Convergence spaces and diagonal conditions

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Abstract

Certain diagonal axioms due to Kowalsky, Cook and Fischer are studied and compared.

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This paper is dedicated to the memory of our friend and colleague, Václav Koutník

Introduction

In 1954, Kowalsky [4] introduced a diagonal condition (that we call K) for convergence spaces such that any convergence space satisfying K has a pretopological modification which is topological. In 1967, Cook and Fischer [2] defined a stronger diagonal condition (that we call F) which, as we show herein, is necessary and sufficient for a convergence structure to be a topology. Furthermore a dual version of F (which we call DF) is necessary and sufficient for a convergence space to be regular, a fact established in [1] and [2]. The dual of Kowalsky's axiom, DK, defines a weaker form of regularity which, to our knowledge, has not been previously studied, and for which we obtain a relatively simple characterization.

All four of the diagonal axioms cited above involve in their definitions a filter selection function σ. If the values of σ are restricted to being ultrafilters, we obtain what appear to be weaker axioms K*, F*, DK*, and DF*. However, we show that F is equivalent to F*, DF is equivalent to DF*, and DK is equivalent to DK*. Only K and K* are distinct, as we show by an example.

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1. Preliminaries

Let $X$ be a set, $\mathcal{F}(X)$ the set of all (proper) filters on $X$, $\mathcal{U}(X)$ the set of all ultrafilters on $X$, and $2^X$ the set of all subsets of $X$. For $x \in X$, let $\hat{x}$ be the fixed ultrafilter generated by $\{x\}$. For $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$, we write $\mathcal{F} \leq \mathcal{G}$ iff $\mathcal{F} \subseteq \mathcal{G}$.

**Definition 1.1.** A convergence structure $q$ on a set $X$ is a function $q : \mathcal{F}(X) \rightarrow 2^X$ satisfying:

1. $x \in q(\hat{x})$, for all $x \in X$;
2. $\mathcal{F} \leq \mathcal{G} \Rightarrow q(\mathcal{F}) \subseteq q(\mathcal{G})$;
3. $x \in q(\mathcal{F}) \Rightarrow x \in q(\mathcal{F} \cap \hat{x})$.

The statement $x \in q(\mathcal{F})$ means "$\mathcal{F}$ q-converges to $x$", which will usually be written "$\mathcal{F} \xrightarrow{q} x$". If $q$ is a convergence structure on $X$, then $(X, q)$ is a convergence space.

Let $\mathcal{C}(X)$ be the set of all convergence structures on $X$, partially ordered by: $p \leq q$ iff $q(\mathcal{F}) \subseteq p(\mathcal{F})$, for all $\mathcal{F} \in \mathcal{F}(X)$. Relative to this order, $\mathcal{C}(X)$ is a complete lattice whose largest member is the discrete topology on $X$ and whose least member is the indiscrete topology.

With each convergence space $(X, q)$, there is an associated closure operator $\text{cl}_q$ and an associated interior operator $\text{I}_q$; these are defined for each $A \in 2^X$ as follows:

$\text{cl}_q A = \{ x \in X : \exists \mathcal{F} \xrightarrow{q} x \text{ such that } A \in \mathcal{F} \}$,

$\text{I}_q A = \{ x \in A : \mathcal{F} \xrightarrow{q} x \Rightarrow A \in \mathcal{F} \}$.

If $\mathcal{F}$ is a filter on $X$, $\text{cl}_q \mathcal{F}$ denotes the filter generated by $\{\text{cl}_q F : F \in \mathcal{F}\}$. At each $x \in X$, let $\mathcal{V}_q(x) = \{ V \subseteq X : x \in \text{I}_q V \}$; $\mathcal{V}_q(x)$ is called the $q$-neighborhood filter at $x$. It can also be described as the intersection of all filters which $q$-converge to $x$.

We consider three additional convergence axioms:

1. $q(\mathcal{F} \cap \mathcal{G}) = q(\mathcal{F}) \cap q(\mathcal{G})$, for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$;
2. for each $\mathcal{F} \in \mathcal{F}(X), x \in q(\mathcal{F})$ iff $x \in q(\mathcal{G})$, for every ultrafilter $\mathcal{G} \geq \mathcal{F}$;
3. $x \in q(\mathcal{V}_q(x))$, for all $x \in X$.

