J. Math. Anal. Appl. 363 (2010) 468–480

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global attractor for a nonlinear plate equation with supported boundary conditions ☆

Danxia Wang, Jianwen Zhang *

Department of Mathematics, Taiyuan University of Technology, Taiyuan, Shanxi 030024, PR China

ARTICLE INFO

Article history:
Received 25 February 2009
Available online 15 September 2009
Submitted by P.J. McKenna

Keywords:
Viscoelastic
Thin rectangular plate
Global solution
Bounded absorbing set
Global attractor

ABSTRACT

In this paper, we consider a two-dimensional nonlinear equation

$$\rho w_{tt} + D\Delta^2 w + \varepsilon\mu w_t - \left(N_1 + \frac{T}{2}\int_\Omega w_x^2\,dx\,dy\right) w_{xx} - \left(N_2 + \frac{T}{2}\int_\Omega w_y^2\,dx\,dy\right) w_{yy} = 0 \qquad (*)$$

which arises from the model of the viscoelastic thin rectangular plate with four edges supported. By virtue of Galerkin method combined with the priori estimates, we prove the existence and uniqueness of the global solution under initial–boundary data for the above equation. Especially the existence of the bounded absorbing set in space E and the existence of the global attractor of system is also obtained.

1. Introduction

In this paper, we consider the nonlinear viscoelastic thin rectangular plate equation $(*)$. For the sake of simplicity, dividing $(*)$ by ρ, we get the equation

$$w_{tt} + \frac{D}{\rho}\Delta^2 w + \frac{\varepsilon\mu}{\rho} w_t - \left(\frac{N_1}{\rho} + \frac{T}{2\rho}\int_\Omega w_x^2\,dx\,dy\right) w_{xx} - \left(\frac{N_2}{\rho} + \frac{T}{2\rho}\int_\Omega w_y^2\,dx\,dy\right) w_{yy} = 0 \qquad (1.1)$$

subject to the boundary conditions

$$w(x,0,t) = w(x,1,t) = w(0,y,t) = w(1,y,t) = 0, \qquad (1.2)$$

$$w_{xx}(0,y,t) = w_{xx}(1,y,t) = w_{yy}(x,0,t) = w_{yy}(x,1,t) = 0 \qquad (1.3)$$

and the initial conditions

$$w(x,y,0) = w^0 \quad \text{and} \quad w_t(x,y,0) = w^1. \qquad (1.4)$$

☆ The project is supported by the National Natural Science Foundation of China (Grant No. 10772131).

* Corresponding author.
E-mail address: jianwen.z2008@163.com (J. Zhang).

0022-247X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2009.09.020

Here $\Omega = (0,1) \times (0,1)$ is a boundary domain of R^2. $w(x, y, t)$ is the vertical displacement of the plate. The coefficients $N_1, N_2, D, \rho, T, \varepsilon$ and μ are all positive constants, where the coefficient N_1 is normal load per unit length in x-direction, N_2 is normal load per unit length in y-direction, $D = \frac{Eh^3}{12(1-\nu^2)}$ is the plate rigidity in which E, h and ν are modulus of elasticity, plate thickness and poisson's ration, respectively, ρ is the material density, T is normal load per unit area, ε is a small parameter, and μ is the damping coefficient of plate. The sign Δ denotes $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

In recent more than 30 years, there have been a lot of works on the stability and the existence of the attractor for nonlinear elastic infinite-dimensional dynamic system.

In 1950, Woinowsky-Krieger [21] proposed the equation

$$u_{tt} + u_{xxxx} - \left(\alpha + \beta \int_0^l \left|u_x(s,t)\right|^2 ds\right) u_{xx} = 0. \tag{1.5}$$

One of the first stability analysis for Eq. (1.5) is done by Ball [2]. Later Eq. (1.5) is extended to an abstract setting by Mederios [14]. Putting $\alpha = \beta = 0$ in (1.5) gives the equation

$$u_{tt} + u_{xxxx} = 0.$$

Feireisl [8] and Feckan [5] study the existence of time-periodic solution for the above equation under the boundary conditions

$$u_{xx}(0,l) = u_{xx}(l,t) = 0,$$
$$u_{xxx}(0,t) = -f\big(u(0,t)\big),$$
$$u_{xxx}(l,t) = f\big(u(l,t)\big).$$

A rather general equation

$$u_{tt} + u_{xxxx} + \mu u_{xxxxt} - \left(\alpha + \beta \int_0^l u_x^2\, dx + 2\delta \int_0^1 u_{xt} u_x\, dx\right) u_{xx} = 0 \tag{1.6}$$

is set up by Ball [3], who presents the existence and uniqueness of the solution under initial data. A larger class of stability of beam is in papers [9,18,19] and references therein. The existence of the absorbing set and the inertial manifolds for Eq. (1.6) under the initial–boundary data is obtained by Zhang [24]. You [23] and Fasangova [4] propose the nonlinear damp beam equation

$$u_{tt} - k\Delta^2 u - \left(a + \int_\Omega |\nabla u|^2\right) \Delta u - \delta u(t) = 0, \tag{1.7}$$

and prove the existence of the finite-dimensional global attractor. For the nonautonomous viscoelastic beam equation

$$u_{tt} - k\Delta^2 u - \left(a(t) + \int_\Omega |\nabla u|^2\right) \Delta u - \delta u(t) = f(t), \tag{1.8}$$

Feireisl [6,7] studies the finite-dimensional behavior. As $k = 0$ in (1.7)–(1.8), we get the string equation and refer to the works [1,11,13,15,16,20].

In the following, we mention some papers on the infinite-dimensional dynamic system determined by plate.

Lu Yang [12] studies the plate equation

$$u_{tt} + a(x)g(u_t) + \Delta^2 u + \lambda u + f(u) = h(x), \quad x \in \Omega \tag{1.9}$$

where $\Omega \subset R^n$ is a bounded domain and proves the existence of a global attractor in the space $H_0^2(\Omega) \times L^2(\Omega)$.

Haibin Xiao [22] considers the long-time behavior of the plate equation

$$\varepsilon u_{tt} + \Delta^2 u + \lambda u_t + \beta(x)u = f(x,u), \quad x \in \Omega = R^n,\ t \geqslant 0$$

on the unbounded domain R^n. Moreover he shows that there exists a compact global attractor for the above equation under certain initial–boundary data.

In 2008, Hao [10] discusses the nonlinear thermoelastic plate equations

$$\begin{cases} u_{tt} - \Delta u_t + \Delta(\Delta u + \theta) + f(u) = 0, \\ \theta_t - \Delta u_t + \displaystyle\int_0^\infty k(s)\big[-\Delta\theta(t-s)\big]\, ds = 0, \quad x \in \Omega \end{cases} \tag{1.10}$$

where $\Omega \subset R^2$ is a bounded domain, and proves the existence and uniqueness of a global solution as well as the existence of a global attractor.

In this paper, our objective is to prove the existence and uniqueness of the global solution and the existence of the global attractor for the system (1.1)–(1.4). The outline of this paper is as follows: the definitions and assumption are presented in Section 2. In Section 3 we give the existence and uniqueness of the global solution. In Section 4 we prove the existence of the bounded absorbing set for the dynamic system of the Cauchy problem (1.1)–(1.4). Finally in Section 5 we prove the existence of the global attractor of the system (1.1)–(1.4).

2. Definitions and assumption

In this paper, we use standard notation $\|\cdot\|$ in $L^2(\Omega)$. And, in the standard L^2 space, the scalar product and norm are denoted by

$$(w, v) = \int_\Omega wv\,dx\,dy, \qquad \|w\|^2 = \int_\Omega |w|^2\,dx\,dy.$$

Sometimes, the function $w = w(x, y, t)$ will simply be denoted by $w(t)$ when the x, y-variable is not in consideration. Our analysis is based on the Sobolev spaces

$$\begin{aligned}
&V = \{w \in H_0^1(\Omega) \cap H^2(\Omega)\},\\
&H = L^2(\Omega),\\
&V_1 = \{w \in H_0^1(\Omega) \cap H^4(\Omega) \mid w_{xx}(0, y) = w_{xx}(1, y) = w_{yy}(x, 0) = w_{yy}(x, 1) = 0\},\\
&E = V \times H,\\
&E_1 = V_1 \times V.
\end{aligned}$$

3. The existence and uniqueness of the global solution

In this section, using Galerkin method we may easily prove the existence and uniqueness of the global weak solution and strong solution for the system (1.1)–(1.4). The main results are as follows.

