Canonical forms for families of anti-commuting diagonalizable linear operators

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ABSTRACT

It is well known that a commuting family of diagonalizable linear operators on a finite dimensional vector space is simultaneously diagonalizable. In this paper, we consider a family $\mathcal{A} = \{A_a\}$, $A_a : V \to V$, $a = 1, \ldots, N$ of anti-commuting (complex) linear operators on a finite dimensional vector space. We prove that if the family is diagonalizable over the complex numbers, then V has an $\mathcal{A}$-invariant direct sum decomposition into subspaces V_α such that the restriction of the family $\mathcal{A}$ to V_α is a representation of a Clifford algebra. Thus unlike the families of commuting diagonalizable operators, diagonalizable anti-commuting families cannot be simultaneously digonalized, but on each subspace, they can be put simultaneously to (non-unique) canonical forms. The construction of canonical forms for complex representations is straightforward, while for the real representations it follows from the results of [A.H. Bilge, Ş. Koçak, S. Uğuz, Canonical bases for real representations of Clifford algebras, Linear Algebra Appl. 419 (2006) 417–439].

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1. Introduction

Simultaneous diagonalization of a family of commuting linear operators on a finite dimensional vector space is a well known result in linear algebra [3]. This result is applicable to an arbitrary (possibly infinite) family and asserts the existence of a basis with respect to which all operators of the family are diagonal. In this paper, we consider an anti-commuting family $\mathcal{A}$ of operators on a finite dimensional vector space V and we show that if the family is diagonalizable over the complex numbers, then the operators in the family can be put simultaneously into canonical forms over both the complex and real numbers.

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Real or complex representations of Clifford algebras are typical examples of anti-commuting families that are diagonalizable over the complex numbers. Our main result is the proof that the finite dimensional vector space V has an $\mathcal{A}$-invariant direct sum decomposition into subspaces V_α, such that, except for the common kernel of the family, the restriction of the family to V_α is either a single nonzero diagonal operator or a representation of some Clifford algebra of dimension larger than 1. This result, presented in Section 3, is derived directly from the fact that if A_a belongs to the family $\mathcal{A}$, then A_a and A_a^2 have the same kernel and the A_a^2's form a commuting diagonalizable family $\mathcal{B}$, hence they are simultaneously diagonalizable. One can then diagonalize the family $\mathcal{B}$ simultaneously, rearrange the basis and obtain subspaces on which there are finite sub-collections of anti-commuting operators whose squares are constants, that is, representations of Clifford algebras. Note that, since a one-dimensional subspace on which there is a single nonzero operator is a representation of a one-dimensional Clifford algebra we could simply state that V has a decomposition to $\mathcal{A}$ invariant subspaces on which the restriction of the family consists of representations of Clifford algebras.

In Section 2, we review basic results for commuting families of diagonalizable operators and we discuss the direct construction of canonical forms, as a motivation for the general case. Then, in Section 3, we give the main theorem for the $\mathcal{A}$ invariant decomposition of V for an arbitrary anti-commuting family leading to canonical forms over the complex numbers. In Section 4, we discuss square diagonalizable operators and describe the construction of real canonical forms for operators that are diagonalizable over the complex numbers.

2. Preliminaries

In Section 2.1, we introduce our notation, present basic properties of commuting families of diagonalizable operators, and give basic definitions related to Clifford algebras. Then in Section 2.2 we discuss the direct construction of canonical forms.

2.1. Notation and basic definitions

In the following V is a finite dimensional real or complex vector space. Linear operators on V will be denoted by upper case Latin letters A, B, etc., and the components of their matrices with respect to some basis will be denoted by A_{ij}, B_{ij}. Labels of operators will be denoted by single indices, for example, A_a, $a = 1, \ldots, n$ denotes elements of a family of operators. A family $\mathcal{A}$ of operators is called an "anti-commuting family" if for every distinct pair of operators A and B in the family, $AB + BA = 0$. The symbol δ_{ij} denotes the Kronecker delta, that is $\delta_{ij} = 1$, if $i = j$ and zero otherwise. When we use partitioning of matrices, scalars will denote sub-matrices of appropriate size.