A convergence structure which satisfies (C4) (respectively (C5), (C6)) is called a limit structure (respectively pseudo-topology, pretopology). Note that pretopology $\Rightarrow$ pseudo-topology $\Rightarrow$ limit structure $\Rightarrow$ convergence structure. A pretopology $q$ is a topology if each neighborhood filter $\mathcal{V}_q(x)$ has a filter base of sets which are $q$-open in the sense the set equals its own interior. It is well known that for any convergence structure $q$ on $X$, there is a finest pretopology $\pi q$ coarser than $q$; $\pi q$ is called the pretopological modification of $q$.

2. The diagonal axioms

Let $(X, q)$ be a convergence space, and let $J$ be any set. If $\mathcal{F} \in \mathcal{F}(J)$ and $\sigma : J \rightarrow \mathcal{F}(X)$ is any "selection function", we define $\kappa \sigma \mathcal{F}$ to be the filter $\bigcup_{\mathcal{F} \in \mathcal{F}} \bigcap_{x \in \mathcal{F}} \sigma(x)$ in $\mathcal{F}(X)$;
K is sometimes called the "compression operator" for \( \sigma \).

We next define four diagonal axioms.

**K**: Let \( \sigma : X \to F(X) \) be any function such that \( \sigma(y) \xrightarrow{q} y \), for all \( y \in X \). If \( F \xrightarrow{q} x \), then \( \kappa \sigma F \xrightarrow{q} x \).

**K***: Let \( \sigma : X \to U(X) \) be any function such that \( \sigma(y) \xrightarrow{q} y \), for all \( y \in X \). If \( F \xrightarrow{q} x \), then \( \kappa \sigma F \xrightarrow{q} x \).

**F**: Let \( J \) be any set, let \( \psi : J \to X \), and let \( \sigma : J \to F(X) \) have the property that \( \sigma(y) \xrightarrow{q} \psi(y) \), for all \( y \in J \). If \( F \in F(J) \) is such that \( \psi(F) \xrightarrow{q} x \), then \( \kappa \sigma F \xrightarrow{q} x \).

**F***: Let \( J \) be any set, let \( \psi : J \to X \), and let \( \sigma : J \to U(X) \) have the property that \( \sigma(y) \xrightarrow{q} \psi(y) \), for all \( y \in J \). If \( F \in F(J) \) is such that \( \psi(F) \xrightarrow{q} x \), then \( \kappa \sigma F \xrightarrow{q} x \).

The axioms \( K \) and \( F \) are those cited in the Introduction; \( K^* \) and \( F^* \) are slightly weaker versions of \( K \) and \( F \), respectively, for which the selection function \( \sigma \) is restricted to selecting ultrafilters. Note also that \( K \) is a special case of \( F \), where \( J = X \) and \( \psi \) is the identity map on \( X \); likewise, \( K^* \) is a special case of \( F^* \). These observations are summarized in the next proposition.

**Proposition 2.1.** For any convergence space \((X, q)\), \( K \Rightarrow K^* \), \( F \Rightarrow F^* \Rightarrow K^* \), and \( F \Rightarrow K \).

In [7], Kowalsky showed that if a convergence space \((X, q)\) satisfies \( K \), then \( \pi q \) is a topology. The next proposition slightly improves this result.

**Proposition 2.2.** If a convergence space \((X, q)\) satisfies \( K^* \), then \( \pi q \) is a topology.

**Proof.** It suffices to show that a convergence structure satisfying \( K^* \) has the property \( \text{cl}_q A \subseteq \text{cl}_q A \) for arbitrary \( A \in 2^X \). Let \( F \) be an ultrafilter on \( X \) containing \( \text{cl}_q A \) such that \( F \xrightarrow{q} x \). For each \( y \in \text{cl}_q A \), choose an ultrafilter \( H_y \xrightarrow{q} y \) such that \( A \in H_y \). We define \( \sigma : X \to U(X) \) as follows:

\[
\sigma(y) = \begin{cases} y, & y \notin \text{cl}_q A, \\ H_y, & y \in \text{cl}_q A. \end{cases}
\]

Then \( \kappa \sigma F \xrightarrow{q} x \), and since \( \text{cl}_q A \in F \) and \( A \in H_y \) for all \( y \in \text{cl}_q A \), \( A \in \kappa \sigma F \). Thus \( x \in \text{cl}_q A \). \( \square \)

Fischer showed (in unpublished notes) that a pseudo-topology satisfying \( F \) is a topology. The next proposition extends this result.