Theorem 1. *Let the initial data $\{w^0, w^1\}$ belongs to $E = V \times H$, then there exists a function $w(t)$ with*

$$w(t) \in L^\infty(0, T; V) \quad \text{and} \quad w_t(t) \in L^\infty(0, T; H)$$

such that $w(t)$ satisfies the initial conditions (1.4) and Eq. (1.1) in the sense that

$$\begin{aligned}
&(w_{tt}, \varphi) + \frac{D}{\rho}(w_{xx}, \varphi_{xx}) + \frac{2D}{\rho}(w_{xy}, \varphi_{xy})\\
&\quad + \frac{D}{\rho}(w_{yy}, \varphi_{yy}) + \frac{\varepsilon\mu}{\rho}(w_t, \varphi) - \left(\frac{N_1}{\rho} + \frac{T}{2\rho}\int_\Omega w_x^2\,dx\,dy\right)(w_{xx}, \varphi)\\
&\quad - \left(\frac{N_2}{\rho} + \frac{T}{2\rho}\int_\Omega w_y^2\,dx\,dy\right)(w_{yy}, \varphi) = 0, \quad \text{for all } \varphi \in V.
\end{aligned} \tag{3.1}$$

In the following we give the proof of the existence of the weak solution.

Proof. This is done with the Galerkin approximations. Let $\{\omega^j(x, y)\}$ be a Galerkin basis of V, and let V_m be the subspace generated by the first m vectors $\omega^1, \dots, \omega^m$. We search for a function

$$w^m(x, y, t) = \sum_{i=1}^m g_{im}(t)\omega^i(x, y) \tag{3.2}$$

satisfying the approximating equation

$$\begin{aligned}
&(w_{tt}^m, \omega^j) + \frac{D}{\rho}(w_{xx}^m, \omega_{xx}^j) + \frac{2D}{\rho}(w_{xy}^m, \omega_{xy}^j)\\
&\quad + \frac{D}{\rho}(w_{yy}^m, \omega_{yy}^j) + \frac{\varepsilon\mu}{\rho}(w_t^m, \omega^j) - \left(\frac{N_1}{\rho} + \frac{T}{2\rho}\int_\Omega (w_x^m)^2\,dx\,dy\right)(w_{xx}^m, \omega^j)
\end{aligned}$$

$$-\left(\frac{N_2}{\rho}+\frac{T}{2\rho}\int_\Omega\left(w_y^m\right)^2dx\,dy\right)\left(w_{yy}^m,\omega^j\right)=0\quad(1\leqslant j\leqslant m)\tag{3.3}$$

with the initial conditions

$$w^m(0)=w^{0m}\to w^0\quad\text{in }V,\qquad w_t^m(0)=w^{1m}\to w^1\quad\text{in }H.\tag{3.4}$$

By standard methods in differential equations, we prove the existence of solution to the approximating problem (3.3)–(3.4) on some interval $[0,t_m)$. After the estimates below the approximating solution $w^m(t)$ will be extended to the interval $[0,T]$, for any given $T>0$.

The following estimates show among other things that $t_m=T$. Multiply (3.3) by $\frac{dg_{jm}(t)}{dt}$ and sum for $j=1,\dots,m$. After using Young inequality, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\bigg\{&\|w_t^m\|^2+\frac{D}{\rho}\|w_{xx}^m\|^2+\frac{2D}{\rho}\|w_{xy}^m\|^2+\frac{D}{\rho}\|w_{yy}^m\|^2\\&+\frac{N_1}{\rho}\|w_x^m\|^2+\frac{T}{4\rho}\|w_x^m\|^4+\frac{N_2}{\rho}\|w_y^m\|^2+\frac{T}{4\rho}\|w_y^m\|^4\bigg\}\leqslant 0.\end{aligned}\tag{3.5}$$

Integrating (3.5) from 0 to t $(<t_m)$ yields the inequality

$$\begin{aligned}&\|w_t^m\|^2+\frac{D}{\rho}\|w_{xx}^m\|^2+\frac{2D}{\rho}\|w_{xy}^m\|^2+\frac{D}{\rho}\|w_{yy}^m\|^2\\&\quad+\frac{N_1}{\rho}\|w_x^m\|^2+\frac{T}{4\rho}\|w_x^m\|^4+\frac{N_2}{\rho}\|w_y^m\|^2+\frac{T}{4\rho}\|w_y^m\|^4\\&\leqslant\|w^{1m}\|^2+\frac{D}{\rho}\|w_{xx}^{0m}\|^2+\frac{2D}{\rho}\|w_{xy}^{0m}\|^2+\frac{D}{\rho}\|w_{yy}^{0m}\|^2\\&\quad+\frac{N_1}{\rho}\|w_x^{0m}\|^2+\frac{T}{4\rho}\|w_x^{0m}\|^4+\frac{N_2}{\rho}\|w_y^{0m}\|^2+\frac{T}{4\rho}\|w_y^{0m}\|^4.\end{aligned}\tag{3.6}$$

Noticing the initial conditions

$$w^m(0)=w^{0m}\to w^0\quad\text{in }V,\qquad w_t^m(0)=w^{1m}\to w^1\quad\text{in }H,$$

we think that there exists a constant $M_1>0$ independent of m and t such that

$$\begin{aligned}&\|w_t^m\|^2+\frac{D}{\rho}\|w_{xx}^m\|^2+\frac{2D}{\rho}\|w_{xy}^m\|^2+\frac{D}{\rho}\|w_{yy}^m\|^2+\frac{N_1}{\rho}\|w_x^m\|^2\\&\quad+\frac{T}{4\rho}\|w_x^m\|^4+\frac{N_2}{\rho}\|w_y^m\|^2+\frac{T}{4\rho}\|w_y^m\|^4\leqslant M_1\end{aligned}\tag{3.7}$$

for all $t\in[0,T]$ and for all $m\in N$. Then the approximating solution $w^m(t)$ can be extended to the whole interval $[0,T]$.

The estimates just derived, together with the Poincaré lemma, show that

$$\begin{aligned}&\{w^m\}\quad\text{is bounded in }L^\infty(0,T;V),\\&\{w_t^m\}\quad\text{is bounded in }L^\infty(0,T;H)\end{aligned}$$

and

$$\begin{aligned}&\{\|w_x^m\|^2w_{xx}^m\}\quad\text{is bounded in }L^\infty(0,T;H),\\&\{\|w_y^m\|^2w_{yy}^m\}\quad\text{is bounded in }L^\infty(0,T;H).\end{aligned}$$

In particular, $\{w^m\}$ is bounded in $H^1(Q)$, where $Q=\Omega\times[0,T]$. Thus we may extract a subsequence $\{w^\mu\}$ of $\{w^m\}$ with the properties

$$\begin{aligned}&w^\mu\to w\quad\text{in }L^\infty(0,T;V)\text{ weak}^*,\\&w_t^\mu\to w_t\quad\text{in }L^\infty(0,T;H)\text{ weak}^*,\\&w^\mu\to w\quad\text{in }L^2(Q)\text{ strongly and a.e.}\end{aligned}$$

and

$$\begin{aligned}&\|w_x^\mu\|^2w_{xx}^\mu\to\|w_x\|^2w_{xx}\quad\text{in }L^\infty(0,T;H)\text{ weak}^*,\\&\|w_y^\mu\|^2w_{yy}^\mu\to\|w_y\|^2w_{yy}\quad\text{in }L^\infty(0,T;H)\text{ weak}^*.\end{aligned}$$

