(x, y)$ by defining*

$$\sigma(x) = x, \qquad \sigma(y) = f(x)/y,$$

where $f(x)$ is some non-zero polynomial in $F[x]$:

(a) *When* $\deg f(x) = 1$, *then $E(x, y)^{\langle\sigma\rangle}$ is always rational over F.*

(b) *When* $\operatorname{char} F \neq 2, \alpha^2 = a \in F \setminus \{0\}, f(x) = b(x^2 - c)$ *for some $b, c \in F$ with $b \neq 0$, then $E(x, y)^{\langle\sigma\rangle}$ is rational over F if and only if $(a, b)_{2,F} \in \mathrm{Br}(F(\sqrt{ac})/F)$ where $(a, b)_{2,F}$ is a norm residue symbol of degree two and $\mathrm{Br}(N/F)$ is the subgroup of Brauer group consisting of similarity classes of central simple F-algebras split by N.*

(c) *When* $\operatorname{char} F = 2, \alpha^2 + \alpha = a \in F, f(x) = b(x^2 + x + c)$ *for some $b, c \in F$ with $b \neq 0$, then $E(x, y)^{\langle\sigma\rangle}$ is rational over F if and only if $[a, b)_{2,F} \in \mathrm{Br}(F(\beta)/F)$, where $\beta^2 + \beta = a + c \in F$ and $[a, b)_{2,F}$ is a 2-symbol $[\cdot,\cdot)_{2,F}$ with the first variable being additive and the second variable multiplicative.*

(d) *When* $\operatorname{char} F = 2, \alpha^2 + \alpha = a \in F, f(x) = b(x^2 + c)$ *for some $b, c \in F \setminus \{0\}$, then $E(x, y)^{\langle\sigma\rangle}$ is rational over F if and only if* $[a, b)_{2,F} \in \mathrm{Br}(F(\sqrt{c})/F)$.

(2.5) The case when $G = W_{11}(187)$:

$$W_{11}(187) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle.$$

The actions of σ and τ on $K(x, y, z)$ are given by

$$\sigma(x) = x, \qquad \sigma(y) = z, \qquad \sigma(z) = x/y,$$
$$\tau(x) = \frac{1}{x}, \qquad \tau(y) = \frac{1}{z}, \qquad \tau(z) = \frac{1}{y}.$$

Define u, v, w as

$$u = y + z + \frac{x}{y} + \frac{x}{z}, \qquad v = \frac{y}{y+z}, \qquad w = \frac{yz}{x+yz}.$$

Note that

$$y + z = \frac{(y+z)(x+yz)}{yz} \cdot \frac{yz}{x+yz} = uw \in K(u,v,w),$$

$$y = \frac{y}{y+z} \cdot (y+z) \in K(u,v,w).$$

Hence

$$K(x,y,z) = K(u,v,w).$$

Recall that the actions of σ, τ, σ^2 on u, v, w are given by

$$\begin{aligned} &\sigma(u) = u, && \sigma(v) = w, && \sigma(w) = 1 - v, \\ &\tau(u) = a/u, && \tau(v) = v, && \tau(w) = 1 - w, \\ &\sigma^2(u) = u, && \sigma^2(v) = 1 - v, && \sigma^2(w) = 1 - w, \end{aligned}$$

where a is defined as

$$a = \frac{1}{vw(1-v)(1-w)}.$$

(i) Suppose that char $K \neq 2$. Note that

$$\sigma^2\left(v - \tfrac{1}{2}\right) = -\left(v - \tfrac{1}{2}\right), \qquad \sigma^2\left(w - \tfrac{1}{2}\right) = -\left(w - \tfrac{1}{2}\right).$$

Hence, we find that

$$K(u,v,w)^{\langle\sigma^2\rangle} = K\left(u, \left(v - \tfrac{1}{2}\right)^2, \left(v - \tfrac{1}{2}\right)\left(w - \tfrac{1}{2}\right)\right).$$

Now the action of σ is given by

$$\sigma\left(\left(v - \tfrac{1}{2}\right)^2\right) = \left(w - \tfrac{1}{2}\right)^2, \qquad \sigma\left(\left(v - \tfrac{1}{2}\right)\left(w - \tfrac{1}{2}\right)\right) = -\left(v - \tfrac{1}{2}\right)\left(w - \tfrac{1}{2}\right).$$

Define p and q as

$$p = \frac{v - \frac{1}{2}}{w - \frac{1}{2}} - \frac{w - \frac{1}{2}}{v - \frac{1}{2}}, \qquad q = \left(v - \frac{1}{2}\right)^2 + \left(w - \frac{1}{2}\right)^2.$$

Since

$$K(u,p,q) \subset K\left(u, \left(v-\tfrac{1}{2}\right)^2, \left(v-\tfrac{1}{2}\right)\left(w-\tfrac{1}{2}\right)\right)^{\langle\sigma\rangle}$$

$$\left[K\left(u, \left(v-\tfrac{1}{2}\right)^2, \left(v-\tfrac{1}{2}\right)\left(w-\tfrac{1}{2}\right)\right) : K(u,p,q)\right] \leq 2,$$

we find that

$$K(u,p,q) = K\left(u, \left(v-\tfrac{1}{2}\right)^2, \left(v-\tfrac{1}{2}\right)\left(w-\tfrac{1}{2}\right)\right)^{\langle\sigma\rangle} = K(u,v,w)^{\langle\sigma\rangle}.$$

The action of τ on u, p, q is given by

$$\tau(u) = \frac{a}{u}, \qquad \tau(p) = -p, \qquad \tau(q) = q,$$

where

$$a = \frac{1}{vw(1-v)(1-w)} = \frac{16(p^2+4)}{p^2 - 4p^2q + 16q^2 - 16q + 4}.$$

Now apply Theorem 2.4 (b) with $F = K(p^2), \alpha = p, x = q - (p^2 + 4)/8$, and $y = u(p^2 - 4p^2q + 16q^2 - 16q + 4), f(x) = 16(p^2+4)(p^2 - 4p^2q + 16q^2 - 16q + 4)$. Write

$$\begin{aligned} &16(p^2+4)(p^2 - 4p^2q + 16q^2 - 16q + 4) \\ &= 16^2(p^2+4)\left\{\left(q - \frac{p^2+4}{8}\right)^2 - \frac{p^4+4p^2}{64}\right\}. \end{aligned}$$

It suffices to show that the symbol

$$(p^2, 16^2(p^2+4))_{2, K(p^2)}$$

is split by $K(p^2\sqrt{p^2+4}/2)$.

Remember that p^2 is transcendental over K. For psychological reasons, we write $X := p^2$. We want to show that the symbol

$$(X, 16^2(X+4))_{2, K(X)}$$

is split by $K(X, X\sqrt{X+4}/2) = K(\sqrt{X+4})$. But this is trivial!

(ii) Suppose that char $K = 2$. Note that

$$\sigma^2(v) = v + 1, \qquad \sigma^2(v+w) = v + w.$$

Hence, we find that

$$K(u,v,w)^{\langle\sigma^2\rangle} = K(u,v(v+1),v+w).$$

Arguing as in the case of char $K \neq 2$, we find that

$$K(u,v,w)^{\langle\sigma\rangle} = K(u,p,q),$$

where p and q are defined by

$$p = \frac{v(v+1)}{(v+w)(v+w+1)} + v + w, q = (v+w)(v+w+1).$$

The action τ on u, p, q is given by

$$\tau(u) = \frac{a}{u}, \qquad \tau(p) = p+1, \qquad \tau(q) = q,$$

where

$$a = \frac{1}{vw(1+v)(1+w)} = \frac{1}{q^2(p^2+p+q)}.$$

Apply Theorem 2.4 (a) with $F = K(p^2+p), \alpha = p, x = q$, and $y = uq(p^2+p+q), f(x) = p^2+p+q$. We are done.

(2.6) The case when $G = W_{12}(187)$,

$$W_{12}(187) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

Apply the same change of variables as in (2.5), i.e.,

$$u = y + z + \frac{x}{y} + \frac{x}{z}, \qquad v = \frac{y}{y+z}, \qquad w = \frac{yz}{x+yz}.$$

The actions of σ and τ are given by

$$\sigma(u) = u, \qquad \sigma(v) = w, \qquad \sigma(w) = 1 - v,$$
$$\tau(u) = u, \qquad \tau(v) = 1 - v, \qquad \tau(w) = w.$$

(i) Suppose that char $K \neq 2$. Define p and q by

$$p = v - \tfrac{1}{2}, \qquad q = w - \tfrac{1}{2}.$$

Then we have

$$\sigma(p) = q, \qquad \sigma(q) = -p,$$
$$\tau(p) = -p, \qquad \tau(q) = q.$$

Hence,

$$K(x, y, z)^G = K(u, p, q)^G = K(p, q)^G(u)$$

is rational over K by the two-dimensional results [8].

(ii) Suppose that char $K = 2$. Again it suffices to show that $K(v, w)^G$ is rational over K. Since

$$\sigma^2(v) = v + 1, \qquad \sigma^2(v + w) = v + w,$$

we find that

$$K(v, w)^{\langle\sigma^2\rangle} = K(v(v + 1), v + w).$$

Define p and q by

$$p = v + w, q = \frac{v(v + 1)}{(v + w)(v + w + 1)} + v + w.$$

Then the actions on p and q are given by

$$\sigma(p) = p + 1, \qquad \sigma(q) = q,$$
$$\tau(p) = p + 1, \qquad \tau(q) = q + 1.$$

Hence,

$$\begin{aligned} K(v, w)^G &= \left\{K(v, w)^{\langle\sigma^2\rangle}\right\}^{\langle\sigma,\tau\rangle} \\ &= K(v(v + 1), v + w)^{\langle\sigma,\tau\rangle} \\ &= K(p, q)^{\langle\sigma,\tau\rangle} \\ &= K(p(p + 1), q)^{\langle\tau\rangle} \\ &= K(p(p + 1), q(q + 1)). \end{aligned}$$

Before proving $W_{13}(187)$ and $W_{14}(187)$, we give another useful lemma. Please compare our proof with that given in the last paragraph of [4].

(2.7) LEMMA. *Let F be any field, $a, b \in F \setminus \{0\}$, and $F(x, y)$ the rational function field of two variables over F. Define an F-automorphism σ on*

$F(x, y)$ by

$$\sigma(x) = \frac{a}{x}, \qquad \sigma(y) = \frac{b}{y}.$$

Let u and v be defined by

$$u := \frac{x - a/x}{xy - ab/xy}, \qquad v := \frac{y - b/y}{xy - ab/xy}.$$

Then

$$F(x, y)^{\langle\sigma\rangle} = F(u, v),$$

$$x + \frac{a}{x} = \frac{-bu^2 + av^2 + 1}{v}, \qquad y + \frac{b}{y} = \frac{bu^2 - av^2 + 1}{u},$$

$$xy + \frac{ab}{xy} = \frac{-bu^2 - av^2 + 1}{uv}.$$

Proof. Since $[F(x, y): F(x, y)^{\langle\sigma\rangle}] = 2$ and $xy \notin F(x, y)^{\langle\sigma\rangle}$, we have

$$F(x, y) = F(x, y)^{\langle\sigma\rangle} + F(x, y)^{\langle\sigma\rangle} xy.$$

Thus we may write x as

$$x = av + uxy \tag{1}$$

for some $u, v \in F(x, y)^{\langle\sigma\rangle}$.