Remark 2.1. If A is a diagonalizable operator on a vector space V and V has an A-invariant direct sum decomposition, then the restriction of A to each invariant subspace is also diagonalizable (Lemma 1.3.10 in [4]). Furthermore if we have a family of commuting (anti-commuting) operators $\mathcal{A}$ on V and V has an $\mathcal{A}$-invariant direct sum decomposition, then the restriction of the family to each summand is again a commuting (anti-commuting) family of diagonalizable operators.

Now, we give Theorem 2.2 whose proof is adopted from [4].

Theorem 2.2. *Let $\mathcal{D}$ be a family of diagonalizable operators on an n-dimensional vector space V and A, B be in $\mathcal{D}$. Then A and B commute if and only if they are simultaneously diagonalizable.*

Proof. Assume that $AB = BA$ holds. By a choice of basis we may assume that A is diagonal, that is $A_{ij} = \lambda_i \delta_{ij}$, $i = 1, \ldots, n$. From the equation $AB = BA$ we have

$$(\lambda_i - \lambda_j)B_{ij} = 0,$$

that is B_{ij} is zero unless $\lambda_i = \lambda_j$. Rearranging the basis, we have a decomposition of V into eigenspaces of A. This decomposition is B invariant, on each subspace A is constant, B is diagonalizable, hence they are simultaneously diagonalizable.

Conversely, assume that A and B are simultaneously diagonalizable. Then, there is a basis with respect to which their matrices are diagonal. Since diagonal matrices commute, it follows that the operators A and B commute. $\square$

Remark 2.3. A commuting family of linear operators can be infinite, since we can always add linear combinations of the elements of the family. However, an anti-commuting family is necessarily finite, unless it contains operators A_a with $A_a^2 = 0$. To see this, let $\mathcal{A} = \{A_1, \ldots, A_N\}$ be a family of anti-commuting diagonalizable linear operators on V. That is $A_a A_b + A_b A_a = 0$, $a \neq b = 1, \ldots, N$. The family $\mathcal{A}$ is necessarily linearly independent and it contains $N < n^2$ elements. Because if B is a linear combination of the A_a, $a = 1, \ldots, k$, i.e., $B = \sum_a^k c_a A_a$ and B anti-commutes with each of the A_b's in this summation, it is necessarily zero. Furthermore, since the anti-commuting family cannot include the identity matrix, it follows that $N < n^2$.

Proposition 2.4. *Let A be a linear operator on a finite dimensional vector space V. Then $Ker(A) \subseteq Ker(A^2)$. If A is diagonalizable over C, $Ker(A^2) = Ker(A)$. If A^2 is diagonalizable over C, then the restriction of A to the complement of its kernel is diagonalizable over C.*

Proof. The first statement is obvious. For the second one, we choose a basis with respect to which A is diagonal. Then eigenvalues of A^2 are squares of eigenvalues of A. Hence, $Ker(A^2) = Ker(A)$. To prove the third statement, note that if A is not diagonalizable over C, then in its Jordan form over C, there is at least one nondiagonal Jordan block whose square is diagonal. But this is possible only when the corresponding eigenvalue is zero. $\square$

Remark 2.5. If A is diagonalizable, then A^2 is also diagonalizable. Also if the pair (A, B) anti-commutes then the pairs (A, B^2) and (A^2, B^2) commute, since

$$AB^2 = -B(AB) = B^2A, \quad A^2B^2 = A(AB^2) = A(B^2A) = (B^2A)A = B^2A^2.$$

Thus given a family $\{A_1, \ldots, A_N\}$ of anti-commuting diagonalizable operators the families $\{A_1, A_2^2, \ldots, A_N^2\}$ and $\{A_1^2, \ldots, A_N^2\}$ are commuting diagonalizable families, hence they are both simultaneously diagonalizable.

We give now the definitions related to Clifford algebras and describe briefly the construction of canonical forms for complex representations.

Definition 2.6. Let V be a vector space over the field k and q be a quadratic form on V. Then the associative algebra with unit, generated by the vector space V and the identity 1 subject to the relations $v \cdot v = -q(v)1$, for any $v \in V$ is called a Clifford algebra and denoted by $Cl(V, q)$.