Then these convergence properties establish the theorem. □

Theorem 2. *The solution $w(x, y, t)$ of Theorem 1 is unique.*

Proof. Let w, v be two solutions of (3.1) with the same initial data. Then writing $p = w - v$, we can obtain $p^0 = p_x^0 = p_y^0 = p_{xx}^0 = p_{yy}^0 = p_{xy}^0 = p^1 = 0$. From (3.1) we have

$$\begin{aligned}
&(p_{tt}, \varphi) + \frac{D}{\rho}(p_{xx}, \varphi_{xx}) + \frac{2D}{\rho}(p_{xy}, \varphi_{xy}) \\
&\quad + \frac{D}{\rho}(p_{yy}, \varphi_{yy}) + \frac{\varepsilon\mu}{\rho}(p_t, \varphi) - \left(\frac{N_1}{\rho}p_{xx} + \frac{N_2}{\rho}p_{yy}, \varphi\right) \\
&= \frac{T}{2\rho}\left(\left(\int_\Omega w_x^2\,dx\,dy\right)w_{xx} - \left(\int_\Omega v_x^2\,dx\,dy\right)v_{xx}, \varphi\right) \\
&\quad + \frac{T}{2\rho}\left(\left(\int_\Omega w_y^2\,dx\,dy\right)w_{yy} - \left(\int_\Omega v_y^2\,dx\,dy\right)v_{yy}, \varphi\right).
\end{aligned} \tag{3.8}$$

Putting $\varphi = p_t$ into (3.8), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|p_t\|^2 + \frac{D}{\rho}\|p_{xx}\|^2 + \frac{2D}{\rho}\|p_{xy}\|^2 + \frac{D}{\rho}\|p_{yy}\|^2 + \frac{N_1}{\rho}\|p_x\|^2 + \frac{N_2}{\rho}\|p_y\|^2\right\} + \frac{\varepsilon\mu}{\rho}\|p_t\|^2 \\
&= \frac{T}{2\rho}\left(\|w_x\|^2 w_{xx} - \|v_x\|^2 v_{xx}, p_t\right) + \frac{T}{2\rho}\left(\|w_y\|^2 w_{yy} - \|v_y\|^2 v_{yy}, p_t\right).
\end{aligned} \tag{3.9}$$

Using Cauchy–Schwartz inequality to the first item of the right hand side in (3.9), we get

$$\begin{aligned}
&\frac{T}{2\rho}\left(\|w_x\|^2 w_{xx} - \|v_x\|^2 v_{xx}, p_t\right) \\
&= \frac{T}{2\rho}\left(\|w_x\|^2\int_\Omega p_{xx}p_t\,dx\,dy + \int_\Omega p_x(w_x + v_x)\,dx\,dy\int_\Omega v_{xx}p_t\,dx\,dy\right) \\
&\leqslant C\left(\|p_{xx}\|^2 + \|p_t\|^2 + \|p_x\|^2 + \|p_t\|^2\right)
\end{aligned}$$

where C is some positive constant in this paper. Similarly,

$$\frac{T}{2\rho}\left(\|w_y\|^2 w_{yy} - \|v_y\|^2 v_{yy}, p_t\right) \leqslant C\left(\|p_{yy}\|^2 + \|p_t\|^2 + \|p_y\|^2 + \|p_t\|^2\right).$$

From (3.9) it is easy checked that for some constant $C > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|p_t\|^2 + \frac{D}{\rho}\|p_{xx}\|^2 + \frac{2D}{\rho}\|p_{xy}\|^2 + \frac{D}{\rho}\|p_{yy}\|^2 + \frac{N_1}{\rho}\|p_x\|^2 + \frac{N_2}{\rho}\|p_y\|^2\right) \\
&\leqslant C\left(\|p_t\|^2 + \frac{D}{\rho}\|p_{xx}\|^2 + \frac{2D}{\rho}\|p_{xy}\|^2 + \frac{D}{\rho}\|p_{yy}\|^2 + \frac{N_1}{\rho}\|p_x\|^2 + \frac{N_2}{\rho}\|p_y\|^2\right).
\end{aligned} \tag{3.10}$$

Then an application of the Gronwall's lemma for (3.10) leads to

$$\begin{aligned}
&\|p_t\|^2 + \frac{D}{\rho}\|p_{xx}\|^2 + \frac{2D}{\rho}\|p_{xy}\|^2 + \frac{D}{\rho}\|p_{yy}\|^2 + \frac{N_1}{\rho}\|p_x\|^2 + \frac{N_2}{\rho}\|p_y\|^2 \\
&\leqslant C\left(\left\|p^1\right\|^2 + \frac{D}{\rho}\left\|p_{xx}^0\right\|^2 + \frac{2D}{\rho}\left\|p_{xy}^0\right\|^2 + \frac{D}{\rho}\left\|p_{yy}^0\right\|^2 + \frac{N_1}{\rho}\left\|p_x^0\right\|^2 + \frac{N_2}{\rho}\left\|p_y^0\right\|^2\right)\exp(CT).
\end{aligned} \tag{3.11}$$

And noticing $p^0 = p_x^0 = p_y^0 = p_{xx}^0 = p_{yy}^0 = p_{xy}^0 = p^1 = 0$, from (3.11) we see that $w = v$. □

Theorem 3. *Suppose $(w^0, w^1) \in E_1 = V_1 \times V$, then we conclude that there exists a unique strong solution $w(t)$ with*

$$w(t) \in L^\infty(0, T; V_1), \qquad w_t(t) \in L^\infty(0, T; V), \qquad w_{tt}(t) \in L^\infty(0, T; H)$$

such that $w(t)$ satisfies the initial condition (1.4) *and the equation*

$$w_{tt} + \frac{D}{\rho}\Delta^2 w + \frac{\varepsilon\mu}{\rho}w_t - \left(\frac{N_1}{\rho} + \frac{T}{2\rho}\int_\Omega w_x^2\,dx\,dy\right)w_{xx} - \left(\frac{N_2}{\rho} + \frac{T}{2\rho}\int_\Omega w_y^2\,dx\,dy\right)w_{yy} = 0 \quad \text{in } L^\infty(0, T; H).$$

Proof. The proof closely follows that of Theorem 1. We use the basic $\omega^j(x,y)$ of V_1. The approximating solution $w^m(t)$ is of the form

$$w^m(t)=\sum_{i=1}^m g_{im}(t)\omega^i$$

and satisfies the equation

$$w^m_{tt}+\frac{D}{\rho}\Delta^2 w^m+\frac{\varepsilon\mu}{\rho}w^m_t-\left(\frac{N_1}{\rho}+\frac{T}{2\rho}\int_\Omega \left(w^m_x\right)^2dx\,dy\right)w^m_{xx}-\left(\frac{N_2}{\rho}+\frac{T}{2\rho}\int_\Omega \left(w^m_y\right)^2dx\,dy\right)w^m_{yy}=0 \tag{3.12}$$

in $[0,t_m]$ subject to the initial conditions

$$w^m(0)=w^{0m}\to w^0\quad\text{in } V_1,\qquad w^m_t(0)=w^{1m}\to w^1\quad\text{in } V.$$

The basic estimates (3.7) hold as before, and show that $t_m=T$. Now let us obtain an estimate for $w^m_{tt}(0)$ in the L^2-norm. Taking the scalar product of (3.12) with $w^m_{tt}(0)$ and $t=0$ and integrating it by parts, we get

$$\begin{aligned}\left\|w^m_{tt}(0)\right\|^2=\Bigg|\Bigg(&-\frac{D}{\rho}\Delta^2 w^{0m}-\frac{\varepsilon\mu}{\rho}w^{1m}+\left(\frac{N_1}{\rho}+\frac{T}{2\rho}\int_\Omega \left(w^{0m}_x\right)^2dx\,dy\right)w^{0m}_{xx}\\&+\left(\frac{N_2}{\rho}+\frac{T}{2\rho}\int_\Omega \left(w^{0m}_y\right)^2dx\,dy\right)w^{0m}_{yy},\,w^m_{tt}(0)\Bigg)\Bigg|.\end{aligned}$$