Apply σ to (1). We obtain

$$\frac{a}{x} = av + u\frac{ab}{xy}. \tag{2}$$

Solve the simultaneous equations (1) and (2), and express u and v in terms of x and xy. Namely, subtracting (2) from (1) we have

$$x - \frac{a}{x} = u\left(xy - \frac{ab}{xy}\right)$$

$$u = \frac{x - a/x}{xy - ab/xy}. \tag{3}$$

Then multiplying (1) by ab/xy and subtracting (2) multiplied by xy gives

$$\frac{ab}{y} - ay = av\left(\frac{ab}{xy} - xy\right)$$

$$v = \frac{y - b/y}{xy - ab/xy}. \tag{4}$$

And multiplying (1) by (2) we have

$$a = a^2v^2 + auv\left(xy + \frac{ab}{xy}\right) + abu^2.$$

Hence, we obtain the expression

$$xy + \frac{ab}{xy} = \frac{-bu^2 - av^2 + 1}{uv}. \tag{5}$$

Note that (5) tells us that

$$[F(xy,u,v) : F(u,v)] \leq 2.$$

By (1), we find that

$$x \in F(xy,u,v).$$

Hence,

$$F(xy,u,v) = F(x,y).$$

Therefore, we have that

$$[F(x,y) : F(u,v)] \leq 2,$$
$$F(u,v) \subset F(x,y)^{\langle\sigma\rangle}.$$

Hence

$$F(x,y)^{\langle\sigma\rangle} = F(u,v)$$

and $u, v, xy + ab/xy$ are given by formulae (3), (4), and (5), respectively. The formulae for $x + a/x$ and $y + b/y$ can be derived similarly or simply by brute force.

(2.8) COROLLARY. *Let F be any field, $a_1, a_2, \ldots, a_n \in F \setminus \{0\}$, and $F(x_1, \ldots, x_n)$ the rational function field of n variables over F. Define an F-automorphism σ on $F(x_1, \ldots, x_n)$ by*

$$\sigma(x_i) = a_i/x_i, \qquad 1 \leq i \leq n.$$

If the subgroup generated by $a_1, \ldots, a_n$ in $F^\times/F^{\times^2}$ is of order ≤ 4, where $F^\times := F \setminus \{0\}$, then $F(x_1, \ldots, x_n)^{\langle\sigma\rangle}$ is rational over F.

Proof. By assumption, we may assume that a_1 and a_2 generate this subgroup. Now suppose that

$$a_3 = a_1 a_2 b^2$$

for some $b \in F^{\times}$. Define y_3 by

$$y_3 = \frac{x_3}{bx_1x_2}.$$

Then we have

$$\sigma(y_3) = 1/y_3.$$

The other situations can be completed similarly. Thus, without loss of generality, we may assume that $a_3 = a_4 = \cdots = a_n = 1$. Now define $z_3, z_4, \ldots, z_n$ by

$$z_i = \frac{1}{1 + x_i}, \qquad 3 \le i \le n.$$

It is clear that

$$\sigma(z_i) = -z_i + 1, \qquad 3 \le i \le n.$$

Now

$$K(x_1, \ldots, x_n)^{\langle\sigma\rangle} = K(x_1, x_2)(z_3, \ldots, z_n)^{\langle\sigma\rangle}$$

is rational over K by Theorem 2.2 and Lemma 2.7.

(2.9) The case when $G = W_{13}(187)$ or $W_{14}(187)$,

$$W_{13}(187) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_{14}(187) = \left\langle \sigma = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

We shall establish the rationality of these two groups simultaneously. Recall the actions of σ and τ on $K(x, y, z)$:

For $W_{13}(187)$,

$$\sigma(x) = \frac{1}{x}, \qquad \sigma(y) = \frac{1}{z}, \qquad \sigma(z) = \frac{y}{x},$$

$$\tau(x) = \frac{1}{x}, \qquad \tau(y) = \frac{1}{z}, \qquad \tau(z) = \frac{1}{y}.$$

For $W_{14}(187)$,

$$\sigma(x) = \frac{1}{x}, \qquad \sigma(y) = \frac{1}{z}, \qquad \sigma(z) = \frac{y}{x},$$
$$\tau(x) = x, \qquad \tau(y) = z, \qquad \tau(z) = y.$$

The action of σ^2 for both groups is the same,

$$\sigma^2(x) = x, \qquad \sigma^2(y) = \frac{x}{y}, \qquad \sigma^2(z) = \frac{x}{z}.$$

Apply Lemma 2.7. Let

$$u = \frac{y - x/y}{yz - x^2/yz}, \qquad v = \frac{z - x/z}{yz - x^2/yz}.$$

Thus we have

$$K(x, y, z)^{\langle\sigma^2\rangle} = K(x, u, v).$$

Now we compute $\sigma(u)$:

$$\begin{aligned}
\sigma(u) &= \frac{1/z - z/x}{y/xz - z/xy} \\
&= -\frac{z - x/z}{y/z - z/y} \\
&= -\frac{z - x/z}{yz - x^2/yz}\,\frac{yz - x^2/yz}{y/z - z/y} \\
&= -v\frac{(yz - x^2/yz)^2}{(y/z - z/y)(yz - x^2/yz)} \\
&= -v\frac{(yz + x^2/yz)^2 - 4x^2}{(y^2 + x^2/y^2) - (z^2 + x^2/z^2)} \\
&= -v\frac{(yz + x^2/yz)^2 - 4x^2}{(y + x/y)^2 - (z + x/z)^2}.
\end{aligned}$$

Apply Lemma 2.7 again for the expression $y + x/y, \ldots$. We find that

$$\sigma(u) = -\frac{v}{u^2 - v^2}.$$

Similarly, we can find $\sigma(v)$, $\tau(u)$, and $\tau(v)$. In summary, we have

(a) for $W_{13}(187)$,

$$\sigma(x) = \frac{1}{x}, \quad \sigma(u) = -\frac{v}{u^2 - v^2}, \quad \sigma(v) = \frac{u}{u^2 - v^2},$$
$$\tau(x) = \frac{1}{x}, \quad \tau(u) = xv, \quad \tau(v) = xu;$$

(b) for $W_{14}(187)$,

$$\sigma(x) = \frac{1}{x}, \quad \sigma(u) = -\frac{v}{u^2 - v^2}, \quad \sigma(v) = \frac{u}{u^2 - v^2},$$
$$\tau(x) = x, \quad \tau(u) = v, \quad \tau(v) = u.$$

(i) Suppose that char $K \neq 2$. Define p and q as

$$p = u + v, \quad q = u - v.$$

For $W_{13}(187)$,

$$\sigma(x) = \frac{1}{x}, \quad \sigma(p) = \frac{1}{p}, \quad \sigma(q) = -\frac{1}{q},$$
$$\tau(x) = \frac{1}{x}, \quad \tau(p) = xp, \quad \tau(q) = -xq.$$

For $W_{14}(187)$,

$$\sigma(x) = \frac{1}{x}, \quad \sigma(p) = \frac{1}{p}, \quad \sigma(q) = -\frac{1}{q}$$
$$\tau(x) = x, \quad \tau(p) = p, \quad \tau(q) = -q.$$

Now we shall solve $W_{13}(187)$. Define r, s, t by

$$r = \frac{1 - x}{1 + x}, \quad s = (1 + x)p, \quad t = (1 - x)q.$$

Then we find that

$$\sigma(r) = -r, \quad \sigma(s) = \frac{4}{1 - r^2}\frac{1}{s}, \quad \sigma(t) = \frac{4r^2}{1 - r^2}\frac{1}{t},$$
$$\tau(r) = -r, \quad \tau(s) = s, \quad \tau(t) = t.$$

It follows that

$$\begin{aligned} K(x, y, z)^G &= \{K(x, y, z)^{\langle \sigma^2 \rangle}\}^{\langle \sigma, \tau \rangle} \\ &= K(x, u, v)^{\langle \sigma, \tau \rangle} \\ &= K(x, p, q)^{\langle \sigma, \tau \rangle} \\ &= \{K(r, s, t)^{\langle \tau \rangle}\}^{\langle \sigma \rangle} \\ &= K(r^2, s, t)^{\langle \sigma \rangle} \end{aligned}$$

is rational over $K(r^2)$ by [8].

The rationality of $W_{14}(187)$ is easier, because

$$\begin{aligned} K(x, y, z)^G &= K(x, p, q)^{\langle \sigma, \tau \rangle} \\ &= \{K(x, p, q)^{\langle \tau \rangle}\}^{\langle \sigma \rangle} \\ &= K(x, p, q^2)^{\langle \sigma \rangle} \end{aligned}$$

is rational over K by Corollary 2.8.

(ii) Suppose that $\operatorname{char} K = 2$. First, we consider the case $W_{13}(187)$. Define r, s, t as

$$r = \frac{1}{1 + x}, \qquad s = (1 + x)(u + v), \qquad t = \frac{1}{1 + x} + \frac{v}{u + v}.$$

The actions of σ and τ on r, s, t are given by

$$\begin{aligned} &\sigma(r) = r + 1, \qquad \sigma(s) = \frac{1}{r^2 + r}\frac{1}{s}, \qquad \sigma(t) = t, \\ &\tau(r) = r + 1, \qquad \tau(s) = s, \qquad \tau(t) = t. \end{aligned}$$

Therefore,

$$\begin{aligned} K(x, y, z)^G &= K(x, u, v)^{\langle \sigma, \tau \rangle} \\ &= \{K(r, s, t)^{\langle \tau \rangle}\}^{\langle \sigma \rangle} \\ &= K(r(r + 1), s, t)^{\langle \sigma \rangle} \end{aligned}$$

is rational over $K(r(r + 1), t)$ by Lüroth's theorem.

Now we turn to the case $W_{14}(187)$. Define p and q by

$$p = u + v, \qquad q = u/v.$$

Then the actions of σ and τ on x, p, q are given by

$$\sigma(x) = \frac{1}{x}, \qquad \sigma(p) = \frac{1}{p}, \qquad \sigma(q) = \frac{1}{q},$$

$$\tau(x) = x, \qquad \tau(p) = p, \qquad \tau(q) = 1/q.$$

Now we have

$$\begin{aligned} K(x, y, z)^G &= K(x, u, v)^{\langle\sigma, \tau\rangle} \\ &= \left\{K(x, p, q)^{\langle\tau\rangle}\right\}^{\langle\sigma\rangle} \\ &= K(x, p, q + 1/q)^{\langle\sigma\rangle} \end{aligned}$$

is rational over $K(q + 1/q)$ by Lemma 2.7.

(2.10) The case $G = W_2(195)$,

$$W_2(195) = \left\langle \sigma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \right.$$

$$\left. \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

Clearly $W_2(195)$ is generated also by $\sigma\lambda$, $\tau\lambda$, and λ. Note also that $\sigma\lambda$ and $\tau\lambda$ are the generators of $W_{14}(187)$ of (2.9). Define u and v as in (2.9),

$$u = \frac{y - x/y}{yz - x^2/yz}, \qquad v = \frac{z - x/z}{yz - x^2/yz}.$$

Then we have that

$$K(x, y, z)^{\langle(\sigma\lambda)^2\rangle} = K(x, u, v).$$

(i) Suppose that char $K \neq 2$. Define p, q, r as

$$p = u + v, \qquad q = u - v, \qquad r = v/u.$$

The actions of $\sigma\lambda$, $\tau\lambda$, and λ are given by

$$\sigma\lambda(x) = \frac{1}{x}, \quad \sigma\lambda(p) = \frac{1}{p}, \quad \sigma\lambda(q) = -\frac{1}{q}, \quad \sigma\lambda(r) = -\frac{1}{r},$$
$$\tau\lambda(x) = x, \quad \tau\lambda(p) = p, \quad \tau\lambda(q) = -q, \quad \tau\lambda(r) = 1/r,$$
$$\lambda(x) = 1/x, \quad \lambda(p) = xp, \quad \lambda(q) = xq, \quad \lambda(r) = r.$$

Now

$$\begin{aligned} K(x,y,z)^G &= K(x,u,v)^{\langle\sigma\lambda,\tau\lambda,\lambda\rangle} \\ &= \left\{K(x,p,r)^{\langle\tau\lambda\rangle}\right\}^{\langle\sigma\lambda,\lambda\rangle} \\ &= K(x,p,r+1/r)^{\langle\sigma\lambda,\lambda\rangle} \end{aligned}$$

is rational over K by Theorem 2.2 and [8].