**Proposition 2.3.** If \((X, q)\) is a convergence space satisfying \( F^* \), then \( q \) is a topology.

**Proof.** Let \( x \in X \), and let \( \{H_\alpha : \alpha \in J\} \) be the set of all ultrafilters \( q \)-converging to \( x \). Define \( \psi : J \to X \) by \( \psi(\alpha) = x \), for all \( \alpha \in J \), and let \( \sigma(\alpha) = H_\alpha \), for all \( \alpha \in J \). Let \( F \) be the filter \( \{J\} \). Since \( \psi(F) = x \xrightarrow{q} x \), \( \kappa \sigma F \xrightarrow{q} x \), by \( F^* \). However

\[
\bigcap \{\sigma(y) : y \in J\} = \bigcap \{H_\alpha : \alpha \in J\} = \nu_q(x) \xrightarrow{q} x.
\]
Thus \( q \) is a pretopology. By Propositions 2.1 and 2.2, \( q \) is also a topology. \( \square \)

It is well known that a topological space satisfies Condition \( F \). Thus we have the following corollaries.

**Corollary 2.4.** For a convergence space \( (X, q) \), the following are equivalent. (1) \( q \) is a topology; (2) \( q \) satisfies \( F \); (3) \( q \) satisfies \( F^* \).

**Corollary 2.5.** For a pretopological space \( (X, q) \), the following are equivalent. (1) \( q \) is a topology; (2) \( q \) satisfies \( K \); (3) \( q \) satisfies \( K^* \); (4) \( q \) satisfies \( F \); (5) \( q \) satisfies \( F^* \).

**Proposition 2.6.** Let \( (X, q) \) be a convergence space.

(a) If \( (X, q) \) satisfies \( K \), then \( q \) is a limit structure.

(b) If \( (X, q) \) satisfies \( K^* \), then a finite intersection of ultrafilters \( q \)-converging to \( x \) must also \( q \)-converge to \( x \).

**Proof.** The proofs of (a) and (b) are essentially the same, so we prove only (a). Let \( F \) and \( G \) \( q \)-converge to \( x \) and assume \( K \). Define

\[
\sigma(y) = \begin{cases} 
  y, & y \neq x, \\
  G \cap \hat{x}, & y = x.
\end{cases}
\]

For \( F \in F \), \( \bigcap \{ \sigma(y): y \in F \cup \{x\} \} = F \cap G \cap \hat{x} \), where \( F \) denotes the filter of oversets of \( F \). Thus \( \kappa \sigma(F \cap \hat{x}) = F \cap G \cap \hat{x} \), which \( q \)-converges to \( x \) by \( K \). \( \square \)

The diagonal property \( F \) is obviously an initial property, since it is equivalent to the property of being topological. The next proposition gives a partial result in this direction for the properties \( K \) and \( K^* \).

**Proposition 2.7.** Let \( (X, q) \) be a convergence space, equipped with the initial convergence structure induced by a family \( \{(Y_\alpha, p_\alpha): \alpha \in A\} \) of spaces and \( \{f_\alpha: \alpha \in A\} \), where each \( f_\alpha: X \to Y_\alpha \) is injective. Then if each \( (Y_\alpha, p_\alpha) \) satisfies condition \( K \) (or \( K^* \)), the same is true of \( (X, q) \).

**Proof.** We prove the result only for \( K \); the proof for \( K^* \) is essentially the same. Let \( F \xrightarrow{q} x \) and let \( \sigma: X \to F(X) \) be any function such that \( \sigma(y) \xrightarrow{q} y \), for all \( y \in X \). We must verify that \( \kappa \sigma F \xrightarrow{q} x \). Let \( \alpha \in A \) be fixed, and let \( \sigma_\alpha: Y_\alpha \to F(Y_\alpha) \) be defined as follows: \( \sigma_\alpha(y) = \hat{y} \) if \( y \in Y_\alpha - f_\alpha(X) \), \( \sigma_\alpha(y) = f_\alpha(\sigma(f_\alpha^{-1}(y))) \) if \( y \in f_\alpha(X) \). One easily verifies that \( f_\alpha(\kappa \sigma F) \geq \kappa \sigma_\alpha f_\alpha(F) \). The latter filter \( p_\alpha \)-converges to \( f_\alpha(x) \) by Condition \( K \), and consequently \( f_\alpha(\kappa \sigma F) \xrightarrow{p_\alpha} f_\alpha(x) \). This holds for all \( \alpha \in A \), and so \( \kappa \sigma F \xrightarrow{q} x \). \( \square \)

We conclude this section with two examples. The first is a limit space which satisfies \( K \) but fails to be pretopological, showing that \( K \) does not imply \( F \). Furthermore, we define
a set $X$ and a surjective function $f : X \to Y$ such that there is no coarsest convergence structure $q$ on $X$ satisfying $K$ such that $f : (X, q) \to (Y, p)$ is continuous. This shows that the assumption of Proposition 2.7 that the $f_{\alpha}$'s be injective cannot be dismissed. In other words, unlike $F$, $K$ is not an initial property.