Using Cauchy–Schwartz inequality and taking into account of the initial conditions, we get that there exists a positive constant $M_2>0$ such that

$$\left\|w^m_{tt}(0)\right\|\leqslant M_2\quad\forall m\in N. \tag{3.13}$$

Differentiating Eq. (3.12) with respect to the time t, and taking the scalar product with $w^m_{tt}(t)$, using Cauchy–Schwartz inequality and Gronwall inequality, and considering initial conditions and the estimates (3.13), we may find a constant $M_3>0$ depending only on T such that

$$\begin{aligned}&\left\|w^m_{tt}\right\|^2+\frac{D}{\rho}\left\|w^m_{xxt}\right\|^2+\frac{2D}{\rho}\left\|w^m_{xyt}\right\|^2+\frac{D}{\rho}\left\|w^m_{yyt}\right\|^2\\&\quad+\frac{N_1}{\rho}\left\|w^m_{xt}\right\|^2+\frac{N_2}{\rho}\left\|w^m_{yt}\right\|^2\leqslant M_3\quad\forall m\in N,\ \forall t\in[0,T].\end{aligned} \tag{3.14}$$

Taking the scalar product of (3.12) with $-(w^m_{xxt}+w^m_{yyt})$, taking into account of the estimates (3.7) and (3.14) and using some inequalities, we conclude that there exists a constant $M_4>0$ such that

$$\begin{aligned}&\left\|w^m_{xt}\right\|^2+\left\|w^m_{yt}\right\|^2+\frac{D}{\rho}\left\|w^m_{xxx}\right\|^2+\frac{3D}{\rho}\left\|w^m_{xxy}\right\|^2+\frac{3D}{\rho}\left\|w^m_{xyy}\right\|^2+\frac{D}{\rho}\left\|w^m_{yyy}\right\|^2\\&\quad+\frac{N_1}{\rho}\left\|w^m_{xx}\right\|^2+\frac{N_1}{\rho}\left\|w^m_{xy}\right\|^2+\frac{N_2}{\rho}\left\|w^m_{yy}\right\|^2+\frac{N_2}{\rho}\left\|w^m_{xy}\right\|^2\leqslant M_4.\end{aligned} \tag{3.15}$$

Also taking the scalar product of (3.12) with $w^m_{xxyyt}+w^m_{xxxxt}+w^m_{yyyyt}$ and with a class of reasoning we think that there exists a constant M_5 independent of m and t such that

$$\begin{aligned}&\left\|w^m_{xxt}\right\|^2+\left\|w^m_{xyt}\right\|^2+\left\|w^m_{yyt}\right\|^2+\frac{D}{\rho}\left\|w^m_{xxxx}\right\|^2+\frac{2D}{\rho}\left\|w^m_{xxyy}\right\|^2\\&\quad+\frac{D}{\rho}\left\|w^m_{yyyy}\right\|^2+\frac{D}{\rho}\left\|w^m_{xxxy}\right\|^2+\frac{2D}{\rho}\left\|w^m_{xxyy}\right\|^2+\frac{D}{\rho}\left\|w^m_{xyyy}\right\|^2+\frac{N_1}{\rho}\left\|w^m_{xxx}\right\|^2\\&\quad+\frac{N_1}{\rho}\left\|w^m_{xyy}\right\|^2+\frac{N_1}{\rho}\left\|w^m_{xxy}\right\|^2+\frac{N_2}{\rho}\left\|w^m_{yyy}\right\|^2+\frac{N_2}{\rho}\left\|w^m_{xxy}\right\|^2+\frac{N_2}{\rho}\left\|w^m_{xyy}\right\|^2\\&\leqslant M_5.\end{aligned} \tag{3.16}$$

Using the estimates (3.7), (3.14), (3.15) and (3.16) just derived, Poincaré inequality and the methods of Theorem 1, it is easy to show the existence of a subsequence $\{w^\mu\}$ of $\{w^m\}$ such that

$$w^{\mu} \to w \quad \text{in } L^{\infty}(0,T;V_1) \text{ weak}^*,$$
$$w_t^{\mu} \to w_t \quad \text{in } L^{\infty}(0,T;V) \text{ weak}^*,$$
$$w_{tt}^{\mu} \to w_{tt} \quad \text{in } L^{\infty}(0,T;H) \text{ weak}^*,$$
$$w^{\mu} \to w \quad \text{in } H^1(Q) \text{ strongly and a.e.},$$
$$\|w_x^{\mu}\|^2 w_{xx}^{\mu} \to \|w_x\|^2 w_{xx} \quad \text{in } L^{\infty}(0,T;V) \text{ weak}^*,$$
$$\|w_y^{\mu}\|^2 w_{yy}^{\mu} \to \|w_y\|^2 w_{yy} \quad \text{in } L^{\infty}(0,T;V) \text{ weak}^*.$$

These convergence properties establish the theorem. □

Remark 3.1. Theorems 1–3 are sufficient to allow us to define the mapping

$$S(t):(w^0,w^1) \mapsto (w,w_t) \quad \text{for all } t \in R^+,$$

where $w(t)$ is the unique generalized solution to problem (1.1)–(1.4) with initial data (w^0,w^1). It maps from $E = V \times H$ into itself and even $E_1 = V_1 \times V$ into itself. Moreover it enjoys the usual semigroup properties

$$S(t+s) = S(t)S(s) \quad \forall t, s \geqslant 0,$$
$$S(0) = I.$$

Hence it organizes a dynamic system. It is easily checked that the semigroup $S(t)$ in E and E_1 is continuous for all $t \geqslant 0$. To prove the attractor of the system, in the following we will prove the existence of an absorbing set.

4. The existence of the bounded absorbing set in space $E = V \times H$

Theorem 4. *Suppose $\varepsilon\mu \geqslant \frac{1}{4}$ and $\frac{D}{\rho} \geqslant \frac{3}{2}$ in* (1.1)*. Then for the dynamic system of the Cauchy problem* (1.1)–(1.4) *there exists the boundary absorbing set in space E, that is; the bounded closed ball $B_E(0,R) = \{(w,w_t) \in E, \|(w,w_t)\|_E \leqslant R\}$ $(R^2 > \frac{17(N_1^2+N_2^2)}{6\rho T})$.*

Usually, proving the existence of absorbing set amounts to proving a priori estimates.

Proof. Take the scalar product of (1.1) with $2w_t$ in H to get

$$\frac{d}{dt}\left\{\|w_t\|^2 + \frac{D}{\rho}\|w_{xx}\|^2 + \frac{2D}{\rho}\|w_{xy}\|^2 + \frac{D}{\rho}\|w_{yy}\|^2 + \frac{N_1}{\rho}\|w_x\|^2 + \frac{T}{4\rho}\|w_x\|^4 + \frac{N_2}{\rho}\|w_y\|^2 + \frac{T}{4\rho}\|w_y\|^4\right\} + \frac{2\varepsilon\mu}{\rho}\|w_t\|^2 = 0. \tag{4.1}$$