(ii) Suppose that char $K = 2$. Define p and q by

$$p = u + v, \qquad q = u/v.$$

The actions of $\sigma\lambda$, $\tau\lambda$, and λ are given by

$$\sigma\lambda(x) = \frac{1}{x}, \quad \sigma\lambda(p) = \frac{1}{p}, \quad \sigma\lambda(q) = \frac{1}{q},$$
$$\tau\lambda(x) = x, \quad \tau\lambda(p) = p, \quad \tau\lambda(q) = 1/q,$$
$$\lambda(x) = 1/x, \quad \lambda(p) = xp, \quad \lambda(q) = q.$$

We have now that

$$\begin{aligned} K(x,y,z)^G &= K(x,u,v)^{\langle\sigma\lambda,\tau\lambda,\lambda\rangle} \\ &= \left\{K(x,p,q)^{\langle\tau\lambda\rangle}\right\}^{\langle\sigma\lambda,\lambda\rangle} \\ &= K(x,p,q+1/q)^{\langle\sigma\lambda,\lambda\rangle} \end{aligned}$$

is rational over K by Theorem 2.2 and [8].

(2.11) The case when char $K \neq 2$ and $G = W_8(191)$, $W_9(191)$, $W_1(198)$, $W_6(198)$, $W_7(198)$, or $W_1(202)$ (the case when char $K = 2$ will be treated in (2.12)):

$$W_8(191) = \left\langle \sigma = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$$

$$W_9(191) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

$$W_1(198) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

$$W_6(198) = \left\langle \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_7(198) = \left\langle \sigma = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

$$W_1(202) = \left\langle \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

We can treat all these six groups by applying exactly the same procedure of changing the variables. We illustrate it by $W_1(198)$.

For $W_1(198)$, the group actions are given by

$$\sigma(x) = z, \quad \sigma(y) = x, \quad \sigma(z) = y,$$
$$\tau(x) = \frac{1}{x}, \quad \tau(y) = y, \quad \tau(z) = \frac{1}{z},$$
$$\lambda(x) = \frac{1}{x}, \quad \lambda(y) = \frac{1}{y}, \quad \lambda(z) = \frac{1}{z}.$$

Define u, v, w by

$$u = \frac{1-x}{1+x}, \qquad v = \frac{1-y}{1+y}, \qquad w = \frac{1-z}{1+z}.$$

Then the actions on u, v, w are given by

$$\begin{aligned} &\sigma(u) = w, \qquad \sigma(v) = u, \qquad \sigma(w) = v, \\ &\tau(u) = -u, \qquad \tau(v) = v, \qquad \tau(w) = -w, \\ &\lambda(u) = -u, \qquad \lambda(v) = -v, \qquad \lambda(w) = -w. \end{aligned}$$

Now define p, q, r by

$$p = u + v + w, \qquad q = \frac{u}{w}, \qquad r = \frac{v}{w}.$$

Then we have

$$u = \frac{pq}{q+r+1}, \qquad v = \frac{pr}{q+r+1}, \qquad w = \frac{p}{q+r+1}.$$

Now it is routine to check that

$$\begin{aligned} &\sigma(p) = p, \qquad \sigma(q) = \frac{1}{r}, \qquad \sigma(r) = \frac{q}{r}, \\ &\tau(p) = \frac{-q+r-1}{q+r+1}p, \qquad \tau(q) = q, \qquad \tau(r) = -r, \\ &\lambda(p) = -p, \qquad \lambda(q) = q, \qquad \lambda(r) = r. \end{aligned}$$

By Theorem 2.2 and [8], we find that $K(p,q,r)^G$ is rational over K.

(2.12) The case when $\operatorname{char} K = 2$ and $G = W_8(191)$, $W_9(191)$, $W_1(198)$, $W_6(198)$, $W_7(198)$, or $W_1(202)$. At the first stage, we can apply exactly the same procedure of changing the variables for all these six groups. We define v, v, w, p, q, r as

$$\begin{aligned} &u = \frac{1}{1+x}, \qquad v = \frac{1}{1+y}, \qquad w = \frac{1}{1+z}, \\ &p = u + v + w, \qquad q = u + w, \qquad r = v + w. \end{aligned}$$

It is routine to check that the group actions on p, q, r are given as

(a) For $W_8(191)$,

$$\sigma(p) = p + 1, \quad \sigma(q) = r, \quad \sigma(r) = q + r,$$
$$\tau(p) = p + 1, \quad \tau(q) = q, \quad \tau(r) = q + r.$$

(b) For $W_9(191)$,

$$\sigma(p) = p, \quad \sigma(q) = r, \quad \sigma(r) = q + r,$$
$$\tau(p) = p, \quad \tau(q) = q, \quad \tau(r) = r + 1.$$

(c) For $W_1(198)$,

$$\sigma(p) = p, \quad \sigma(q) = r, \quad \sigma(r) = q + r,$$
$$\tau(p) = p, \quad \tau(q) = q, \quad \tau(r) = r + 1,$$
$$\lambda(p) = p + 1, \quad \lambda(q) = q, \quad \lambda(r) = r.$$

(d) For $W_6(198)$,

$$\sigma(p) = p + 1, \quad \sigma(q) = q + 1, \quad \sigma(r) = q + r,$$
$$\tau(p) = p + 1, \quad \tau(q) = q + r, \quad \tau(r) = r.$$

(e) For $W_7(198)$,

$$\sigma(p) = p, \quad \sigma(q) = q + 1, \quad \sigma(r) = r,$$
$$\tau(p) = p, \quad \tau(q) = q + r, \quad \tau(r) = r.$$

(f) For $W_1(202)$,

$$\sigma(p) = p + 1, \quad \sigma(q) = q + 1, \quad \sigma(r) = q + r,$$
$$\tau(p) = p + 1, \quad \tau(q) = q + r, \quad \tau(r) = r,$$
$$\lambda(p) = p + 1, \quad \lambda(q) = q, \quad \lambda(r) = r.$$

By Theorem 2.2 it suffices to establish the rationality of $K(q, r)^G$. When G is restricted to $K(q, r)$, $W_9(191)$, and $W_1(198)$ are the same; also $W_6(198)$ and $W_1(202)$ are the same. Moreover, since r is kept fixed in $W_7(198)$, $K(q, r)^G$ is rational over K by Theorem 2.2.

Now consider the case $W_8(191)$. It is easy to see that

$$K(q, r)^G = K(q^2 + qr + r^2, qr(q + r)),$$

where these two generators are the elementary symmetric functions of q, r, and $q + r$.

Now consider the case $W_9(191)$. Let H be the subgroup of $G|_{K(q,r)}$ generated by τ and $\sigma^{-1}\tau\sigma$. H is isomorphic to the Klein four group. Note that

$$\sigma^{-1}\tau\sigma(q) = q + 1, \qquad \sigma^{-1}\tau\sigma(r) = r + 1.$$

Hence,

$$K(q,r)^H = K(A,B),$$

where A and B are defined by

$$A = q^2 + q, \qquad B = r^2 + r.$$

Note that

$$\sigma(A) = B, \sigma(B) = A + B.$$

Write

$$C := B/A.$$

Then we have

$$\sigma(C) = \frac{C+1}{C}, \qquad \sigma(A) = CA.$$

By Theorem 2.2 $K(C, A)^{\langle\sigma\rangle}$ is rational over K. (We can apply Corollary 2.15 also.)

It remains to consider $W_6(198)$. As pointed out in [20, p. 198], $G = W_6(198)$ is isomorphic to S_4, the symmetric group of degree four. It is easy to check that $G|_{K(q,r)}$ is isomorphic to S_4. Hence,

$$\left[K(q,r) : K(q,r)^G\right] = 24.$$

On the other hand, consider elements Q and R defined by

$$Q = q(q+1), \qquad R = r(r+1).$$

Then we find that

$$\sigma\tau(Q) = R, \qquad \sigma\tau(R) = Q + R,$$
$$\sigma(Q) = Q, \qquad \sigma(R) = Q + R.$$

Thus

$$K(Q,R)^{\langle\sigma,\tau\rangle} = K(Q,R)^{\langle\sigma\tau,\sigma\rangle}$$
$$= K(Q^2+QR+R^2, QR(Q+R)).$$

Note that

$$\left[K(q,r):K(Q,R)^{\langle\sigma,\tau\rangle}\right]$$
$$= [K(q,r):K(Q,R)]\left[K(Q,R):K(Q,R)^{\langle\sigma,\tau\rangle}\right]$$
$$= 4\cdot 6 = 24.$$

Hence,

$$K(q,r)^G = K(Q^2+QR+R^2, QR(Q+R)).$$

(2.13) The case $G = W_{11}(191)$,

$$W_{11}(191) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

The actions of σ and τ on $K(x,y,z)$ are given by

$$\sigma(x) = z, \qquad \sigma(y) = x, \qquad \sigma(z) = y,$$
$$\tau(x) = \frac{1}{x}, \qquad \tau(y) = \frac{z}{x}, \qquad \tau(z) = \frac{y}{x}.$$

(i) Suppose that char $K \neq 2$. Define u, v, w as

$$u = \frac{y+z}{1+x}, \qquad v = \frac{x+z}{1+y}, \qquad w = \frac{x+y}{1+z}.$$

Then we find that

$$\frac{1}{u+1} = \frac{1+x}{1+x+y+z}, \qquad \frac{1}{v+1} = \frac{1+y}{1+x+y+z},$$
$$\frac{1}{w+1} = \frac{1+z}{1+x+y+z}$$
$$\frac{1}{u+1} + \frac{1}{v+1} + \frac{1}{w+1} = \frac{2}{1+x+y+z} + 1.$$

Hence,

$$1 + x + y + z \in K(u, v, w).$$

It follows that

$$K(x, y, z) = K(u, v, w).$$

Note that the actions of σ and τ are given by

$$\sigma(u) = w, \qquad \sigma(v) = u, \qquad \sigma(w) = v,$$
$$\tau(u) = u, \qquad \tau(v) = \frac{1}{v}, \qquad \tau(w) = \frac{1}{w}.$$

Since $G = \langle \sigma, \tau \rangle = \langle \sigma, \sigma^{-1}\tau\sigma \rangle$ and the action of $\sigma^{-1}\tau\sigma$ is given by

$$\sigma^{-1}\tau\sigma(u) = \frac{1}{u}, \qquad \sigma^{-1}\tau\sigma(v) = v, \qquad \sigma^{-1}\tau\sigma(w) = \frac{1}{w},$$

we find that the actions of σ and $\sigma^{-1}\tau\sigma$ on $K(u, v, w)$ are the same as those of σ and τ on $K(x, y, z)$ in $W_9(191)$ of (2.11). Hence we are done.

(ii) Suppose that char $K = 2$. Define u, v, w as

$$u = \frac{1 + y}{1 + x + y + z}, \qquad v = \frac{1 + z}{1 + x + y + z}, \qquad w = \frac{1 + y + z}{1 + x + y + z}.$$

The actions of σ and τ are given by

$$\sigma(u) = u + v + 1, \qquad \sigma(v) = u, \qquad \sigma(w) = u + w + 1,$$
$$\tau(u) = u + 1, \qquad \tau(v) = v + 1, \qquad \tau(w) = u + v + w + 1.$$

By Theorem 2.2 it suffices to establish the rationality of $K(u, v)^G$. Note that

$$\sigma^{-1}\tau\sigma(u) = u, \qquad \sigma^{-1}\tau\sigma(v) = v + 1.$$

Moreover, $\langle \tau, \sigma^{-1}\tau\sigma \rangle$ is isomorphic to the Klein four group. Hence,

$$K(u, v)^{\langle \tau, \sigma^{-1}\tau\sigma \rangle} = K(A, B),$$

where A and B are defined by

$$A = u(u + 1), \qquad B = v(v + 1).$$

Note that the action of σ on A and B is given by

$$\sigma(A) = A + B, \qquad \sigma(B) = A.$$

Hence,

$$K(A,B)^{\langle\sigma\rangle}$$

is rational by Corollary 2.15 given below.