Definition 2.7. Let W be a real or complex vector space and $Cl(V, q)$ be a Clifford algebra. A representation of $Cl(V, q)$ on W is an algebra homomorphism $\rho : \ Cl(V, q) \rightarrow End(W)$.

The construction of complex representations of an N-dimensional Clifford algebra on an n-dimensional vector space is a straightforward process. We first construct canonical forms for a pair of anti-commuting operators, then use the requirement that the remaining $N - 2$ operators anti-commute with these two to show that their canonical forms are given by the representation of an $N - 2$-dimensional Clifford algebra on an $n/2$-dimensional vector space. If the Clifford algebra contains at least one element with positive square, the construction of real representations is similar to the complex case. But if all Clifford algebra elements have negative squares, the construction of simultaneous canonical forms is nontrivial [1].

In the next subsection, we shall discuss the case of two anti-commuting operators and outline a proof of Theorem 3.1.

2.2. A pair of anti-commuting operators: outline of a direct proof

Theorem 3.1. given in Section 3 states that the anti-commuting family is essentially a direct sum of representations of Clifford algebras. The proof of the theorem is almost trivial but it is nonconstructive; the difficulty of the construction is in a sense transferred to the construction of the simultaneous canonical forms of Clifford algebras. In this section, we describe in detail the construction of canonical forms for two anti-commuting operators and outline an alternative proof of Theorem 3.1.

Let $\mathcal{A} = \{A_1, \ldots, A_N\}$ be a finite family of anti-commuting diagonalizable operators. One can always choose a basis with respect to which any member of the family is diagonal, hence, without loss of generality we may assume that $A = A_1$ is diagonal and write $A_{ij} = \lambda_i \delta_{ij}$. If $B = B_{ij}$ is any other member of the family, substituting these in equation $AB + BA = 0$, we obtain

$$(\lambda_i + \lambda_j)B_{ij} = 0$$

Thus B_{ij} is zero unless $\lambda_i + \lambda_j = 0$. This happens either when λ_i and λ_j are both zero, or when they are a pair of eigenvalues with equal magnitude and opposite sign. This suggests that we should group the eigenvalues of A in three sets

$$\{0\}, \quad \{\mu_1, \ldots, \mu_l\}, \quad \{\lambda_1, -\lambda_1, \ldots, \lambda_k, -\lambda_k\},$$

where $\mu_i + \mu_j \neq 0$ for $i, j = 1, \ldots, l$. Let the direct sum of the eigenvectors for each group be $Ker(A)$, U_A and W_A respectively. That is

$$V = Ker(A) \oplus U_A \oplus W_A.$$

From the last equation it can easily be seen that these subspaces are $\mathcal{A}$ invariant and for any other member of the family B, $B|_{Ker(A)}$ is free and $B|_{U_A} = 0$. On $Ker(A)$ we have a family of $N - 1$ operators and on U_A, only A is nonzero. Thus we have nontrivially an N element anti-commuting family only on the subspace W_A.

Let $W_{A,i}^{\pm}$ be the eigenspaces corresponding to the eigenvalues $\pm\lambda_i$ and let $W_{A,i} = W_{A,i}^{+} \oplus W_{A,i}^{-}$. Since if $AX = \lambda X$, then $A(BX) = -BAX = -\lambda(BX)$ and it follows that B maps $W_{A,i}^{+}$ into $W_{A,i}^{-}$ and vice versa, that is,

$$B\left(W_{A,i}^{+}\right) \subset W_{A,i}^{-}, \quad B\left(W_{A,i}^{-}\right) \subset W_{A,i}^{+},$$

hence $W_{A,i}$'s are B invariant.