For η fixed (arbitrary at the moment), also take the scalar product of (1.1) with ηw in H to get

$$\frac{d}{dt}\big(\eta(w_t,w)\big) - \eta\|w_t\|^2 + \frac{\eta D}{\rho}\big(\|w_{xx}\|^2 + \|w_{yy}\|^2 + 2\|w_{xy}\|^2\big) + \frac{\eta\varepsilon\mu}{2\rho}\frac{d}{dt}\|w\|^2 + \frac{\eta N_1}{\rho}\|w_x\|^2 + \frac{\eta T}{2\rho}\|w_x\|^4 + \frac{\eta N_2}{\rho}\|w_y\|^2 + \frac{\eta T}{2\rho}\|w_y\|^4 = 0 \tag{4.2}$$

where $0 < \eta \leqslant 1$. Then (4.1) plus (4.2) is as follows

$$\begin{aligned}
&\frac{d}{dt}\left\{\|w_t\|^2 + \frac{D}{\rho}\|w_{xx}\|^2 + \frac{2D}{\rho}\|w_{xy}\|^2 + \frac{D}{\rho}\|w_{yy}\|^2 + \frac{N_1}{\rho}\|w_x\|^2 \right.\\
&\quad\left. + \frac{T}{4\rho}\|w_x\|^4 + \frac{N_2}{\rho}\|w_y\|^2 + \frac{T}{4\rho}\|w_y\|^4 + \eta(w_t,w) + \frac{\eta\varepsilon\mu}{2\rho}\|w\|^2\right\}\\
&\quad + \frac{2\varepsilon\mu}{\rho}\|w_t\|^2 - \eta\|w_t\|^2 + \frac{\eta D}{\rho}\big(\|w_{xx}\|^2 + \|w_{yy}\|^2 + 2\|w_{xy}\|^2\big)\\
&\quad + \frac{\eta N_1}{\rho}\|w_x\|^2 + \frac{\eta T}{2\rho}\|w_x\|^4 + \frac{\eta N_2}{\rho}\|w_y\|^2 + \frac{\eta T}{2\rho}\|w_y\|^4 = 0.
\end{aligned}$$

Writing

$$L(t) = \frac{D}{\rho}\left(\|w_{xx}\|^2 + \|w_{yy}\|^2 + 2\|w_{xy}\|^2\right) + \|w_t\|^2 + \frac{N_1}{\rho}\|w_x\|^2 + \frac{T}{4\rho}\|w_x\|^4 + \frac{N_2}{\rho}\|w_y\|^2 + \frac{T}{4\rho}\|w_y\|^4 + \eta(w_t, w) + \frac{\eta\varepsilon\mu}{2\rho}\|w\|^2$$

and

$$Y(t) = \frac{2\varepsilon\mu}{\rho}\|w_t\|^2 - \eta\|w_t\|^2 + \frac{\eta D}{\rho}\left(\|w_{xx}\|^2 + \|w_{yy}\|^2 + 2\|w_{xy}\|^2\right) + \frac{\eta N_1}{\rho}\|w_x\|^2 + \frac{\eta T}{2\rho}\|w_x\|^4 + \frac{\eta N_2}{\rho}\|w_y\|^2 + \frac{\eta T}{2\rho}\|w_y\|^4,$$

we have

$$\begin{aligned}
\frac{2}{\eta}Y(t) - L(t) &= \frac{D}{\rho}\left(\|w_{xx}\|^2 + \|w_{yy}\|^2 + 2\|w_{xy}\|^2\right) + \frac{N_1}{\rho}\|w_x\|^2 + \frac{N_2}{\rho}\|w_y\|^2 \\
&\quad + \frac{3T}{4\rho}\|w_x\|^4 + \frac{3T}{4\rho}\|w_y\|^4 + \left(\frac{4\varepsilon\mu}{\eta\rho} - 3\right)\|w_t\|^2 - \eta(w_t, w) - \frac{\eta\varepsilon\mu}{2\rho}\|w\|^2 \\
&\geqslant \frac{D}{\rho}\left(\|w_{xx}\|^2 + \|w_{yy}\|^2 + 2\|w_{xy}\|^2\right) + \frac{3T}{4\rho}\left[\left(\|w_x\|^2 + \frac{2N_1}{3T}\right)^2 + \left(\|w_y\|^2 + \frac{2N_2}{3T}\right)^2\right. \\
&\quad \left. - \frac{4N_1^2}{9T^2} - \frac{4N_2^2}{9T^2}\right] + \left(\frac{4\varepsilon\mu}{\eta\rho} - 3\right)\|w_t\|^2 - \frac{\eta}{2}\|w_t\|^2 - \frac{\eta}{2}\|w\|^2 - \frac{\eta\varepsilon\mu}{2\rho}\|w\|^2.
\end{aligned}$$

Because of $w(0, y, t) = w(1, y, t) = w(x, 0, t) = w(x, 1, t) = 0$, there exist points α, β such that $w_x(\alpha, y, t) = w_y(x, \beta, t) = 0$ from the Roll theorem. By Poincaré inequality we get

$$\|w\|^2 \leqslant \frac{1}{4}\|w_{xx}\|^2, \qquad \|w\|^2 \leqslant \frac{1}{4}\|w_{yy}\|^2.$$

Also as $w_x(x, 0, t) = w_x(x, 1, t) = 0$, there exists a point γ such that $w_{xy}(x, \gamma, t) = 0$ from the Roll theorem. By Poincaré inequality we also have

$$\|w\|^2 \leqslant \frac{1}{4}\|w_{xy}\|^2.$$

So it follows that

$$\frac{2}{\eta}Y(t) - L(t) \geqslant \left(\frac{16D}{\rho} - \frac{\eta}{2} - \frac{\eta\varepsilon\mu}{2\rho}\right)\|w\|^2 + \left(\frac{4\varepsilon\mu}{\eta\rho} - 3 - \frac{\eta}{2}\right)\|w_t\|^2 - \frac{N_1^2}{3T\rho} - \frac{N_2^2}{3T\rho}.$$

As $0 < \eta \leqslant \min(\frac{32D}{\rho+\varepsilon\mu}, \sqrt{9 + \frac{8\varepsilon\mu}{\rho}} - 3, \frac{1}{4\rho}, 1)$, we have

$$\frac{2}{\eta}Y(t) - L(t) \geqslant -\frac{N_1^2}{3T\rho} - \frac{N_2^2}{3T\rho}.$$

Furthermore

$$\frac{d}{dt}L(t) + \frac{\eta}{2}L(t) \leqslant \frac{\eta}{2}\left(\frac{N_1^2}{3T\rho} + \frac{N_2^2}{3T\rho}\right). \tag{4.3}$$

On the one hand, using the classical Gronwall inequality, we deduce from (4.3)

$$\begin{aligned}
L(t) &\leqslant L(0)\exp\left(-\frac{\eta}{2}t\right) + \left(\frac{N_1^2}{3T\rho} + \frac{N_2^2}{3T\rho}\right)\left[1 - \exp\left(-\frac{\eta}{2}t\right)\right] \\
&\leqslant L(0)\exp\left(-\frac{\eta}{2}t\right) + \left(\frac{N_1^2}{3T\rho} + \frac{N_2^2}{3T\rho}\right).
\end{aligned} \tag{4.4}$$

On the other hand, considering $\frac{D}{\rho} \geqslant \frac{3}{2}$, $\eta < \frac{1}{4\rho}$ and $\varepsilon\mu \geqslant \frac{1}{4}$, we have

$$\begin{aligned}
L(t) &= \frac{D}{\rho}\big(\|w_{xx}\|^2+\|w_{yy}\|^2+2\|w_{xy}\|^2\big)+\|w_t\|^2+\frac{N_1}{\rho}\|w_x\|^2 \\
&\quad+\frac{T}{4\rho}\|w_x\|^4+\frac{N_2}{\rho}\|w_y\|^2+\frac{T}{4\rho}\|w_y\|^4+\eta(w_t,w)+\frac{\eta\varepsilon\mu}{2\rho}\|w\|^2 \\
&\geqslant \frac{D}{\rho}\big(\|w_{xx}\|^2+\|w_{yy}\|^2+2\|w_{xy}\|^2\big)-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T}+(1-2\rho\eta)\|w_t\|^2+\eta\left(\frac{\varepsilon\mu-\frac{1}{4}}{2\rho}\right)\|w\|^2 \\
&\geqslant \frac{D}{\rho}\big(\|w_{xx}\|^2+\|w_{yy}\|^2+2\|w_{xy}\|^2\big)-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T}+\frac{1}{2}\|w_t\|^2 \\
&\geqslant \frac{3}{2}\big(\|w_{xx}\|^2+\|w_{yy}\|^2+2\|w_{xy}\|^2\big)-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T}+\frac{1}{2}\|w_t\|^2 \\
&\geqslant \frac{1}{2}\big(\|w_{xx}\|^2+\|w_{yy}\|^2+2\|w_{xy}\|^2+\|w_t\|^2\big)-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T} \\
&\geqslant \frac{8}{17}\big(\|w_{xx}\|^2+\|w_{yy}\|^2+\|w_{xy}\|^2+\|w_x\|^2+\|w_y\|^2+\|w\|^2+\|w_t\|^2\big)-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T} \\
&= \frac{8}{17}\big\|(w,w_t)\big\|_E^2-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T}.
\end{aligned} \tag{4.5}$$