(2.14) LEMMA. *Let F be any field and let $F(x,y,z)$ be the rational function field of three variables over F. Define an F-automorphism σ on $F(x,y,z)$ by*

$$\sigma(x)=y, \qquad \sigma(y)=z, \qquad \sigma(z)=x.$$

(i) Suppose that char $F\neq 3$. Let ζ be a primitive third root of 1. (We do not assume that $\zeta\in F$.) Define u,v,w as

$$u=x+y+z, \qquad v=x+\zeta y+\zeta^2 z, \qquad w=x+\zeta^2 y+\zeta z.$$

Then

$$F(x,y,z)^{\langle\sigma\rangle}=F\left(u,\frac{v^2}{w}+\frac{w^2}{v},\zeta\frac{v^2}{w}+\zeta^2\frac{w^2}{v}\right).$$

(ii) Suppose that char $F=3$. Define u,s,t as

$$u=x+y+z, \qquad s=xy+yz+zx, \qquad t=(x-y)(y-z)(z-x).$$

Then

$$xyz=-\frac{s^3-u^2s^2+t^2}{u^3},$$

$$F(x,y,z)^{\langle\sigma\rangle}=F(u,s,t).$$

Proof. (i) The case when char $K\neq 3$. If $\zeta\in F$, then

$$\sigma(u)=u, \qquad \sigma(v)=\zeta^2 v, \qquad \sigma(w)=\zeta w.$$

Hence, we have

$$F(x,y,z)^{\langle\sigma\rangle}=F\left(u,\frac{v^2}{w},\frac{w^2}{v}\right)$$

$$=F\left(u,\frac{v^2}{w}+\frac{w^2}{v},\zeta\frac{v^2}{w}+\zeta^2\frac{w^2}{v}\right).$$

On the other hand, suppose that $\zeta \notin F$. Define F-automorphisms σ and λ on $F(\zeta)(x, y, z)$ by

$$\sigma(\zeta) = \zeta, \quad \sigma(x) = y, \quad \sigma(y) = z, \quad \sigma(z) = x,$$
$$\lambda(\zeta) = \zeta^2, \quad \lambda(x) = x, \quad \lambda(y) = y, \quad \lambda(z) = z.$$

Note that $\sigma\lambda = \lambda\sigma$. Moreover, we have

$$\begin{aligned} F(x, y, z)^{\langle\sigma\rangle} &= \left\{F(\zeta)(x, y, z)^{\langle\lambda\rangle}\right\}^{\langle\sigma\rangle} \\ &= F(\zeta)(x, y, z)^{\langle\sigma, \lambda\rangle} \\ &= \left\{F(\zeta)(x, y, z)^{\langle\sigma\rangle}\right\}^{\langle\lambda\rangle} \\ &= F(\zeta)\left(u, \frac{v^2}{w}, \frac{w^2}{v}\right)^{\langle\lambda\rangle} \\ &= F\left(u, \frac{v^2}{w} + \frac{w^2}{v}, \zeta\frac{v^2}{w} + \zeta^2\frac{w^2}{v}\right). \end{aligned}$$

(ii) The case when char $K = 3$. In addition to u, s, t defined above, define v and w as

$$v = \frac{y + z}{x + y + z} - \frac{(y - z)^2}{(x + y + z)^2}, \qquad w = \frac{y - z}{x + y + z}.$$

We have

$$\sigma(u) = u, \qquad \sigma(v) = v, \qquad \sigma(w) = w - 1.$$

Hence, it follows that

$$\begin{aligned} F(x, y, z)^{\langle\sigma\rangle} &= F(u, v, w)^{\langle\sigma\rangle} \\ &= F(u, v, w(w + 1)(w + 2)). \end{aligned}$$

It is routine to verify that

$$s = u^2 v, \qquad t = u^3 w(w + 1)(w + 2),$$
$$y + z = u(v + w^2), \qquad y - z = uw.$$

Therefore,

$$\begin{aligned} xyz &= \{u - u(v + w^2)\}\{u(v + w^2) + uw\}\{u(v + w^2) - uw\} \\ &= -u^3(v^3 - v^2 + w^6 + w^4 + w^2) \\ &= -\frac{s^3 - u^2 s^2 + t^2}{u^3}. \end{aligned}$$

(2.15) COROLLARY. *Let F be any field and let $F(y, z)$ be the rational function field of two variables over F. Define an F-automorphism σ on $F(y, z)$ by*

$$\sigma(y) = z, \qquad \sigma(z) = -y - z.$$

If $\operatorname{char} F \neq 3$, *then*

$$F(y, z)^{\langle\sigma\rangle} = F\left(\frac{yz(y+z)}{y^2 + yz + z^2}, \frac{-y^3 + 3yz^2 + z^3}{y^2 + yz + z^2}\right).$$

If $\operatorname{char} F = 3$, *then*

$$F(y, z)^{\langle\sigma\rangle} = F(y - z, yz(y + z)).$$

Proof. If $\operatorname{char} F \neq 3$, set $x = -y - z$ in the proof of Lemma 2.14. The case when $\operatorname{char} F = 3$ is easy to verify.

(2.16) *Remarks.* From the proof of the case $\operatorname{char} K \neq 2$ of (2.13), we find that the groups in $W_9(191)$ and $W_{11}(191)$ are conjugate as subgroups of the group of K-automorphisms on $K(x, y, z)$, although they are not conjugate as subgroups in $GL(3, Z)$.

Kuniyoshi proves a general theorem for the case $\operatorname{char} F = 3$ of Lemma 2.14. He shows that if $\operatorname{char} F = p > 0$ and σ is defined by $\sigma(x_i) = x_{i+1}$ for $1 \leq i \leq p^n - 1$ and $\sigma(x_{p^n}) = x_1$, then $F(x_1, x_2, \ldots, x_{p^n})^{\langle\sigma\rangle}$ is rational over F [12; 13; 7].

(2.17) The case when $\operatorname{char} K \neq 2$ and $G = W_{10}(191), W_2(198)$; the case when $\operatorname{char} K = 2$ will be treated in (2.18).

$$W_{10}(191) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \right\rangle$$

$$W_2(198) = \left\langle \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

Define w by

$$w := 1/xyz. \tag{6}$$

The actions σ, τ, λ on $K(x, y, z)$ are given by

$$\sigma(x) = z, \quad \sigma(y) = x, \quad \sigma(z) = y, \quad \sigma(w) = w,$$
$$\tau(x) = z, \quad \tau(y) = w, \quad \tau(z) = x, \quad \tau(w) = y,$$
$$\lambda(x) = \frac{1}{x}, \quad \lambda(y) = \frac{1}{y}, \quad \lambda(z) = \frac{1}{z}, \quad \lambda(w) = \frac{1}{w}.$$

Note that the action of $\sigma^{-1}\tau\sigma$ is given by

$$\sigma^{-1}\tau\sigma(x) = y, \quad \sigma^{-1}\tau\sigma(y) = x, \quad \sigma^{-1}\tau\sigma(z) = w, \quad \sigma^{-1}\tau\sigma(w) = z.$$

Both τ and $\sigma^{-1}\tau\sigma$ generate a subgroup, the Klein four group, in $W_{10}(191)$ and $W_2(198)$.

Define X, Y, Z, W as

$$X = \frac{1-x}{1+x}, \quad Y = \frac{1-y}{1+y}, \quad Z = \frac{1-z}{1+z}, \quad W = \frac{1-w}{1+w}.$$

Then the relation (6) becomes

$$(1-X)(1-Y)(1-Z)(1-W) = (1+X)(1+Y)(1+Z)(1+W),$$
$$X + Y + Z + W + XYZ + YZW + ZWX + WXY = 0. \qquad (7)$$

The actions on X, Y, Z, W are given by

$$\sigma(X) = Z, \quad \sigma(Y) = X, \quad \sigma(Z) = Y, \quad \sigma(W) = W,$$
$$\tau(X) = Z, \quad \tau(Y) = W, \quad \tau(Z) = X, \quad \tau(W) = Y,$$
$$\sigma^{-1}\tau\sigma(X) = Y, \quad \sigma^{-1}\tau\sigma(Y) = X, \quad \sigma^{-1}\tau\sigma(Z) = W, \sigma^{-1}\tau\sigma(W) = Z,$$
$$\lambda(X) = -X, \quad \lambda(Y) = -Y,$$
$$\lambda(Z) = -Z, \quad \lambda(W) = -W.$$

Define u, v, s, t as

$$4u = X + Y + Z + W,$$
$$4v = X + Y - Z - W,$$
$$4s = X - Y + Z - W,$$
$$4t = X - Y - Z + W.$$

We find that $K(x, y, z) = K(X, Y, Z) = K(u, v, s, t)$ and the relation (7) becomes

$$\begin{aligned}
&-4u \\
&\quad = -(X + Y + Z + W) \\
&\quad = XY(Z + W) + ZW(X + Y) \\
&\quad = 2(u + v + s + t)(u + v - s - t)(u - v) \\
&\qquad + 2(u - v + s - t)(u - v - s + t)(u + v) \\
&\quad = 4\{u(u^2 - v^2) - u(s^2 + t^2) + 2stv\}, \\
&\qquad -1 = u^2 - (v^2 + s^2 + t^2) + 2stv/u. \qquad (8)
\end{aligned}$$

The actions on u, v, s, t are given by

$$\begin{array}{llll}
\sigma(u) = u, & \sigma(v) = s, & \sigma(s) = -t, & \sigma(t) = -v, \\
\tau(u) = u, & \tau(v) = v, & \tau(s) = s, & \tau(t) = -t, \\
\sigma^{-1}\tau\sigma(u) = u, & \sigma^{-1}\tau\sigma(v) = v, & \sigma^{-1}\tau\sigma(s) = -s, & \sigma^{-1}\tau\sigma(t) = -t, \\
\lambda(u) = -u, & \lambda(v) = -v, & \lambda(s) = -s, & \lambda(t) = -t.
\end{array}$$

Define A, B, C, D by

$$A = -\frac{vs}{tu}, \qquad B = -\frac{tv}{su}, \qquad C = -\frac{st}{uv}, \qquad D = \frac{1}{u}.$$

It is easy to see that

$$K(u, v, s, t)^{\langle \tau, \sigma^{-1}\tau\sigma \rangle} = K(A, B, C, D).$$

The relation on A, B, C, D follows from (8), i.e.,

$$1 + D^2 = AB + BC + CA + 2ABC. \qquad (9)$$

Recall the actions of σ and λ on A, B, C, D. We have

$$\begin{array}{llll}
\sigma(A) = C, & \sigma(B) = A, & \sigma(C) = B, & \sigma(D) = D, \\
\lambda(A) = A, & \lambda(B) = B, & \lambda(C) = C, & \lambda(D) = -D.
\end{array}$$

Now consider the case $G = W_2(198)$ first,

$$\begin{aligned}
K(A, B, C, D)^{\langle \sigma, \lambda \rangle} &= \{K(A, B, C, D)^{\langle \lambda \rangle}\}^{\langle \sigma \rangle} \\
&= K(A, B, C, D^2)^{\langle \sigma \rangle} \\
&= K(A, B, C)^{\langle \sigma \rangle},
\end{aligned}$$

where the last equality follows from formula (9). The rationality of $K(A,B,C)^{\langle\sigma\rangle}$ follows from Lemma 2.14.

It remains to establish the rationality of $G = W_{10}(191)$, namely, the rationality of $K(A,B,C,D)^{\langle\sigma\rangle}$.