If the dimensions of the subspaces $W_{A,i}^{\pm}$ are not equal, then the restriction of B to $W_{A,i} = W_{A,i}^{+} \oplus W_{A,i}^{-}$ is necessarily singular. Because if B were nonsingular on either $W_{A,i}^{\pm}$, it would map a linearly independent set to a linearly independent set, but this is impossible if the dimensions are different. However, the restriction of B can be singular even if the dimensions are equal. On the other hand if B is nonsingular, then necessarily $dim(W_{A,i}^{+}) = dim(W_{A,i}^{-})$ since bases of $W_{A,i}^{+}$ are mapped to bases of $W_{A,i}^{-}$ and vice versa. Thus we can refine the direct sum decomposition of $W_{A,i}$ and we arrive to the direct sum decomposition

$$W_{A,i} = \left(Ker(B) \cap W_{A,i}^{+}\right) \oplus \left(Ker(B) \cap W_{A,i}^{-}\right) \oplus \tilde{W}_{A,i}^{+} \oplus \tilde{W}_{A,i}^{-},$$

where the $\tilde{W}_{A,i}^{\pm}$ are subspaces of equal dimension on which B is nonsingular. It follows that the restrictions of A and B to $W_{A,i}$ have the following block diagonal form

$$A_{|W_{A,i}} = \begin{pmatrix} \lambda_i & 0 & 0 & 0 \\ 0 & -\lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & -\lambda_i \end{pmatrix}, \quad B_{|W_{A,i}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & B_2 & 0 \end{pmatrix},$$

where the first two diagonal blocks may have different dimensions but the last two diagonal blocks in the restriction of A and the sub-matrices B_1 and B_2 in the restriction of B are square matrices of the same dimension. Incorporating this decomposition to the previous one, we have a direct sum decomposition of V adopted to the pair of anti-commuting diagonalizable operators A and B as follows

$$V = (Ker(A) \cap Ker(B)) \oplus U_A \oplus U_B \oplus W_1^+ \oplus W_1^- \oplus \cdots \oplus W_k^+ \oplus W_k^-,$$

where $B|_{U_A} = 0$, $A|_{U_B} = 0$, and both A and B are nonsingular on the $W_i^{\pm}$, and $dim(W_i^+) = dim(W_i^-)$ for $i = 1, \ldots, k$. Thus we have now subspaces $W_i = W_i^+ \oplus W_i^-$ on which A^2 is a nonzero constant and B is nonsingular.

To determine the forms of B_1 and B_2, we start with the following observation. By Remark 2.5 (A, B^2) and (A^2, B^2) are simultaneously diagonalizable and by Proposition 2.4 $Ker(B) = Ker(B^2)$. Recall that A and B are both nonsingular on the subspace W_i and diagonalize A and B^2 simultaneously. We choose a basis $\{X_1, \ldots, X_m\}$ for W_i^+, the $+\lambda$ eigenspace of A. Thus $AX_i = \lambda X_i$, $B^2 X_i = \eta_i X_i$, for $i = 1 \ldots, m$ and we define $Y_i = BX_i$. Then $AY_i = A(BX_i) = -B(AX_i) = -\lambda(BX_i) = -\lambda Y_i$, hence Y_i belongs to the $-\lambda$ eigenspace of A. Furthermore $BY_i = B^2 X_i = \eta_i X_i$. It follows that with respect to the basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m\}$, the matrices of A, B and B^2 are as below.

$$A|_{W_i} = \begin{pmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{pmatrix}, \quad B|_{W_i} = \begin{pmatrix} 0 & D \\ I & 0 \end{pmatrix}, \quad B^2|_{W_i} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

where all sub-matrices are square, I is the identity matrix and D is a diagonal matrix. If B^2 has q distinct eigenvalues $d_1, \ldots, d_q$ with eigenspaces of dimensions m_i, we can rearrange the basis so that

$$A|_{W_i} = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix}, \quad B|_{W_i} = \begin{pmatrix} 0 & d_1 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_q \\ 0 & 0 & \cdots & I & 0 \end{pmatrix},$$

$$B^2|_{W_i} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_q & 0 \\ 0 & 0 & \cdots & 0 & d_q \end{pmatrix},$$

where λ and d_i's denote constant matrices of appropriate size. Hence each W_i has a direct sum decomposition

$$W_i = W_{i,1} \oplus \cdots \oplus W_{i,q}$$

such that both A^2 and B^2 restricted to each summand are constant matrices. Thus on the $W_{i,j}$ we have a representation of a two-dimensional Clifford algebra. This is possible in particular if the dimensions of $W_{i,j}$'s are even.