**Example 2.8.** Let $Y$ be an infinite set, and choose $a \in Y$. Let $\{F_n: n \in \mathbb{N}\}$ be a set of distinct, free ultrafilters on $Y$, and let $G_n = F_n \cap \mathfrak{a}$, for all $n \in \mathbb{N}$. We define $p$ to be the finest limit structure on $Y$ such that each $G_n$ $p$-converges to $a$; thus $p$ is not pretopological since $G = \bigcap\{G_n: n \in \mathbb{N}\}$ does not $p$-converge.

To check that $(Y, p)$ satisfies $K$, assume $\sigma : Y \to F(Y)$ is such that $\sigma(y) \xrightarrow{p} y$, for all $y \in Y$, and let $\mathcal{H} \xrightarrow{p} x$. If $x \neq a$, then $\mathcal{H} = \mathfrak{x}$ and $\kappa \sigma \mathcal{H} = \sigma(x) = \mathfrak{x}$. If $x = a$, then $\mathcal{H} \supseteq \bigcap\{G_n: i = 1, \ldots, k\}$, and one easily checks that $\kappa \sigma \mathcal{H} \supseteq \mathcal{H} \cap \sigma(a)$, which $p$-converges to $a$. However, since $p$ is not pretopological, it follows by Proposition 2.3 that $(Y, p)$ does not satisfy $F$.

Next, let $X_n = Y \times \{n\}$, and let $X = \bigcup\{X_n: n \in \mathbb{N}\}$. Let $f : X \to Y$ be defined by $f(y, n) = y$, for all $(y, n) \in X$. Let $x_n = (a, n)$, for all $n \in \mathbb{N}$, and let $q_n$ be the finest limit structure on $X$ such that $f^{-1}(G_k) \xrightarrow{q_n} x_n$, for all $k \in \mathbb{N}$. The argument of the preceding paragraph shows that $q_n$ satisfies $K$, for all $n \in \mathbb{N}$. Also note that $f : (X, q_n) \to (Y, p)$ is continuous, for all $n \in \mathbb{N}$.

Finally, suppose there is a coarsest convergence structure $q$ on $X$ satisfying $K$ such that $f : (X, q) \to (Y, p)$ is continuous. Obviously, $q \leq q_n$ for all $n$, and so $f^{-1}(G_n) \xrightarrow{q} x_n$, for all $n \in \mathbb{N}$. To see that $q$ does not satisfy $K$, define $\sigma : X \to F(X)$ as follows:

$$
\sigma(z) = \begin{cases} 
  f^{-1}(G_n), & \text{if } z = x_n \text{ for some } n \in \mathbb{N}, \\
  \mathfrak{x}, & \text{otherwise}.
\end{cases}
$$

Let $F = f^{-1}(G_1)$; then $F \xrightarrow{q} x_1$. Let $F = f^{-1}(G) \in F$, where $G \in G_1$. Since $x_n \in f^{-1}(G)$, for all $n \in \mathbb{N}$,

$$
K = \bigcap\{f^{-1}(G_n): n \in \mathbb{N}\} = \bigcap\{\sigma(z): z \in F\},
$$

and therefore $K \supseteq \kappa \sigma F$. If $\kappa \sigma F \xrightarrow{q} x_1$, then $f(K) = G \xrightarrow{p} a$, a contradiction. Thus $\kappa \sigma F$ fails to $q$-converge to $x$, and therefore $q$ does not satisfy $K$.

The second example describes a convergence space which satisfies $K^*$ but not $K$, showing that (unlike $F$ and $F^*$) the axioms $K$ and $K^*$ are distinct.