(i) Suppose that char $K = 3$. Define L, M, N by

$$L = A + B + C, \qquad M = AB + BC + CA,$$
$$N = (A - B)(B - C)(C - A).$$

By Lemma 2.14

$$K(A,B,C,D)^{\langle\sigma\rangle} = K(L,M,N,D)$$

and the relations on L, M, N, D follows from (9), i.e.,

$$1 + D^2 = M - 2\frac{M^3 - L^2M^2 + N^2}{L^3}$$
$$L^3(1 + D^2) = L^3M - 2M^3 + 2L^2M^2 - 2N^2$$
$$\frac{1}{L}(1 + D^2) = \frac{M}{L} - 2\left(\frac{M}{L}\right)^3\frac{1}{L} + 2\left(\frac{M}{L}\right)^2 - 2\left(\frac{N}{L^2}\right)^2. \tag{10}$$

Hence,

$$K(A,B,C,D)^{\langle\sigma\rangle} = K(L,M,N,D) = K\left(\frac{M}{L},\frac{N}{L^2},D\right)$$

is rational over K because of formula (10).

(ii) Suppose that char $K \neq 2, 3$. Define P, Q, R as

$$P = A + 1, \qquad Q = B + 1, \qquad R = C + 1.$$

The relation (9) becomes now

$$1 + D^2 = 2PQR - (PQ + QR + RP) + 1. \tag{11}$$

The action of σ is given by

$$\sigma(P) = R, \qquad \sigma(Q) = P, \qquad \sigma(R) = Q, \qquad \sigma(D) = D.$$

Apply Lemma 2.14 on P, Q, R. We find that

$$K(A,B,C,D)^{\langle\sigma\rangle} = K(P,Q,R,D)^{\langle\sigma\rangle} = K(L,M,N,D),$$

where L, M, N, S, T are defined by

$$L = P + Q + R, S = P + \zeta Q + \zeta^2 R, T = P + \zeta^2 Q + \zeta R,$$

$$M = \frac{S^2}{T} + \frac{T^2}{S}, \qquad N = \zeta\frac{S^2}{T} + \zeta^2\frac{T^2}{S}$$

with ζ being a primitive third root of 1.

Note that the relation (11) becomes

$$1 + D^2 = \tfrac{2}{27}(L^3 + S^3 + T^3 - 3LST) - \tfrac{1}{3}(L^2 - ST) + 1$$

$$\begin{aligned}27D^2 &= 2\left\{L^3 + \left(\frac{S^2}{T}\right)^2\left(\frac{T^2}{S}\right) + \left(\frac{S^2}{T}\right)\left(\frac{T^2}{S}\right)^2 - 3L\left(\frac{S^2}{T}\right)\left(\frac{T^2}{S}\right)\right\} \\ &\quad - 9\left\{L^2 - \left(\frac{S^2}{T}\right)\left(\frac{T^2}{S}\right)\right\} \\ &= \left(\frac{S^2}{T}\right)\left(\frac{T^2}{S}\right)\left\{9 + 2\left(\frac{S^2}{T} + \frac{T^2}{S}\right) - 6L\right\} - 9L^2 + 2L^3 \\ &= \tfrac{1}{3}(M^2 + MN + N^2)(9 + 2M - 6L) - 9L^2 + 2L^3,\end{aligned}$$

where the last equality follows from

$$ST = \tfrac{1}{3}(M^2 + MN + N^2).$$

Thus the relation on L, M, N, D is

$$81D^2 = (M^2 + MN + N^2)(9 + 2M - 6L) - 27L^2 + 6L^3.$$

$$\begin{aligned}81\left(\frac{D}{L}\right)^2\left(\frac{1}{L}\right) &= \left\{\left(\frac{M}{L}\right)^2 + \frac{M}{L}\cdot\frac{N}{L} + \left(\frac{N}{L}\right)^2\right\} \\ &\quad\times\left\{\frac{9}{L} + 2\left(\frac{M}{L}\right) - 6\right\} - 27\left(\frac{1}{L}\right) + 6.\end{aligned} \tag{12}$$

The above formula (12) tells us that

$$\frac{1}{L} \in K\left(\frac{M}{L}, \frac{N}{L}, \frac{D}{L}\right).$$

Hence,

$$K(A,B,C,D)^{\langle\sigma\rangle} = K(L,M,N,D) = L\left(\frac{1}{L},\frac{M}{L},\frac{N}{L},\frac{D}{L}\right) = K\left(\frac{M}{L},\frac{N}{L},\frac{D}{L}\right)$$

is rational over K.

(2.18) The case when char $K = 2$ and $G = W_{10}(191)$ or $W_2(198)$. As in (2.17), introduce w by

$$w := 1/xyz.$$

Define X, Y, Z, W as

$$X = \frac{1}{1+x}, \qquad Y = \frac{1}{1+y}, \qquad Z = \frac{1}{1+z}, \qquad W = \frac{1}{1+w}.$$

The relation of X, Y, Z, W is given by

$$(1+X)(1+Y)(1+Z)(1+W) = XYZW,$$
$$1 + (X+Y+Z+W) + (XY+XZ+XW+YZ+YW+ZW) + (XYZ+XYW+XZW+YZW) = 0. \tag{13}$$

The actions on X, Y, Z, W are given by

$$\sigma(X) = Z, \quad \sigma(Y) = X, \quad \sigma(Z) = Y, \quad \sigma(W) = W,$$
$$\tau(X) = Z, \quad \tau(Y) = W, \quad \tau(Z) = X, \quad \tau(W) = Y,$$
$$\sigma^{-1}\tau\sigma(X) = Y, \quad \sigma^{-1}\tau\sigma(Y) = X,$$
$$\sigma^{-1}\tau\sigma(Z) = W, \quad \sigma^{-1}\tau\sigma(W) = Z,$$
$$\lambda(X) = X+1, \quad \lambda(Y) = Y+1, \quad \lambda(Z) = Z+1,$$
$$\lambda(W) = W+1.$$

Define u, v, s, t, d as

$$u = X, \qquad v = X+Z, \qquad s = X+Y,$$
$$t = X+Y+Z+W, \qquad d = XW+YZ.$$

Note that

$$K(X,Y,Z) = K(u,v,s,t) = K(d,v,s,t)$$

because

$$d = ut + vs.$$

Now the actions on u, v, s, t, d are given by

$$\sigma(u) = u + v, \quad \sigma(v) = v + s, \quad \sigma(s) = v,$$
$$\sigma(t) = t, \quad \sigma(d) = d + v(v + t),$$
$$\tau(u) = u + v, \quad \tau(v) = v, \quad \tau(s) = s + t,$$
$$\tau(t) = t, \quad \tau(d) = d,$$
$$\sigma^{-1}\tau\sigma(u) = u + s, \quad \sigma^{-1}\tau\sigma(v) = v + t, \quad \sigma^{-1}\tau\sigma(s) = s,$$
$$\sigma^{-1}\tau\sigma(t) = t, \quad \tau^{-1}\tau\sigma(d) = d,$$
$$\lambda(u) = u + 1, \quad \lambda(v) = v, \quad \lambda(s) = s,$$
$$\lambda(t) = t, \quad \lambda(d) = d + t.$$

First, consider the case $G = W_2(198)$. We have

$$K(d, v, s, t)^{\langle \tau, \sigma^{-1}\tau\sigma, \lambda \rangle} = K(D, V, S, t),$$

where D, V, S are defined by

$$D = d(d + t), \quad V = v(v + t), \quad S = s(s + t).$$

We now compute the relation of D, V, S, t. Since

$$X = u, \quad Y = u + s, \quad Z = u + v, \quad W = u + v + s + t,$$

we have

$$X + Y + Z + W = t,$$
$$XY + XZ + XW + YZ + YW + ZW = (v + s)(v + s + t) + ut + vs,$$
$$XYZ + XYW + XZW + YZW = vs(v + s + t) + u^2 t.$$

Substitute these identities into formula (13). We obtain

$$1 + t + (v + s)(v + s + t) + ut + vs + vs(v + s + t) + u^2 t = 0$$
$$t(u^2 + u + 1) + 1 + (v^2 + vt) + (s^2 + st) + vs(1 + v + s + t) = 0 \tag{14}$$
$$t^2(u^2 + u) + t^2 + t\{1 + V + S + vs(1 + v + s + t)\} = 0$$
$$(d + vs)^2 + t(d + vs) + t^2 + t\{1 + V + S + vs(1 + v + s + t)\} = 0$$
$$d(d + t) + v^2 s^2 + t\{1 + V + S + vs(v + s + t)\} + t^2 = 0$$
$$D + t + (V + t)(S + t) = 0. \tag{15}$$

Therefore, we find that

$$K(d,v,s,t)^{\langle \tau, \sigma^{-1}\tau\sigma, \lambda \rangle} = K(D,V,S,t) = K(V,S,t).$$

The action of σ on V, S, t is given by

$$\sigma(V) = V + S, \qquad \sigma(S) = V, \qquad \sigma(t) = t.$$

By Corollary 2.15

$$K(V,S,t)^{\langle \sigma \rangle} = K(V,S)^{\langle \sigma \rangle}(t)$$

is rational over K.

It remains to establish the rationality of the case $G = W_{10}(191)$. Define A, B, C as

$$A = v(v + t) + t, \qquad B = s(s + t) + t, \qquad C = A + B + t.$$

It is easy to see that

$$K(d,v,s,t)^{\langle \tau, \sigma^{-1}\tau\sigma \rangle} = K(A,B,C,d).$$

Now we compute the relation of A, B, C, d. Note that formula (14) is still valid. So we start with it:

$$t(u^2 + u + 1) + 1 + (v^2 + vt) + (s^2 + st) + vs(1 + v + s + t) = 0$$

$$t^2(u^2 + u) + t^2 + t(v^2 + vt + t) + t(s^2 + st + t) + t + tvs(1 + v + s + t) = 0$$

$$(d + vs)^2 + t(d + vs) + t^2 + t\{1 + A + B + vs(1 + v + s + t)\} = 0$$

$$d(d + t) + v^2s^2 + t\{A + B + vs(v + s + t)\} + t^2 + t = 0$$

$$d(d + t) + (A + t)(B + t) + t(A + B) + t^2 + t = 0$$

$$d(d + A + B + C) + AB + A + B + C = 0$$

$$(d + A + B)^2 + A^2 + B^2 + (d + A + B)(A + B + C) + A^2 + B^2 + AC + BC + AB + A + B + C = 0 \tag{16}$$

Define P, Q, R as

$$P = d + A + B, \qquad Q = A + B + C, \qquad R = AB + BC + CA.$$

Then formula (14) becomes

$$P^2 + PQ + Q + R = 0. \tag{17}$$

Now the action of σ is given by

$$\sigma(A) = C, \qquad \sigma(B) = A, \qquad \sigma(C) = B, \qquad \sigma(P) = P.$$

Let ζ be a primitive third root of 1. As in Lemma 2.14 let M, N, I, J be defined by

$$M = A + \zeta B + \zeta^2 C, \qquad N = A + \zeta^2 B + \zeta C,$$
$$I = \frac{M^2}{N} + \frac{N^2}{M}, \qquad J = \zeta\frac{M^2}{N} + \zeta^2\frac{N^2}{M}.$$

Then we have

$$\begin{aligned} K(A,B,C,d)^{\langle\sigma\rangle} &= K(A,B,C,P)^{\langle\sigma\rangle} \\ &= K(A,B,C)^{\langle\sigma\rangle}(P) \\ &= K(Q,I,J,P). \end{aligned}$$

Since

$$AB + BC + CA = Q^2 + MN,$$

it follows that

$$\begin{aligned} R &= AB + BC + CA \\ &= Q^2 + MN \\ &= Q^2 + \frac{M^2}{N}\cdot\frac{N^2}{M} \\ &= Q^2 + (\zeta I + J)(\zeta^2 I + J) \\ &= Q^2 + I^2 + IJ + J^2. \end{aligned}$$

Hence, formula (17) becomes

$$P^2 + PQ + Q + Q^2 + I^2 + IJ + J^2 = 0 \tag{18}$$

$$\left(\frac{P}{Q}\right)^2 + \frac{P}{Q} + \frac{1}{Q} + 1 + \left(\frac{I}{Q}\right)^2 + \frac{I}{Q}\frac{J}{Q} + \left(\frac{J}{Q}\right)^2 = 0. \tag{19}$$

Therefore,

$$\begin{aligned} K(A,B,C,d)^{\langle\sigma\rangle} &= K(Q,I,J,P) \\ &= K\left(\frac{1}{Q},\frac{I}{Q},\frac{J}{Q},\frac{P}{Q}\right) \\ &= K\left(\frac{I}{Q},\frac{J}{Q},\frac{P}{Q}\right) \end{aligned}$$

is rational over K.