By (4.4) and (4.5) we get

$$\frac{8}{17}\big\|(w,w_t)\big\|_E^2-\frac{N_1^2}{\rho T}-\frac{N_2^2}{\rho T}\leqslant L(0)\exp\left(-\frac{\eta}{2}t\right)+\frac{N_1^2}{3\rho T}+\frac{N_2^2}{3\rho T}.$$

Hence we have

$$\big\|(w,w_t)\big\|_E^2\leqslant\frac{17}{8}L(0)\exp\left(-\frac{\eta}{2}t\right)+R_0^2 \tag{4.6}$$

where $R_0^2=\frac{17N_1^2+17N_2^2}{6T\rho}$. So

$$\limsup_{t\to\infty}\big\|(w,w_t)\big\|^2\leqslant R_0^2. \tag{4.7}$$

The balls $B_E(0,R)$ of E centered at 0 of radius $R>R_0$ are absorbing in E for the semigroup $S(t)$, $t\geqslant 0$. We choose $R_0'>R_0$ and set $ß_0=B_E(0,R_0')$. If ß is any bounded set of E, $S(t)ß\subset ß_0$ for $t\geqslant t_0(ß,R_0')$; the time t_0 is easily computed from (4.6)

$$t_0=\frac{2}{\eta}\ln\frac{\frac{17}{8}L(0)}{(R_0')^2-R_0^2}. \tag{4.8}$$

Thus

$$B_E(0,R)=\big\{(w,w_t)\in E;\ \big\|(w,w_t)\big\|_E\leqslant R\big\}\quad\left(R^2>\frac{17N_1^2+17N_2^2}{6T\rho}\right)$$

is the bounded absorbing set of $S(t)$ in E. □

Remark 4.1. The existence of an absorbing set of system is an evidence of the dissipative property.

5. The existence of the global attractor of the system

In order to prove the existence of the global attractor, we introduce the following lemma.

Lemma 1. *(See [17].) We assume that E' is a Banach space and that operator $S(t)$ is given and enjoys the following conditions*:

(1) *The usual semigroup properties*

$$S(t+s)=S(t)S(s)\quad\forall t,s\geqslant 0,$$
$$S(0)=I\quad(\textit{identity in } E).$$

(2) *$S(t)$ is continuous nonlinear operator from E' into itself.*

(3) *$S(t)$ exists a bounded absorbing set in E'.*
(4) *The operator $S(t)$ is uniformly compact for t large. By this we mean that for every bounded set ß there exists t_0 which may depend on ß such that*

$$\bigcup_{t\geqslant t_0} S(t)ß$$

is relatively compact in E'. that is, $S(t)$ is a completely continuous operator.

Then $S(t)$ exists a compact attractor which attracts the bounded sets of E'.

From Lemma 1, we now need to prove the existence of a bounded absorbing set in E_1 and the uniform compactness of the $S(t)$.

Theorem 5. *The semigroup $S(t)$ is a completely continuous operator.*

Proof. We proceed as in Theorem 4, we begin with the analog of (4.1) which is obtained by taking the scalar product of (1.1) with $2(w_{xxxxt} + w_{yyyyt})$ in H to get

$$\begin{aligned}
&\frac{d}{dt}\Big\{\|w_{xxt}\|^2 + \|w_{yyt}\|^2 + \frac{D}{\rho}\|w_{xxxx}\|^2 + \frac{D}{\rho}\|w_{yyyy}\|^2 + \frac{2D}{\rho}\|w_{xxyy}\|^2 \\
&\quad + \frac{2D}{\rho}\|w_{xxxy}\|^2 + \frac{2D}{\rho}\|w_{xyyy}\|^2 + \frac{N_1}{\rho}\big(\|w_{xxx}\|^2 + \|w_{xyy}\|^2\big) + \frac{N_2}{\rho}\big(\|w_{yyy}\|^2 + \|w_{xxy}\|^2\big)\Big\} \\
&\quad + \frac{2\varepsilon\mu}{\rho}\big(\|w_{xxt}\|^2 + \|w_{yyt}\|^2\big) - \frac{T}{2\rho}\|w_x\|^2(w_{xx}, 2w_{xxxxt} + 2w_{yyyyt}) \\
&\quad - \frac{T}{2\rho}\|w_y\|^2(w_{yy}, 2w_{xxxxt} + 2w_{yyyyt}) = 0.
\end{aligned} \tag{5.1}$$

For η fixed, also take the scalar product of (1.1) with $\eta(w_{xxxx} + w_{yyyy})$ in H to get

$$\begin{aligned}
&\frac{d}{dt}(w_t, \eta w_{xxxx} + \eta w_{yyyy}) - (w_t, \eta w_{xxxxt} + \eta w_{yyyyt}) + \frac{\eta\varepsilon\mu}{2\rho}\frac{d}{dt}\big(\|w_{xx}\|^2 + \|w_{yy}\|^2\big) \\
&\quad + \frac{\eta D}{\rho}\big(\|w_{xxxx}\|^2 + \|w_{yyyy}\|^2 + 2\|w_{xxyy}\|^2 + 2\|w_{xxxy}\|^2 + 2\|w_{xyyy}\|^2\big) \\
&\quad + \frac{\eta N_1}{\rho}\big(\|w_{xxx}\|^2 + \|w_{xyy}\|^2\big) + \frac{\eta N_2}{\rho}\big(\|w_{xxy}\|^2 + \|w_{yyy}\|^2\big) - \frac{T}{2\rho}\|w_x\|^2(w_{xx}, \eta w_{xxxx} + \eta w_{yyyy}) \\
&\quad - \frac{T}{2\rho}\|w_y\|^2(w_{yy}, \eta w_{xxxx} + \eta w_{yyyy}) = 0.
\end{aligned} \tag{5.2}$$

Then (5.1) plus (5.2) is as follows

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{D}{\rho}\|w_{xxxx}\|^2 + \frac{2D}{\rho}\|w_{xxyy}\|^2 + \frac{2D}{\rho}\|w_{xxxy}\|^2 + \frac{2D}{\rho}\|w_{xyyy}\|^2 \\
&\quad + \frac{D}{\rho}\|w_{yyyy}\|^2 + \|w_{xxt}\|^2 + \|w_{yyt}\|^2 + + \frac{N_1}{\rho}\big(\|w_{xxx}\|^2 + \|w_{xyy}\|^2\big) + \frac{N_2}{\rho}\big(\|w_{xxy}\|^2 + \|w_{yyy}\|^2\big) \\
&\quad + \frac{\eta\varepsilon\mu}{2\rho}\big(\|w_{xx}\|^2 + \|w_{yy}\|^2\big) + \big(w_t, \eta w_{xxxx} + \eta w_{yyyy}\big)\Big\} - (w_t, \eta w_{xxxxt} + \eta w_{yyyyt}) \\
&\quad + \frac{\eta D}{\rho}\big(\|w_{xxxx}\|^2 + \|w_{yyyy}\|^2 + 2\|w_{xxyy}\|^2 + 2\|w_{xxxy}\|^2 + 2\|w_{xyyy}\|^2\big) \\
&\quad + \frac{\eta N_1}{\rho}\big(\|w_{xxx}\|^2 + \|w_{xyy}\|^2\big) + \frac{\eta N_2}{\rho}\big(\|w_{xxy}\|^2 + \|w_{yyy}\|^2\big) \\
&\quad + \frac{2\varepsilon\mu}{\rho}\big(\|w_{xxt}\|^2 + \|w_{yyt}\|^2\big) - \frac{T}{2\rho}\|w_x\|^2(w_{xx}, 2w_{xxxxt} + 2w_{yyyyt}) \\
&\quad - \frac{T}{2\rho}\|w_y\|^2(w_{yy}, 2w_{xxxxt} + 2w_{yyyyt}) - \frac{T}{2\rho}\|w_x\|^2(w_{xx}, \eta w_{xxxx} + \eta w_{yyyy}) \\
&\quad - \frac{T}{2\rho}\|w_y\|^2(w_{yy}, \eta w_{xxxx} + \eta w_{yyyy}) = 0.
\end{aligned}$$