**Example 2.9.** Let $X$ be any infinite set, and let $\mathcal{F}$ and $\mathcal{G}$ be two distinct, free filters on $X$ such that neither is a finite intersection of free ultrafilters. Fix $x_0 \in X$, and define $q$ to be the finest convergence structure on $X$ such that:

$\mathcal{H} \xrightarrow{q} x_0$ iff either there is a finite set of free ultrafilters $\mathcal{G}_1, \ldots, \mathcal{G}_n$, all finer than $\mathcal{G}$, such that $\mathcal{H} \supseteq \mathcal{F} \cap \mathcal{G}_i \cap \cdots \cap \mathcal{G}_n \cap \mathfrak{x}_0$, or else there is a finite set of free ultrafilters $\mathcal{F}_1, \ldots, \mathcal{F}_k$, all finer than $\mathcal{F}$, such that $\mathcal{H} \supseteq \mathcal{G} \cap \mathcal{F}_i \cap \cdots \cap \mathcal{F}_k \cap \mathfrak{x}_0$. 
Note that if \( \sigma : X \to U(X) \) is such that \( \sigma(x) \nrightarrow x \) for all \( x \), then
\[
\kappa \sigma(\mathcal{F} \cap \mathcal{G}_1 \cap \cdots \cap \mathcal{G}_n) \supseteq \mathcal{F} \cap \mathcal{G}_1 \cap \cdots \cap \mathcal{G}_n \cap \mathcal{K},
\]
where \( \mathcal{K} \) is some free ultrafilter finer than \( \mathcal{G} \); a similar observation applies to \( \kappa \sigma(\mathcal{G} \cap \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_k) \). Thus \( \mathcal{H} \nrightarrow x_0 \) implies \( \kappa \sigma \mathcal{H} \nrightarrow x_0 \), and it follows that \( (X, \mathcal{Q}) \) satisfies \( \mathbf{K}^* \). But \( (X, \mathcal{Q}) \) is not a limit space, so \( (X, \mathcal{Q}) \) fails to satisfy \( \mathbf{K} \), by Proposition 2.6.

Finally, we remark that none of the diagonal properties are preserved under final structures, since every convergence space is the image of a topological space under a convergence quotient map.

3. The dual axioms

Corresponding to the axioms \( \mathbf{K} \) and \( \mathbf{F} \) for a convergence space \( (X, \mathcal{Q}) \) are the following dual axioms.

**DK**: Let \( \sigma : X \to \mathcal{F}(X) \) be any function such that \( \sigma(y) \nrightarrow y \), for all \( y \in X \). If \( \kappa \sigma \mathcal{F} \nrightarrow x \), then \( \mathcal{F} \nrightarrow x \).

**DF**: Let \( J \) be any set, let \( \psi : J \to X \), and let \( \sigma : J \to \mathcal{F}(X) \) have the property that \( \sigma(y) \nrightarrow \psi(y) \), for all \( y \in J \). If \( \mathcal{F} \in \mathcal{F}(J) \) is such that \( \kappa \sigma \mathcal{F} \nrightarrow x \), then \( \psi(\mathcal{F}) \nrightarrow x \).

If \( \sigma \) is restricted to range in \( U(X) \) in each of the above axioms, we obtain the axioms \( \mathbf{DK}^* \) and \( \mathbf{DF}^* \), respectively.

A convergence space \( (X, \mathcal{Q}) \) is regular if \( \mathcal{Cl}_\mathcal{Q} \mathcal{F} \nrightarrow x \) whenever \( \mathcal{F} \nrightarrow x \). If \( \mathcal{Q} \) and \( \mathcal{P} \) are convergence structures on the same set \( X \), we say that \( (X, \mathcal{Q}) \) is \( \mathcal{P} \)-regular if \( \mathcal{Cl}_\mathcal{P} \mathcal{F} \nrightarrow x \) whenever \( \mathcal{F} \nrightarrow x \). This notion of \( \mathcal{P} \)-regularity was introduced by the authors in [3].

In [2], Cook and Fischer showed that \( \mathbf{DF} \) implies regularity, and in [1], Biesterfeldt showed that regularity implies \( \mathbf{DF} \). Furthermore, the proofs used to establish the equivalence of regularity and the condition \( \mathbf{DF} \) can be adapted to prove that regularity is equivalent to \( \mathbf{DF}^* \). Thus we obtain the following result.

**Theorem 3.1.** For a convergence space \( (X, \mathcal{Q}) \), the following are equivalent.

1. \( (X, \mathcal{Q}) \) is regular,
2. \( (X, \mathcal{Q}) \) satisfies \( \mathbf{DF} \),
3. \( (X, \mathcal{Q}) \) satisfies \( \mathbf{DF}^* \).