(2.19) Presentations of $W_3(198)$ and $W_3(203)$:

$$W_3(203) = \left\langle \sigma_1 = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \tau_1 = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

Let μ be the matrix defined by

$$\mu = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in GL(3, Z).$$

The $\langle \mu^{-1}\sigma_1\mu, \mu^{-1}\tau_1\mu, \mu^{-1}\lambda_1\mu \rangle$ is a group conjugate to $\langle \sigma_1, \tau_1, \lambda_1 \rangle$ in $GL(3, Z)$. Define σ, τ, λ as

$$\sigma = \mu^{-1}\sigma_1\mu, \qquad \tau = \mu^{-1}\tau_1\sigma_1^{-1}\mu, \qquad \lambda = \mu^{-1}\lambda_1\mu.$$

From now on, we shall use $\{\sigma, \tau, \lambda\}$ as the generating set of $W_3(203)$. Note that

$$\sigma = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, $\{\tau, \sigma^2, \lambda\}$ is the generating set of $W_3(198)$ given in [20, p. 198]. In conclusion, from now on we shall write

$$W_3(198) = \left\langle \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

$$W_3(203) = \left\langle \sigma = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

(2.20) The case when char $k \neq 2$ and $G = W_3(198), W_3(203)$; the case when char $K = 2$ will be treated in (2.21).

By (2.19), the actions of σ, τ, λ on x, y, z are given by

$$\sigma(x) = \frac{z}{y}, \qquad \sigma(y) = z, \qquad \sigma(z) = \frac{z}{x},$$
$$\tau(x) = z, \qquad \tau(y) = x, \qquad \tau(z) = y,$$
$$\lambda(x) = \frac{1}{x}, \qquad \lambda(y) = \frac{1}{y}, \qquad \lambda(z) = \frac{1}{z}.$$

Let X, Y, Z be defined by

$$X = \frac{1-x}{1+x}, \qquad Y = \frac{1-y}{1+y}, \qquad Z = \frac{1-z}{1+z}.$$

The actions of σ, τ, λ on X, Y, Z are given by

$$\sigma(X) = \frac{Z-Y}{1-ZY}, \qquad \sigma(Y) = Z, \qquad \sigma(Z) = \frac{Z-X}{1-ZX},$$
$$\tau(X) = Z, \qquad \tau(Y) = X, \qquad \tau(Z) = Y,$$
$$\lambda(X) = -X, \qquad \lambda(Y) = -Y, \qquad \lambda(Z) = -Z.$$

It is easy to see that

$$K(X, Y, Z)^{\langle\lambda\rangle} = K(U, V, W),$$

where U, V, W are defined by

$$U = XZ, \qquad V = XY, \qquad W = YZ.$$

The actions of σ, σ^2, τ are given by

$$\sigma(U) = \frac{(U-V)(W-V)}{V(U-1)(W-1)}, \qquad \sigma(V) = \frac{W(V-U)}{V(W-1)},$$
$$\sigma(W) = \frac{U(V-W)}{V(U-1)},$$
$$\sigma^2(U) = \frac{V(W-U)}{W(V-1)}, \qquad \sigma^2(V) = \frac{U(W-V)}{W(U-1)},$$
$$\sigma^2(W) = \frac{(W-U)(W-V)}{W(U-1)(V-1)},$$
$$\tau(U) = W, \qquad \tau(V) = U, \qquad \tau(W) = V.$$

Define u, v, w as

$$u = U - 1, \qquad v = V - 1, \qquad w = W - 1.$$

Then the actions on u, v, w are given by

$$\sigma(u) = av, \qquad \sigma(v) = au, \qquad \sigma(w) = aw, \tag{20}$$

$$\sigma^2(u) = cu, \qquad \sigma^2(v) = cv, \qquad \sigma^2(w) = cw, \tag{21}$$

$$\tau(u) = w, \qquad \tau(v) = u, \qquad \tau(w) = v,$$

where a, b, c are defined by

$$a = \frac{-u + v - w - uw}{uw(v + 1)}, \qquad b = \frac{u - v - w - vw}{vw(u + 1)},$$

$$c = \frac{-u - v + w - uv}{uv(w + 1)}.$$

Note that

$$b = \tau(a) \qquad \text{and} \qquad c = \tau^2(a). \tag{22}$$

Now we shall find the invariants of $K(U, V, W) = K(u, v, w)$ under the action of $\langle \sigma^2, \tau^{-1}\sigma^2\tau \rangle$, which is isomorphic to a Klein four group contained in $W_3(198)$ and $W_3(203)$. Recall that

$$\tau^{-1}\sigma^2\tau(u) = bu, \qquad \tau^{-1}\sigma^2\tau(v) = bv, \qquad \tau^{-1}\sigma^2\tau(w) = bw$$

because of formulae (21) and (22). Define p, q, r, s by

$$p = \frac{u}{v}, \qquad q = \frac{v}{w}, \qquad r = s + as + bs + cs, \qquad s = u + v + w.$$

Then we have

$$K(u, v, w) = K(p, q, s).$$

Note that

$$\sigma^2 \cdot \tau^{-1}\sigma^2\tau = \tau\sigma^2\tau^{-1}.$$

Thus we have

$$\begin{aligned} \sigma^2(bu) &= \sigma^2(\tau^{-1}\sigma^2\tau(u)) \\ &= \sigma^2\tau^{-1}\sigma^2\tau(u) \\ &= \tau\sigma^2\tau^{-1}(u) \\ &= au. \end{aligned}$$

Similarly, we find that

$$\begin{aligned} \sigma^2(bu) &= au, & \sigma^2(bv) &= av, & \sigma^2(bw) &= aw, \\ \tau^{-1}\sigma^2\tau(bu) &= u, & \tau^{-1}\sigma^2\tau(bv) &= v, & \tau^{-1}\sigma^2\tau(bw) &= w. \end{aligned} \tag{23}$$

Hence, r is fixed by both σ^2 and $\tau^{-1}\sigma^2\tau$. It follows that

$$K(p,q,r) \subset K(p,q,s)^{\langle \sigma^2, \tau^{-1}\sigma^2\tau \rangle}.$$

We claim that

$$K(p,q,r) = K(p,q,s)^{\langle \sigma^2, \tau^{-1}\sigma^2\tau \rangle}. \tag{24}$$

Assume this. It is easy to deduce the rationality for $G = W_3(198)$ and $W_3(203)$. First, we compute $\sigma(au), \sigma(bu), \sigma(cu), \ldots$. We apply σ to formula (20) and substitute $\sigma^2(u)$ by formula (21). We obtain

$$\sigma(au) = cv, \qquad \sigma(av) = cu, \qquad \sigma(aw) = cw. \tag{25}$$

We apply σ^{-1} to formula (23) and substitut $\sigma^{-1}(au)$ by formula (20). We obtain

$$\sigma(bu) = v, \qquad \sigma(bv) = u, \qquad \sigma(bw) = w.$$

We apply σ to formula (25) and substitute $\sigma^2(au)$ by formula (23) noting that the order of σ is 4. We obtain

$$\sigma(cu) = bv, \qquad \sigma(cv) = bu, \qquad \sigma(cw) = bw.$$

Now it is easy to verify the actions of σ and τ on p, q, r are given by

$$\begin{aligned} \sigma(p) &= 1/p, & \sigma(q) &= pq, & \sigma(r) &= r, \\ \tau(p) &= 1/pq, & \tau(q) &= p, & \tau(r) &= r. \end{aligned}$$

For $G = W_3(198)$,

$$\begin{aligned} K(x,y,z)^G &= K(p,q,r)^{\langle \tau \rangle} \\ &= K(p,q)^{\langle \tau \rangle}(r) \end{aligned}$$

is rational over K by [8].

For $G = W_3(203)$,

$$\begin{aligned} K(x,y,z)^G &= K(p,q,r)^{\langle \sigma,\tau \rangle} \\ &= K(p,q)^{\langle \sigma,\tau \rangle}(r) \end{aligned}$$

is rational over K again by [8].

Hence, it remains to establish formula (24). It suffices to show that

$$[K(p,q,s):K(p,q,r)] \leq 4.$$

Since $K(p,q,s) = K(p,q,w)$, we shall show that w satisfies an equation of degree ≤ 4 with coefficients over $K(p,q,r)$. We first simplify the form of $1 + a + b + c$. Let s_1, s_2, s_3, t be defined by

$$s_1 = u + v + w, \qquad s_2 = uv + vw + wu, \qquad s_3 = uvw,$$
$$t = (1+u)(1+v)(1+w) = 1 + s_1 + s_2 + s_3.$$

Now, we have

$$\begin{aligned}
&s_3t(1+a+b+c)\\
&\quad = s_3t + v(u+1)(w+1)(-u+v-w-uw) + u(v+1)\\
&\qquad \times(w+1)(u-v-w-vw)\\
&\qquad + w(u+1)(v+1)(-u-v+w-uv)\\
&\quad = s_3t + s_1t - u(v+1)^2(w+1)^2 - v(u+1)^2(w+1)^2\\
&\qquad - w(u+1)^2(v+1)^2\\
&\quad = s_3t + s_1t - t\{(v+1)(w+1) + (u+1)(w+1)\\
&\qquad\qquad + (u+1)(v+1)\}\\
&\qquad + (v+1)^2(w+1)^2 + (u+1)^2(w+1)^2 + (u+1)^2(v+1)^2\\
&\quad = t(s_3 - s_2 - 3s_1 - 9) + (s_2 + 2s_1 + 3)^2\\
&\quad = (s_3 + s_2 + s_1 + 1)(s_3 - s_2 - 3s_1 - 9) + (s_2 + 2s_1 + 3)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&rs_3t\\
&\quad = s_1(1+a+b+c)s_3t\\
&\quad = s_1\{(s_3 + s_2 + s_1 + 1)(s_3 - s_2 - 3s_1 - 9) + (s_2 + 2s_1 + 3)^2\}.
\end{aligned}$$

We write

$$r = s_1E_1/E_2,$$

where E_1 and E_2 are defined by

$$E_1 = (s_3 + s_2 + s_1 + 1)(s_3 - s_2 - 3s_1 - 9) + (s_2 + 2s_1 + 3)^2,$$
$$E_2 = s_3t = s_3(s_3 + s_2 + s_1 + 1).$$

Note that all of s_1, s_2, s_3, E_1, E_2 are polynomials in u, v, w. We shall count the degrees, i.e., the total degrees in u, v, w, of terms in E_1 and E_2.