If $N \geq 2$ and C_a is any other element of the family then it will be free on the common kernel of A and B, and it will be zero on U_A and U_B. From the anti-commutativity of C_a with A and B, on each $W_{i,j}$, it will be of the form

$$C_a|_{W_{i,j}} = \begin{pmatrix} 0 & -d_j E_a \\ E_a & 0 \end{pmatrix}$$

where E_a is a square matrix. By change of basis on $W_{i,j}^+$ we can diagonalize the restriction of C_a^2 to $W_{i,j}^+$, rearrange the basis to obtain subspaces on which A, B and C_a^2's are constants. Here, by construction, the nonzero eigenvalues of C_a occur in pairs with equal absolute value and opposite sign and the dimensions of the zero eigenspaces in $W_{i,j}^+$ and $W_{i,j}^-$ are the same.

It is possible to continue with this construction by adding operators one by one, but this procedure gets more and more complicated. Instead, we note that the C_a^2's restricted to $W_{i,j}^+$ form a commuting diagonalizable family, and instead of adding new operators one by one, we could diagonalize them simultaneously, and find subspaces on which C_a^2's are constants. This remark suggests that one could do the same trick at the beginning and diagonalize simultaneously the squared family. This leads to the proof given below.

3. Anti-commuting families of diagonalizable linear operators: proof

The construction of simultaneous canonical forms for a family $\mathcal{A} = \{A_1, \ldots, A_N\}$, of anti-commuting of diagonalizable operators on a finite dimensional (real or complex) vector space V, is based on the fact that the family of squared operators $\mathcal{B} = \{A_1^2, \ldots, A_N^2\}$ is a commuting diagonalizable family, hence it is simultaneously diagonalizable. One can then find a basis with respect to which the family of squared operators is diagonal and rearrange this basis in such a way that the vector space V is a direct sum of subspaces on which the operators of the family $\mathcal{B}$ are either zero or constant. This is nothing but a representation of a Clifford algebra which exists in specific dimensions. We state this result below.

Theorem 3.1. *Let $\mathcal{A}$ be a family of N diagonalizable anti-commuting operators on an n-dimensional vector space V. Then V has a direct sum decomposition*

$$V = U_0 \oplus U_1 \oplus \ldots U_j \oplus W_2^1 \oplus \cdots \oplus W_2^{k_2} \oplus \cdots \oplus W_N^1 \oplus \cdots \oplus W_N^{k_N},$$

where U_0 is the common kernel of the family, U_i's are subspaces of arbitrary dimensions on which only A_i is nonzero and it is nonsingular with $j \leq N$, and W_i^j's are subspaces on which the restriction of the family $\mathcal{A}$ is a representation of an i-dimensional Clifford algebra with $k_i \leq \binom{N}{i}$.

Proof. Let $\mathcal{B} = \{A_1^2, \ldots, A_N^2\}$ be the family of squared operators. Since $\mathcal{A}$ is an anti-commuting family, $\mathcal{B}$ is a commuting diagonalizable family hence it is simultaneously diagonalizable. Let $\{X_1, \ldots, X_n\}$ be a basis with respect to which $\mathcal{B}$ is diagonal. We can group the eigenvectors in such a way that on the subspace spanned by each group the A_i^2's are constant. It follows that on each of these subspaces the A_i's belong to a representation of a Clifford algebra. □

As an example consider a family of $N = 5$ anti-commuting diagonalizable operators A_i on an $n = 20$ dimensional vector space and let $\{X_j\}_{j=1}^{20}$ be a basis with respect to which the squared family is diagonal. The ranges of the operators A_i are given below.

$$\text{Range}(A_1) = \text{Span}\{X_5, X_6, X_8, X_{12}, X_{16}, X_{19}\},$$

$$\text{Range}(A_2) = \text{Span}\{X_3, X_7, X_9, X_{11}, X_{13}, X_{15}, X_{17}\},$$

$$\text{Range}(A_3) = \text{Span}\{X_2, X_4, X_5, X_6, X_8, X_{10}, X_{12}, X_{14}, X_{16}, X_{19}\},$$

$$\text{Range}(A_4) = \text{Span}\{X_1, X_3, X_5, X_7, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, X_{19}, X_{20}\},$$

$$\text{Range}(A_5) = \text{Span}\{X_2, X_4, X_5, X_6, X_8, X_{10}, X_{12}, X_{14}, X_{16}, X_{19}\},$$