The conditions \( \mathbf{DK} \) and \( \mathbf{DK}^* \) are obviously weaker than \( \mathbf{DF} \), and consequently they define weaker versions of regularity, which we will call \( \mathbf{K} \)-regularity and \( \mathbf{K}^* \)-regularity, respectively. For the purpose of studying these new concepts, it will be convenient to introduce some new notation.

Given a convergence space \( (X, \mathcal{Q}) \), let \( \Sigma \) denote the set of all selection functions \( \sigma : X \to \mathcal{F}(X) \) such that \( \sigma(y) \nrightarrow y \), for every \( y \in F \), and let \( \Sigma^* \) be the subset consisting of all \( \sigma \in \Sigma \) such that \( \sigma(y) \in U(X) \), for all \( y \in X \). If \( A \subseteq X \) and \( \sigma \in \Sigma \), let \( A^\sigma = \{ y \in X : A \in \sigma(y) \} \); note that \( (A \cap B)^\sigma = A^\sigma \cap B^\sigma \). If \( \mathcal{F} \in \mathcal{F}(X) \) and \( F^\sigma \neq \emptyset \), for all \( F \in \mathcal{F} \), then \( \mathcal{F}^\sigma \) denotes the (proper) filter generated by \( \{ F^\sigma : F \in \mathcal{F} \} \); however,
may sometimes fail to be a proper filter. We omit the straightforward proof of the next lemma.

**Lemma 3.2.** Let \((X, \tau)\) be a convergence space, \(\mathcal{F} \in \mathcal{F}(X)\), and \(\sigma \in \Sigma\). Then:

1. \((\kappa \sigma \mathcal{F})^\sigma\) is a proper filter and \(\mathcal{F} \supseteq (\kappa \sigma \mathcal{F})^\sigma\).
2. If \(\mathcal{F}^\sigma\) is a proper filter, then \(\kappa \sigma (\mathcal{F})^\sigma \supseteq \mathcal{F}\).

**Theorem 3.3.** Let \((X, \tau)\) be a convergence space. Then \((X, \tau)\) is K-regular (respectively K*-regular) iff for each \(\sigma \in \Sigma\) (respectively \(\sigma \in \Sigma^*\)), \(\mathcal{F}^\sigma \nrightarrow x\) whenever \(\mathcal{F}^\sigma\) is a proper filter and \(\mathcal{F} \nrightarrow x\).

**Proof.** We give the proof only for K-regularity, the proof for K*-regularity being similar.

Assume the given condition, and let \(\sigma \in \Sigma\) and \(\kappa \sigma \mathcal{F} \nrightarrow x\). Then \((\kappa \sigma \mathcal{F})^\sigma \nrightarrow x\), and by Lemma 3.2, \(\mathcal{F} \supseteq (\kappa \sigma \mathcal{F})^\sigma\), which implies \(\mathcal{F} \nrightarrow x\), and so DK holds and \((X, \tau)\) is K-regular.

Conversely, suppose that \(\mathcal{F} \nrightarrow x\), \(\sigma \in \Sigma\), and \(\mathcal{F}^\sigma\) is proper filter. By Lemma 3.2, \(\kappa \sigma (\mathcal{F}^\sigma) \supseteq \mathcal{F}\), and hence \(\kappa \sigma (\mathcal{F}^\sigma) \nrightarrow x\). It follows by DK that \(\mathcal{F}^\sigma \nrightarrow x\), and so the given condition is satisfied. \(\Box\)

**Theorem 3.4.** For a convergence space \((X, \tau)\), the conditions DK and DK* are equivalent.

**Proof.** Let \((X, \tau)\) be K*-regular. Let \(\sigma \in \Sigma\) and define \(\sigma^*\) to be any member of \(\Sigma^*\) such that \(\sigma(y) \subseteq \sigma^*(y)\), for all \(y \in X\). Assume that \(\mathcal{F} \nrightarrow x\), and that \(\mathcal{F}^\sigma\) is a proper filter. If \(F \in \mathcal{F}\), then \(F^\sigma \subseteq F^{\sigma^*}\); thus \(\mathcal{F}^{\sigma^*} \subseteq \mathcal{F}^\sigma\). By Theorem 3.3, \(\mathcal{F}^{\sigma^*} \nrightarrow x\), and therefore \(\mathcal{F}^\sigma \nrightarrow x\). Thus \((X, \tau)\) is K-regular. The converse is clear. \(\Box\)

We next consider the relationship between K-regularity and p-regularity. A pretopology \(p\) on a set \(X\) will be called an ultrapretopology if, for each \(y \in X\), there is \(\mathcal{H}_y \in \mathcal{U}(X)\) such that \(\mathcal{V}_p(y) = \mathcal{H}_y \cap \bar{y}\).