Noting that

$$L_1(t)=\frac{D}{\rho}\|w_{xxxx}\|^2+\frac{2D}{\rho}\|w_{xxyy}\|^2+\frac{2D}{\rho}\|w_{xxxy}\|^2+\frac{2D}{\rho}\|w_{xyyy}\|^2 + \frac{D}{\rho}\|w_{yyyy}\|^2+\|w_{xxt}\|^2+\|w_{yyt}\|^2+\frac{N_1}{\rho}\big(\|w_{xxx}\|^2+\|w_{xyy}\|^2\big) + \frac{N_2}{\rho}\big(\|w_{xxy}\|^2+\|w_{yyy}\|^2\big)+\frac{\eta\varepsilon\mu}{2\rho}\big(\|w_{xx}\|^2+\|w_{yy}\|^2\big)+(w_t,\eta w_{xxxx}+\eta w_{yyyy})$$

and

$$Y_1(t)=\frac{\eta D}{\rho}\big(\|w_{xxxx}\|^2+\|w_{yyyy}\|^2+2\|w_{xxyy}\|^2+2\|w_{xxxy}\|^2+2\|w_{xyyy}\|^2\big) - (w_t,\eta w_{xxxxt}+\eta w_{yyyyt})+\frac{\eta N_1}{\rho}\big(\|w_{xxx}\|^2+\|w_{xyy}\|^2\big) + \frac{\eta N_2}{\rho}\big(\|w_{xxy}\|^2+\|w_{yyy}\|^2\big)+\frac{2\varepsilon\mu}{\rho}\big(\|w_{xxt}\|^2+\|w_{yyt}\|^2\big) - \frac{T}{2\rho}\|w_x\|^2(w_{xx},2w_{xxxxt}+2w_{yyyyt})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},2w_{xxxxt}+2w_{yyyyt}) - \frac{T}{2\rho}\|w_x\|^2(w_{xx},\eta w_{xxxx}+\eta w_{yyyy})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},\eta w_{xxxx}+\eta w_{yyyy}),$$

we have

$$\frac{2}{\eta}Y_1(t)-L_1(t)=\frac{D}{\rho}\|w_{xxxx}\|^2+\frac{2D}{\rho}\|w_{xxyy}\|^2+\frac{2D}{\rho}\|w_{xxxy}\|^2+\frac{2D}{\rho}\|w_{xyyy}\|^2 + \frac{D}{\rho}\|w_{yyyy}\|^2+\left(\frac{4\varepsilon\mu}{\eta\rho}-3\right)\big(\|w_{xxt}\|^2+\|w_{yyt}\|^2\big)+\frac{N_1}{\rho}\big(\|w_{xxx}\|^2+\|w_{xyy}\|^2\big) + \frac{N_2}{\rho}\big(\|w_{xxy}\|^2+\|w_{yyy}\|^2\big)-\frac{\eta\varepsilon\mu}{2\rho}\big(\|w_{xx}\|^2+\|w_{yy}\|^2\big)-(w_t,\eta w_{xxxx}+\eta w_{yyyy}) + \frac{2}{\eta}\left\{-\frac{T}{2\rho}\|w_x\|^2(w_{xx},2w_{xxxxt}+2w_{yyyyt})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},2w_{xxxxt}+2w_{yyyyt}) - \frac{T}{2\rho}\|w_x\|^2(w_{xx},\eta w_{xxxx}+\eta w_{yyyy})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},\eta w_{xxxx}+\eta w_{yyyy})\right\}.$$

With $-(w_t,\eta w_{xxxx}+\eta w_{yyyy})\geqslant-\frac{\eta^2}{2}(\|w_{xx}\|^2+\|w_{yy}\|^2)-\frac{1}{2}(\|w_{xxt}\|^2+\|w_{yyt}\|^2)$ and $\frac{D}{\rho}\geqslant\frac{3}{2}$, we have

$$\frac{2}{\eta}Y_1(t)-L_1(t)\geqslant\frac{3}{2}\big(\|w_{xxxx}\|^2+\|w_{yyyy}\|^2+2\|w_{xxyy}\|^2+2\|w_{xxxy}\|^2+2\|w_{xyyy}\|^2\big) + \left(\frac{4\varepsilon\mu}{\eta\rho}-\frac{7}{2}\right)\big(\|w_{xxt}\|^2+\|w_{yyt}\|^2\big)+\frac{N_1}{\rho}\big(\|w_{xxx}\|^2+\|w_{xyy}\|^2\big) + \frac{N_2}{\rho}\big(\|w_{xxy}\|^2+\|w_{yyy}\|^2\big)+\left(-\frac{\eta\varepsilon\mu}{2\rho}-\frac{\eta^2}{2}\right)\big(\|w_{xx}\|^2+\|w_{yy}\|^2\big) + \frac{2}{\eta}\left\{-\frac{T}{2\rho}\|w_x\|^2(w_{xx},2w_{xxxxt}+2w_{yyyyt})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},2w_{xxxxt}+2w_{yyyyt}) - \frac{T}{2\rho}\|w_x\|^2(w_{xx},\eta w_{xxxx}+\eta w_{yyyy})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},\eta w_{xxxx}+\eta w_{yyyy})\right\}.$$

Because of

$$\frac{2}{\eta}\left\{-\frac{T}{2\rho}\|w_x\|^2(w_{xx},2w_{xxxxt}+2w_{yyyyt})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},2w_{xxxxt}+2w_{yyyyt}) - \frac{T}{2\rho}\|w_x\|^2(w_{xx},\eta w_{xxxx}+\eta w_{yyyy})-\frac{T}{2\rho}\|w_y\|^2(w_{yy},\eta w_{xxxx}+\eta w_{yyyy})\right\} \geqslant-\frac{T}{\eta\rho}R^2\big(\|w_{xxxx}\|^2+\|w_{yyyy}\|^2+2\|w_{xxyy}\|^2+2\|w_{xxt}\|^2+2\|w_{yyt}\|^2\big)-\|w_{xxxx}\|^2-\|w_{yyyy}\|^2-\frac{2T}{\rho}R^6$$

we have

$$\frac{2}{\eta}Y_1(t)-L_1(t)\geqslant\left(\frac{1}{2}-\frac{TR^2}{\eta\rho}\right)\left(\|w_{xxxx}\|^2+\|w_{yyyy}\|^2+2\|w_{xxyy}\|^2\right)$$
$$+\left(\frac{4\varepsilon\mu}{\eta\rho}-\frac{7}{2}-\frac{2TR^2}{\eta\rho}\right)\left(\|w_{xxt}\|^2+\|w_{yyt}\|^2\right)$$
$$+\left(12-\frac{\eta\varepsilon\mu}{2\rho}-\frac{\eta^2}{2}\right)\left(\|w_{xx}\|^2+\|w_{yy}\|^2\right)-\frac{2T}{\rho}R^6.$$