We can see that the common kernel of the family is $U_0 = \text{Span}\{X_{18}\}$. There are two subspaces on which there is a single nonzero operator. These are $U_1 = \text{Span}\{X_1, X_{20}\}$ where A_4 is nonzero and $U_2 = \text{Span}\{X_9\}$ where A_2 is nonzero. Since all other operators are zero, $A_4(X_1) = \mu_1$, $A_4(X_{20}) = \mu_2$, where μ_1 and μ_2 are arbitrary. There are two subspaces on which there are two nonzero operators. The first one is $W_2^1 = \text{Span}\{X_2, X_4\}$ where A_3 and A_5 are nonzero. On this subspace we have a

representation of a two-dimensional Clifford algebra. Thus $A_3(X_2) = \lambda_1 X_2$, $A_3(X_4) = -\lambda_1 X_4$, and $A_5(X_2) = X_4$, $A_5(X_4) = dX_2$. The other subspace on which there is a representation of a two-dimensional Clifford algebra is $W_2^2 = \text{Span}\{X_3, X_7, X_{11}, X_{13}, X_{15}, X_{17}\}$ where A_2 and A_4 are nonzero. This is a six-dimensional subspace, the eigenvalues of A_2 and A_4 can be the same or different. Hence we may have a combination of reducible and irreducible representations. There are two subspaces with three nonzero operators. These are $W_3^1 = \text{Span}\{X_6, X_8\}$ on which A_1, A_3 and A_5 are nonzero and $W_3^1 = \text{Span}\{X_{10}, X_{14}\}$ on which A_1, A_3 and A_5 are nonzero. On these subspaces we have representations of three-dimensional Clifford algebras. Finally on $W_4^1 = \text{Span}\{X_5, X_{12}, X_{16}, X_{19}\}$ the operators A_1, A_3, A_4 and A_5 are nonzero. There is a representation of a four-dimensional Clifford algebra which can exist only on a four-dimensional subspace.

4. Square diagonalizable anti-commuting families of linear operators

The construction in the previous section suggests that we may only require the diagonalizability of the squared family in order to obtain canonical forms for an anti-commuting family. This will not be quite true, because there will be difficulties when the kernels of the operators in the original and the squared family are different. In this section we consider now an anti-commuting family $\mathcal{A}$ of operators whose squares are diagonalizable.

In the easiest case, the family $\mathcal{A}$ is a family of real operators that are diagonalizable over C but not diagonalizable over R. In this case A^2 is diagonalizable over R and $Ker(A) = Ker(A^2)$. The construction above works with the exception that the dimensions of the representation spaces are determined by the real representations of Clifford algebras [1]. We have thus the analog of Theorem 3.1, where the only difference is that the representations of the Clifford algebras are real.

Theorem 4.1. *Let $\mathcal{A}$ be a family of N real, anti-commuting operators on an n-dimensional vector space V. If the operators $A_a \in \mathcal{A}$ are diagonaliazable over C, then V has a direct sum decomposition*

$$V = U_0 \oplus U_1 \oplus \ldots U_j \oplus W_2^1 \oplus \cdots \oplus W_2^{k_2} \oplus \cdots \oplus W_N^1 \oplus \cdots \oplus W_N^{k_N},$$

where U_0 is the common kernel of the family, U_i's are subspaces of arbitrary dimensions on which only A_i is nonzero and it is nonsingular with $j \leq N$, and W_i^j's are subspaces on which the restriction of the family $\mathcal{A}$ is a real representation of an i-dimensional Clifford algebra with $k_i \leq \binom{N}{i}$.

If $Ker(A) \neq Ker(A^2)$, then in the minimal polynomial $m_A(t)$ of A, the only nonlinear factor is t^2. As in the previous section, we can diagonalize the squared family, arrange the eigenspaces so that V is a direct sum of subspaces on which A_a^2's are constant. But as opposed to the previous case, if A_a^2 is zero on some subspace, A_a need not be zero, hence we may have a family of anti-commuting matrices whose squares are positive, negative or zero. This is just the representation of some degenerate Clifford algebra [2] for which the construction of canonical forms is not known.

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