**Proposition 3.5.** Let \((X, \tau)\) be a convergence space which is p-regular relative to every ultrapretopology \(p \supseteq \tau\). Then \((X, \tau)\) is K-regular.

**Proof.** By Theorem 3.4, it is sufficient to show that \((X, \tau)\) satisfies DK*. Let \(\sigma \in \Sigma^*\), and let \(p\) be the ultrapretopology defined by \(\mathcal{V}_p(y) = \sigma(y) \cap \bar{y}\), for all \(y \in X\). Let \(\mathcal{F} \in \mathcal{F}(X)\) be such that \(\kappa \sigma \mathcal{F} \nrightarrow x\). Given \(F \in \mathcal{F}\), choose \(A_y \in \sigma(y)\), for all \(y \in F\), so that \(A = \bigcup_{y \in F} A_y\) is a basic set in \(\kappa \sigma \mathcal{F}\). Note that \(F \subseteq \text{cl}_p A\), and thus \(\text{cl}_p(\kappa \sigma \mathcal{F}) \subseteq \mathcal{F}\). By p-regularity, \(\mathcal{F} \nrightarrow x\), and therefore DK* holds. \(\Box\)

**Theorem 3.6.** A topological space \((X, \tau)\) is K-regular iff it is p-regular for every ultrapretopology \(p \supseteq \tau\).

**Proof.** The proof in one direction follows by Proposition 3.5. For the converse argument, it suffices to show that if \((X, \tau)\) is K*-regular, then \((X, \tau)\) is p-regular for an arbitrary
ultrapretopology $p \geq q$. Let $x \in X$, and assume $\mathcal{V}_p(y) = \mathcal{H}_y \cap \check{y}$, where $\mathcal{H}_y \in U(X)$, and $\mathcal{H}_y \not\rightarrow y$, for all $y \in X$. Let $\mathcal{G} = \text{cl}_p \mathcal{V}_q(x)$. Let $\sigma(y) = \mathcal{H}_y$, for all $y \in X$. Using the fact that $\mathcal{V}_q(x)$ has a base of $q$-open sets, one easily verifies that $\kappa \sigma \mathcal{G} \geq \mathcal{V}_q(x)$, implying that $\kappa \sigma \mathcal{G} \not\rightarrow x$. Thus, by $DK^*$, $\text{cl}_p \mathcal{V}_q(x) \not\rightarrow x$, and therefore $\text{cl}_p \mathcal{V}_q(x) = \mathcal{V}_q(x)$. Since this holds for arbitrary $x \in X$, $p$-regularity is established. \( \square \)

It is easy to verify that $K$-regularity is an initial property relative to any family of injective maps; the proof is similar to that of Proposition 2.7. We conclude with a simple example to show that regularity and $K$-regularity are distinct notions.

**Example 3.7.** Let $X$ be an infinite set, $\mathcal{H}$ a free ultrafilter on $X$, and $a, b \in X$. Define the convergence structure $q$ on $X$ as follows:

- $\mathcal{F} \not\rightarrow a$ iff $\mathcal{F} \not\supseteq \mathcal{H} \cap \check{a}$,
- $\mathcal{F} \not\rightarrow b$ iff $\mathcal{F} \not\supseteq \mathcal{G} \cap \check{b}$, where $\mathcal{G}$ is any free ultrafilter on $X$ distinct from $\mathcal{H}$,
- $\mathcal{F} \not\rightarrow x$, for $x \notin \{a, b\}$, iff $\mathcal{F} = \check{x}$.

Note that $(X, q)$ is not regular, since $\mathcal{H} \not\rightarrow a$, $\check{b} \not\geq \text{cl}_q \mathcal{H}$, and $\check{b}$ does not $q$-converge to $a$. However it is clear that $(X, q)$ is $p$-regular for every ultrapretopology $p \geq q$, and consequently $(X, q)$ is $K$-regular by Proposition 3.5.

**References**