The constant term and the linear term of E_1 vanish. Hence the degrees of non-vanishing monomials in $s_1 E_1$ can only be 3, 4, 5, 6, 7. On the other hand, the degrees of monomials in E_2 can only be 3, 4, 5, 6. We now write s_1, s_2, s_3 in terms of p, q, w, i.e.,

$$s_1 = u + v + w = w(1 + q + pq),$$

$$s_2 = uv + vw + wu = w^2 q(1 + p + pq),$$

$$s_3 = uvw = w^3 pq^2.$$

We may regard p, q as homogeneous forms in u, v, w of degree zero. Hence, after clearing the factor w^3, the denominator of

$$r = s_1 E_1 / E_2$$

is a polynomial in w of degree 3 with coefficients in $K[p, q]$, while its numerator is a polynomial in w of degree 4 with coefficients in $K[p, q]$. Hence, we have found the polynomial $\Phi(T)$, of degree 4 with coefficients in $K[p, q, r]$ such that $\Phi(w) = 0$.

(2.21) The case when char $K = 2$ and $G = W_3(198), W_3(203)$. Let X, Y, Z be defined by

$$X = \frac{1}{1 + x}, \qquad Y = \frac{1}{1 + y}, \qquad Z = \frac{1}{1 + z}.$$

By (2.19), the actions of σ, τ, λ on X, Y, Z are given by

$$\sigma(X) = \frac{Z(Y + 1)}{Y + Z}, \qquad \sigma(Y) = Z, \qquad \sigma(Z) = \frac{Z(X + 1)}{X + Z},$$

$$\tau(X) = Z, \qquad \tau(Y) = X, \qquad \tau(Z) = Y,$$

$$\lambda(X) = X + 1, \qquad \lambda(Y) = Y + 1, \qquad \lambda(Z) = Z + 1.$$

Define U, V, W as

$$U = X + Z, \qquad V = X + Y, \qquad W = X(X + 1).$$

Then it is easy to see that

$$K(X, Y, Z)^{\langle \lambda \rangle} = K(U, V, W).$$

The actions of σ, σ^2, τ are given by

$$\sigma(U) = \frac{V\{U(U+1)+W\}}{U(U+V)}, \qquad \sigma(V) = \frac{U(U+1)+W}{U+V},$$
$$\sigma(W) = \frac{\{U(U+1)+W\}\{V(V+1)+W\}}{(U+V)^2},$$
$$\sigma^2(U) = \frac{W}{V}, \qquad \sigma^2(V) = \frac{W}{U}, \qquad \sigma^2(W) = W,$$
$$\tau(U) = U+V, \qquad \tau(V) = U, \qquad \tau(W) = U(U+1)+W.$$

Define u, v, w as

$$u = W + UV, \qquad v = W + UV + V, \qquad w = W + UV + U.$$

Then the actions on u, v, w are given by

$$\sigma(u) = au, \qquad \sigma(v) = aw, \qquad \sigma(w) = av,$$
$$\sigma^2(u) = cu, \qquad \sigma^2(v) = cv, \qquad \sigma^2(w) = cw,$$
$$\tau(u) = w, \qquad \tau(v) = u, \qquad \tau(w) = v,$$

where a, b, c are defined by

$$a = 1 + \frac{w}{(u+w)(v+w)}, \qquad b = 1 + \frac{v}{(u+v)(v+w)},$$
$$c = 1 + \frac{u}{(u+v)(u+w)}.$$

Note that

$$b = \tau(a), \qquad c = \tau^2(a), \qquad c = W/UV.$$

In the same way as was used in (2.20), we can deduce the effect of σ on au, bu, cu. We summarize the results as

$$\tau^{-1}\sigma^2\tau(u) = bu, \qquad \tau^{-1}\sigma^2\tau(v) = bv, \qquad \tau^{-1}\sigma^2\tau(w) = bw,$$
$$\tau^2(bu) = au, \qquad \sigma^2(bv) = av, \qquad \sigma^2(bw) = aw,$$
$$\sigma(au) = cu, \qquad \sigma(av) = cw, \qquad \sigma(aw) = cv,$$
$$\sigma(bu) = u, \qquad \sigma(bv) = w, \qquad \sigma(bw) = v,$$
$$\sigma(cu) = bu, \qquad \sigma(cv) = bw, \qquad \sigma(cw) = bv.$$

Now we shall compute the invariants of $K(u, v, w)$ under the actions of $\langle \sigma^2, \tau^{-1}\sigma^2\tau \rangle$. Define p, q, r, s by

$$p = \frac{v}{u + v + w}, \qquad q = \frac{w}{u + v + w}, \qquad r = abcs^4, \qquad s = u + v + w.$$

Be careful that $1 + a + b + c = 0$ in this case. Therefore we cannot imitate the char $K \neq 2$ case.

Now it is easy to find that

$$K(p, q, r) \subset K(p, q, s)^{\langle \sigma^2, \tau^{-1}\sigma^2\tau \rangle}.$$

Suppose that

$$K(p, q, r) = K(p, q, s)^{\langle \sigma^2, \tau^{-1}\sigma^2\tau \rangle}. \tag{26}$$

The actions of σ and τ are given by

$$\sigma(p) = q, \qquad \sigma(q) = p, \qquad \sigma(r) = r,$$
$$\tau(p) = p + q + 1, \qquad \tau(q) = p, \qquad \tau(r) = r.$$

For $G = W_3(198)$,

$$K(x, y, z)^G = K(p, q, r)^{\langle \tau \rangle} = K(p, q)^{\langle \tau \rangle}(r)$$

is rational over K by setting $x = p + q + 1, y = q + 1, z = p + 1$ in Corollary 2.15.

For $G = W_3(203)$,

$$\begin{aligned} K(x, y, z)^G &= K(p, q, r)^{\langle \sigma, \tau \rangle} \\ &= K(p, q)^{\langle \sigma, \tau \rangle}(r) \\ &= K(pq(p + q + 1), p^2 + pq + q^2 + p + q, r) \end{aligned}$$

because $p + q + (p + q + 1) = 1, p^2 + pq + q^2 + p + q, pq(p + q + 1)$ are the elementary symmetric functions of $p, q, p + q + 1$ and, therefore,

$$\left[K(p, q) : K(pq(p + q + 1), p^2 + pq + q^2 + p + q)\right] \leq 6.$$

Therefore, it remains to establish formula (26). Since

$$a = 1 + \frac{w}{(u+w)(v+w)} = \frac{w(s+1)+uv}{(u+w)(v+w)},$$

it follows that

$$abc = \frac{\{u(s+1)+vw\}\{v(s+1)+uw\}\{w(s+1)+uv\}}{(u+v)^2(v+w)^2(w+u)^2}.$$

Plugging in

$$u = ts, v = ps, w = qs,$$

where $t = 1 + p + q$, we find that

$$\begin{aligned} r &= abcs^4 \\ &= \frac{s\{t(s+1)+pqs\}\{p(s+1)+tqs\}\{q(s+1)+tps\}}{(t+p)^2(p+q)^2(q+t)^2}. \end{aligned}$$

Hence, s satisfies an equation of degree 4 with coefficients in $K(t, p, q, r) = K(p, q, r)$, hence, the result.

(2.22) The case when $G = W_8(198), W_2(203)$,

$$W_8(198) = \left\langle \sigma_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \tau_1 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

$$W_2(203) = \left\langle \sigma_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \tau_1 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \lambda_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

We may use the techniques in (2.17) and (2.18) to solve the rationality of these two groups. Before we start the proof, we transform the generating sets of these groups into convenient ones. Let μ be the matrix defined by

$$\mu = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \in GL(3, Z).$$

Then $\langle \mu^{-1}\sigma_1\mu, \mu^{-1}\tau_1\mu\rangle$ and $\langle \mu^{-1}\sigma_1\mu, \mu^{-1}\tau_1\mu, \mu^{-1}\lambda_1\mu\rangle$ are conjugate to $\langle \sigma_1, \tau_1\rangle$ and $\langle \sigma_1, \tau_1, \lambda_1\rangle$, respectively. Define $\rho, \sigma, \tau, \lambda$ as

$$\rho = \mu^{-1}\tau_1\mu, \qquad \sigma = \mu^{-1}\sigma_1^{-1}\tau_1\mu, \qquad \tau = \sigma(\sigma\rho)^2\sigma^{-1}, \qquad \lambda = \mu^{-1}\lambda_1\mu.$$

From now on, we shall use $\{\rho, \sigma\}$ and $\{\rho, \sigma, \lambda\}$ as the generating sets of $W_8(198)$ and $W_2(203)$, respectively. Note that

$$\rho = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, $\{\sigma, \tau\}$ and $\{\sigma, \tau, \lambda\}$ are the generating sets of $W_{10}(191)$ and $W_2(198)$ in (2.17) and (2.18), respectively. Hence, it is easy to adapt the techniques in (2.17) and (2.18) to this situation. Define w by

$$w := 1/xyz. \tag{27}$$

The actions on $K(x, y, z)$ are given by

$$\begin{aligned}
&\sigma(x) = z, \quad \sigma(y) = x, \quad \sigma(z) = y, \quad \sigma(w) = w,\\
&\tau(x) = z, \quad \tau(y) = w, \quad \tau(z) = x, \quad \tau(w) = y,\\
&\sigma^{-1}\tau\sigma(x) = y, \quad \sigma^{-1}\tau\sigma(y) = x, \quad \sigma^{-1}\tau\sigma(z) = w, \quad \sigma^{-1}\tau\sigma(w) = z,\\
&\rho(x) = \frac{1}{x}, \quad \rho(y) = \frac{1}{w}, \quad \rho(z) = \frac{1}{z}, \quad \rho(w) = \frac{1}{y},\\
&\lambda(x) = \frac{1}{x}, \quad \lambda(y) = \frac{1}{y}, \quad \lambda(z) = \frac{1}{z}, \quad \lambda(w) = \frac{1}{w}.
\end{aligned}$$

(i) Suppose that char $K \neq 2$. We shall imitate the process of (2.17) and leave the details to the readers. Define $X, Y, Z, W, u, v, s, t, A, B, C, D$ as

$$\begin{aligned}
&X = \frac{1-x}{1+x}, \quad Y = \frac{1-y}{1+y}, \quad Z = \frac{1-z}{1+z}, \quad W = \frac{1-w}{1+w},\\
&4u = X + Y + Z + W,\\
&4v = X + Y - Z - W,\\
&4s = X - Y + Z - W,\\
&4t = X - Y - Z + W,\\
&A = -\frac{vs}{tu}, \quad B = -\frac{tv}{su}, \quad C = -\frac{st}{uv}, \quad D = \frac{1}{u}.
\end{aligned}$$

Then we have

$$K(x,y,z)^{\langle \tau, \sigma^{-1}\tau\sigma \rangle} = K(A,B,C,D)$$

with the relation

$$1 + D^2 = AB + BC + CA + 2ABC \tag{28}$$

and the group actions

$$\begin{array}{llll} \sigma(A) = C, & \sigma(B) = A, & \sigma(C) = B, & \sigma(D) = D, \\ \rho(A) = C, & \rho(B) = B, & \rho(C) = A, & \rho(D) = -D \\ \lambda(A) = A, & \lambda(B) = B, & \lambda(C) = C, & \lambda(D) = -D. \end{array}$$

For $G = W_2(203)$,

$$\begin{aligned} K(A,B,C,D)^{\langle \sigma,\rho,\lambda \rangle} &= \left\{K(A,B,C,D)^{\langle \lambda \rangle}\right\}^{\langle \sigma,\rho \rangle} \\ &= K(A,B,C,D^2)^{\langle \sigma,\rho \rangle} \\ &= K(A,B,C)^{\langle \sigma,\rho \rangle} \\ &= K(A+B+C, AB+BC+CA, ABC), \end{aligned}$$

where the third equality holds because of formula (28).