As $\frac{2T}{\rho}R^2\leqslant\eta\leqslant\min\{\frac{8\varepsilon\mu-4TR^2}{7\rho},-\frac{\varepsilon\mu}{2\rho}+\sqrt{\frac{\varepsilon^2\mu^2}{4\rho^2}+24},\eta_1\}$, we have

$$\frac{2}{\eta}Y_1(t)-L_1(t)\geqslant-\frac{2T}{\rho}R^6$$

where $\eta_1=\min(\frac{32D}{\rho+\varepsilon\mu},\sqrt{9+\frac{8\varepsilon\mu}{\rho}}-3,\frac{1}{4\rho},1)$. It follows that

$$\frac{dL_1(t)}{dt}+\frac{\eta}{2}L_1(t)\leqslant\frac{\eta}{2}\frac{2T}{\rho}R^6. \tag{5.3}$$

If $\{w^0,w^1\}$ belongs to a bounded ß of E_1, since ß is also bounded in E, then there exists a time t_0 given by (4.7) such that for $t\geqslant t_0$, $S(t)ß\subset ß_0$, which implies that

$$\left\|(w,w_t)\right\|_E^2\leqslant\left(R_0'\right)^2.$$

On the one hand thanks to the classical Gronwall lemma we infer from (5.3) that

$$L_1(t)\leqslant L_1(0)\exp\left(-\frac{\eta}{2}(t-t_0)\right)+\frac{2T}{\rho}R^6\left(1-\exp\left(-\frac{\eta}{2}(t-t_0)\right)\right) \tag{5.4}$$

for $t\geqslant t_0$. On the other hand, considering that at least $\eta\leqslant1$, we have

$$(w_t,\eta w_{xxxx}+\eta w_{yyyy})\geqslant-\frac{1}{4}\left(\|w_{xxxt}\|^2+2\|w_{yyyt}\|^2\right)-\frac{1}{2}\left(\|w_{xxx}\|^2+2\|w_{yyy}\|^2\right).$$

With a class of reasoning, we get

$$L_1(t)\geqslant\frac{3}{2}\left(\|w_{xxxx}\|^2+2\|w_{xxyy}\|^2+2\|w_{xxxy}\|^2+2\|w_{xyyy}\|^2+\|w_{yyyy}\|^2\right)$$
$$+\|w_{xxt}\|^2+\|w_{yyt}\|^2+\frac{N_1}{\rho}\left(\|w_{xxx}\|^2+\|w_{xyy}\|^2\right)+\frac{N_2}{\rho}\left(\|w_{xxy}\|^2+\|w_{yyy}\|^2\right)$$
$$+\frac{\eta\varepsilon\mu}{2\rho}\left(\|w_{xx}\|^2+\|w_{yy}\|^2\right)+(w_t,\eta w_{xxxx}+\eta w_{yyyy})$$
$$\geqslant\frac{1}{2}\left\|(w,w_t)\right\|_{E_1}^2. \tag{5.5}$$

From (5.4) and (5.5) we have

$$\left\|(w,w_t)\right\|_{E_1}^2\leqslant2L_1(0)\exp\left(-\frac{\eta}{2}(t-t_0)\right)+\frac{4T}{\rho}R^6\left[1-\exp\left(-\frac{\eta}{2}(t-t_0)\right)\right] \tag{5.6}$$

for $t\geqslant t_0$. Defining R_1 by $R_1^2=\frac{4TR^6}{\rho}$, we see that

$$\limsup_{t\to\infty}\left\|(w,w_t)\right\|_{E_1}^2\leqslant R_1^2. \tag{5.7}$$

It follows that the ball $ß_1=B_{E_1}(0,R_1')$ centered at (0) of radius $R_1'>R_1$ of E_1 is absorbing in E_1 for the semigroup $S(t)$, $t\geqslant0$. The time $t_1=t_1(ß,R_1')$ after which $S(t)ß$ is included in $ß_1$, the time $t_1=t_1(ß,R_1')\geqslant t_0+t_1'$, $t_1'=\frac{2}{\eta}\ln\frac{2L_1(0)}{(R_1')^2-R_1^2}$. This result provides the uniform compactness of $S(t)$. That is, $S(t)$ is completely continuous operator. The proof is complete. □

From Lemma 1 and Theorems 1–5, we may push out the following Theorem 6.

Theorem 6. *The dynamic system $S(t)$ associated with boundary-value problem* (1.1)–(1.4) *possesses a compact attractor $\Re$ which is bounded in E_1.*

Proof. We apply Lemma 1 with E' replaced by E. The necessary assumptions of Lemma 1 have been proved above, namely in Theorems 1–5. □

References

[1] M. Aassila, Global existence and energy decay for a damped quasi-linear wave equation, Math. Methods Appl. Sci. 21 (1998) 1185–1194.
[2] J.M. Ball, Initial–boundary value problems for an extensible beam, J. Math. Anal. Appl. 42 (1973) 61–90.
[3] J.M. Ball, Stability theory for an extensible beam, J. Differential Equations 14 (1973) 0463–0486.
[4] E. Fasangova, Attractor for a beam equation with weak damping, Appl. Anal. 59 (1995) 1–13.
[5] M. Feckan, Free vibrations of beams on bearings with nonlinear elastic responses, J. Differential Equations 154 (1999) 55–72.
[6] E. Feireisl, Exponential attractors for non-autonomous systems long-time behavior of vibrating beams, Math. Methods Appl. Sci. 15 (1992) 1287–1297.
[7] E. Feireisl, Finite dimensional behavior of a nonautomous partial differential equation: Forced oscillations of an extensible beam, J. Differential Equations 101 (1993) 302–312.
[8] E. Feireisl, Non-zero time periodic solutions to an equation of Petrovsky type with nonlinear boundary condition: Show oscillation of beams on elastic bearing, Ann. Sc. Norm. Super. Pisa 20 (1993) 133–146.
[9] J. Ferreira, Stability for the beam equation with memory in non-cylindrical domains, Math. Methods Appl. Sci. 27 (2004) 1493–1506.
[10] W. Hao, Long time behavior for a nonlinear plate equation with thermal memory, J. Math. Anal. Appl. 348 (2008) 650–670.
[11] S. Kouemou-Patcheu, Global existence and exponential decay estimates for a damped quasilinear equation, Comm. Partial Differential Equations 22 (1997) 2007–2024.
[12] L. Yang, C.K. Zhong, Global attractor for plate equation with nonlinear damping, Nonlinear Anal. 69 (2008) 3802–3810.
[13] P. Massatt, Limiting behavior for strongly damped nonlinear wave equations, J. Differential Equations 48 (1983) 334–349.
[14] L.A. Mederios, On a new class of nonlinear wave equations, J. Math. Appl. 69 (1979) 252–262.
[15] K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation, Math. Methods Appl. Sci. 20 (1997) 151–177.
[16] K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, J. Math. Anal. Appl. 216 (1997) 321–342.
[17] R. Teman, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 2000, pp. 15–329.
[18] F.M. To, Boundary stabilization for a non-linear beam on elastic bearings, Math. Methods Appl. Sci. 24 (2001) 583–594.
[19] M. Tucsnak, Semi-internal stabilization for a nonlinear Euler–Bernoulli equation, Math. Methods Appl. Sci. 9 (1996) 897–907.
[20] G.F. Webb, Existence and asymptotic behavior for a strongly damped nonlinear wave equation, Canad. J. Math. 32 (1980) 631–643.
[21] S. Woinowsky-Krieger, The effect of axial force on the vibration of hinged bars, J. Appl. Mech. 17 (1950) 35–36.
[22] H.B. Xiao, Asymptotic dynamics of plate equations with a critical exponent on unbounded domain, Nonlinear Anal. 70 (2009) 1288–1301.
[23] Y. You, Spillover problem and global dynamics of nonlinear beam equations, in: Proceedings of the 34th Conference on Decision and Control New Orleans, December 1995, pp. 3231–3236.
[24] J.W. Zhang, Global dynamics on the equations of nonlinear viscous-elastic beam, J. Engrg. Math. 20 (2003) 38–42.