It remains to consider the case of $G = W_8(198)$. If char $K = 3$, define L, M, N by

$$L = A + B + C, \qquad M = AB + BC + CA,$$
$$N = (A - B)(B - C)(C - A).$$

By Lemma 2.14

$$K(A,B,C,D)^{\langle \sigma \rangle} = K(L,M,N,D)$$

with the relation

$$L^3(1 + D^2) = L^3M - 2M^3 + 2L^2M^2 - 2N^2 \tag{29}$$

and the action of ρ on L, M, N, D:

$$\rho(L) = L, \qquad \rho(M) = M, \qquad \rho(N) = -N, \qquad \rho(D) = -D.$$

Define S and T by

$$S = N/D, \qquad T = D^2.$$

Then the relation (29) becomes

$$L^3(1 + T) = L^3M - 2M^3 + 2L^2M^2 - 2S^2T. \tag{30}$$

Hence, T is linear in formula (30). Therefore,

$$\begin{aligned} K(L, M, N, D)^{\langle\rho\rangle} &= K(L, M, S, D)^{\langle\rho\rangle} \\ &= K(L, M, S, T) \\ &= K(L, M, S). \end{aligned}$$

It remains to consider the case char $K \neq 2, 3$ and $G = W_8(198)$. Define P, Q, R, U, V by

$$P = B + A + C, \qquad Q = B + \zeta A + \zeta^2 C, \qquad R = B + \zeta^2 A + \zeta C,$$

$$U = \frac{Q^2}{R} + \frac{R^2}{Q}, \qquad V = \zeta\frac{Q^2}{R} + \zeta^2\frac{R^2}{Q},$$

where ζ is a primitive third root of 1.

Then we have, by Lemma 2.14

$$K(A, B, C, D)^{\langle\sigma\rangle} = K(P, U, V, D)$$

and the relation (28) becomes

$$1 + D^2 = \tfrac{1}{3}(P^2 - QR) + \tfrac{2}{27}(P^3 + Q^3 + R^3 - 3PQR).$$

The action of ρ is given by

$$\rho(P) = P, \qquad \rho(U) = U, \qquad \rho(V) = -U - V, \qquad \rho(D) = -D.$$

Define E by

$$E = V + \tfrac{1}{2}U.$$

Note that

$$\rho(E) = -E.$$

Substitute the identities

$$\begin{aligned} QR &= \tfrac{1}{3}(U^2 + UV + V^2) = \tfrac{1}{4}U^2 + \tfrac{1}{3}E^2 \\ Q^3 + R^3 &= \tfrac{1}{3}(U^3 + U^2V + UV^2) = \tfrac{1}{4}U^3 + \tfrac{1}{3}UE^2 \end{aligned}$$

into formula (31). We find that

$$D^2 \in K(P, U, E/D).$$

Now it is clear that

$$\begin{aligned} K(P,U,V,D)^{\langle \rho \rangle} &= K(P,U,E,D)^{\langle \rho \rangle} \\ &= K\left(P, U, \frac{E}{D}, D^2\right) \\ &= K\left(P, U, \frac{E}{D}\right). \end{aligned}$$

(ii) Suppose that char $K = 2$. We shall imitate the process of (2.18). As in the above (i), introduce w by

$$w := 1/xyz.$$

For the case $G = W_2(203)$, define $X, Y, Z, W, u, v, s, t, d, D, V, S$ just the same as the proof of $G = W_2(198)$ in (2.18). We find that

$$K(d, v, s, t)^{\langle \tau, \sigma^{-1}\tau\sigma, \lambda \rangle} = K(V, S, t)$$

and the actions of σ and ρ are given by

$$\begin{aligned} &\sigma(V) = V + S, \qquad \sigma(S) = V, \qquad \sigma(t) = t, \\ &\rho(V) = V, \qquad \rho(S) = S + V, \qquad \rho(t) = t. \end{aligned}$$

Therefore,

$$\begin{aligned} K(V,S,t)^{\langle \sigma, \rho \rangle} &= K(V,S)^{\langle \sigma, \rho \rangle}(t) \\ &= K\big(VS + (V+S)^2, VS(V+S), t\big). \end{aligned}$$

Now we turn to the case $G = W_8(198)$. Define $X, Y, Z, W, u, v, s, t, d$ the same as those at the beginning of (2.18). Define V and S by

$$V = v(v + t), \qquad S = s(s + t).$$

We find that

$$K(d, v, s, t)^{\langle \tau, \sigma^{-1}\tau\sigma \rangle} = K(V, S, d, t)$$

and the relation $xyzw = 1$ becomes

$$d^2 + dt + t^2 + t + t(V + S) + VS = 0. \tag{32}$$

Recall that the actions of σ and ρ are given by

$$\sigma(V) = V + S, \quad \sigma(S) = V, \quad \sigma(d) = d + V, \quad \sigma(t) = t,$$
$$\rho(V) = V, \quad \rho(S) = V + S, \quad \rho(d) = d + t + V, \quad \rho(t) = t.$$

Define P, Q, R by

$$P = d, \quad Q = d + V, \quad R = d + S.$$

Then we find that

$$\sigma(P) = Q, \quad \sigma(Q) = R, \quad \sigma(R) = P, \quad \sigma(t) = t.$$

Define A, B, C, M, N by

$$A = R + P + Q, \quad B = R + \zeta P + \zeta^2 Q, \quad C = R + \zeta^2 P + \zeta Q,$$
$$M = \frac{B^2}{C} + \frac{C^2}{B}, \quad N = \zeta \frac{B^2}{C} + \zeta^2 \frac{C^2}{B},$$

where ζ is a primitive third root of 1.

By Lemma 2.14

$$K(V, S, d, t)^{\langle \sigma \rangle} = K(A, M, N, t).$$

Now formula (32) becomes

$$t^2 + t + tA + A^2 + M^2 + MN + N^2 = 0. \tag{33}$$

However, ρ acts on A, M, N, t by

$$\rho(A) = A + t, \quad \rho(M) = M, \quad \rho(N) = M + N, \quad \rho(t) = t.$$

Let L be defined by

$$L = A + tN/M.$$

Then

$$\rho(L) = L.$$

Hence,

$$K(A, M, N, t)^{\langle \rho \rangle} = K(L, M, N(M + N), t)$$
$$= K(L, M, t),$$

since $N(M + N)$ can be expressed by L, M, t because of formula (33).

(2.23) The case when $G = w_9(198)$,

$$W_9(198) = \left\langle \sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \tau_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let μ be the matrix defined by

$$\mu = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\langle \mu^{-1}\sigma_1\mu, \mu^{-1}\tau_1\mu \rangle$ is conjugate to $\langle \sigma_1, \tau_1 \rangle$ in $GL(3, Z)$. Define σ and τ by

$$\sigma = \mu^{-1}\sigma_1\mu, \qquad \tau = \mu^{-1}\tau_1\mu.$$

From now on, we shall use $\{\sigma, \tau\}$ as a generating set of $W_9(198)$. Note that

$$\sigma = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Define w by

$$w := 1/xyz.$$

The actions of σ and τ on x, y, z, w are given by

$$\sigma(x) = y, \quad \sigma(y) = z, \quad \sigma(z) = w, \quad \sigma(w) = x,$$
$$\tau(x) = x, \quad \tau(y) = z, \quad \tau(z) = y, \quad \tau(w) = w.$$

Hence, the fixed field of G is generated by the elementary symmetric polynomials in x, y, z, w, i.e.,

$$x + y + z + w,$$
$$xy + xz + xw + yz + yw + zw,$$
$$xyz + xyw + xzw + yzw,$$
$$xyzw = 1.$$

Thus $K(x, y, z)^G$ is rational over K.

(2.24) The case when $G = W_{11}(198)$. Note that there is a misprint on page 198 of Tahara's paper [20]. The correct formulation can be found in the proof of Proposition 9 in the same paper. In fact, from page 201 of [20] we find the correct generators for $G = W_{11}(198)$ as

$$W_{11}(198) = \left\langle \sigma_1 = \begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tau_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle.$$

Let μ be the matrix defined by

$$\mu = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then $\langle \mu^{-1}\sigma_1\mu, \mu^{-1}\tau_1\mu \rangle$ is conjugate to $\langle \sigma_1, \tau_1 \rangle$. Define ρ, σ, τ by

$$\rho = \mu^{-1}\tau_1\sigma_1^2\mu, \qquad \sigma = \mu^{-1}\sigma_1\tau_1\sigma_1^2\mu, \qquad \tau = \rho^2.$$

Since σ_1 and τ_1 are of orders 4 and 2, respectively, we may as well take $\{\rho,\sigma\}$ as a generating set of $W_{11}(198)$. Note that

$$\rho = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\tau = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Moreover, $\{\sigma, \tau\}$ is the generating set of $W_{11}(191)$ in (2.13).

(i) Suppose that char $K \neq 2$. As in (2.13), define u, v, w by

$$u = \frac{y+z}{1+x}, \qquad v = \frac{x+z}{1+y}, \qquad w = \frac{x+y}{1+z}.$$

Define U, V, W by

$$U = \frac{1-u}{1+u}, \qquad V = \frac{1-v}{1+v}, \qquad W = \frac{1-w}{1+w}.$$

The actions on U, V, W are given by

$$\begin{aligned} \rho(U) &= -U, & \rho(V) &= W, & \rho(W) &= -V, \\ \sigma(U) &= W, & \sigma(V) &= U, & \sigma(W) &= V, \\ \tau(U) &= U, & \tau(V) &= -V, & \tau(W) &= -W, \\ \sigma^{-1}\tau\sigma(U) &= -U, & \sigma^{-1}\tau\sigma(V) &= V, & \sigma^{-1}\tau\sigma(W) &= -W. \end{aligned}$$

Define A, B, C by

$$A = \frac{U}{VW}, \qquad B = \frac{V}{UW}, \qquad C = \frac{W}{UV}.$$

Since

$$[K(U,V,W) : K(A,B,C)] = 4,$$

it follows that

$$K(x,y,z)^{\langle \tau, \sigma^{-1}\tau\sigma \rangle} = K(A,B,C).$$

Now ρ and σ act on A, B, C by

$$\begin{aligned} \rho(A) &= A, & \rho(B) &= C, & \rho(C) &= B, \\ \sigma(A) &= C, & \sigma(B) &= A, & \sigma(C) &= B. \end{aligned}$$

Hence, $K(A,B,C)^{\langle \rho, \sigma \rangle}$ is rational over K.

(ii) Suppose that char $K = 2$. Again imitate the process of (2.13). Define u, v, w by

$$u = \frac{1+y}{1+x+y+z}, \qquad v = \frac{1+z}{1+x+y+z}, \qquad w = \frac{1+y+z}{1+x+y+z}.$$

The action on u, v, w are given by

$$\begin{aligned} \rho(u) &= v, & \rho(v) &= u+1, & \rho(w) &= w+v+1, \\ \sigma(u) &= u+v+1, & \sigma(v) &= u, & \sigma(w) &= w+u+1, \\ \tau(u) &= u+1, & \tau(v) &= v+1, & \tau(w) &= w+u+v+1, \\ \sigma^{-1}\tau\sigma(u) &= u, & \sigma^{-1}\tau\sigma(v) &= v+1, & \sigma^{-1}\tau\sigma(w) &= w+u+1. \end{aligned}$$

By Theorem 2.2 it suffices to establish the rationality of $K(u,v)^G$. Now define A and B by

$$A = u(u+1), \qquad B = v(v+1).$$

The same as in (2.13), we find that

$$K(u,v)^{\langle \tau, \sigma^{-1}\tau\sigma\rangle} = K(A,B).$$

The actions of ρ and σ on A, B are given by

$$\rho(A) = B, \qquad \rho(B) = A,$$
$$\sigma(A) = A + B, \qquad \sigma(B) = A.$$

Hence, we have

$$K(A,B)^{\langle \rho, \sigma\rangle} = K\big(AB + (A+B)^2, AB(A+B)\big).$$

(2.25) There remains only one case, $G = W_{10}(198)$,

$$W_{10}(198) = \left\langle \sigma = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

We do not know whether $K(x, y, z)^G$ is rational over K.

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