Properties of the generalized $f$-projection operator and its applications in Banach spaces

Ke-qing Wu, Nan-jing Huang *

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

Received 16 June 2006; received in revised form 20 January 2007; accepted 22 January 2007

Abstract

In this paper, we prove some properties of the generalized $f$-projection operator in Banach spaces and show an interesting relation between the generalized $f$-projection operator and the resolvent operator for the subdifferential of a proper, convex and lower semi-continuous functional in reflexive and smooth Banach spaces. Moreover, we prove that the generalized $f$-projection operator is maximal monotone. As applications, we propose an iterative method of approximating solutions for a class of generalized variational inequalities and give a convergence result for the iterative method in uniformly convex and uniformly smooth Banach spaces.

Keywords: Generalized $f$-projection operator; Generalized variational inequality; Maximal monotone operator; Lower semi-continuity; Iterative sequence; Convergence

1. Introduction

It is well known that the metric projection operators in Hilbert and Banach spaces are widely used in different areas of mathematics such as functional and numerical analysis, theory of optimization and approximation, and also for the problems of optimal control and operations research, nonlinear and stochastic programming, variational inequality and complementarity problem and game theory, etc. (see, for example, [1–10] and the references therein).

In 1994, Alber [1] introduced the generalized projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [2], Alber presented some applications of the generalized projections to approximate solving variational inequalities and Von-Neumann intersection problem in Banach spaces. Furthermore, Alber [11] proved the convergence and stability of Cesàro averages generated by the proximal projection method applied to nonlinear equations and variational inequalities in uniformly convex and uniformly smooth Banach spaces. The iterative subgradient methods for nonsmooth convex constrained optimization problems in uniformly convex and uniformly smooth Banach spaces can be found in the papers due to Alber [12] and Alber and Iusem [13].
Recently, Li [9] extended the definition of the generalized projection operator \( \pi_K : B^* \to K \), where \( B \) is a reflexive Banach space with dual space \( B^* \) and \( K \) is a nonempty, closed and convex subset of \( B \) and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces. For some related works, we refer to [14–17] and the references therein.

Very recently, by employing the generalized projection operators, Zeng and Yao [18] establish some existence results for the variational inequality problem and the quasivariational inequality problem in reflexive, strictly convex and smooth Banach spaces and convergence results for the variational inequality.

On the other hand, Wu and Huang [19] introduced and studied a new class of generalized \( f \)-projection operator in Banach spaces, which extends the definition of the generalized projection operators introduced by Alber [1] and Li [9]. They proved some properties of the generalized \( f \)-projection operator. As applications, some existence theorems of solutions for a class of variational inequalities in Banach spaces were also obtained in [19].

Motivated and inspired by the above works, in this paper, we continue to study some properties of the generalized \( f \)-projection operator introduced by Wu and Huang [19]. We establish some properties of the generalized \( f \)-projection operator in Banach spaces and show an interesting relation between the generalized \( f \)-projection operator and the resolvent operator for the subdifferential of a proper, convex and lower semi-continuous functional in reflexive and smooth Banach spaces. Moreover, we prove that the generalized \( f \)-projection operator is maximal monotone. As applications, we propose an iterative method of approximating solutions for the variational inequality problem: find \( x^* \in K \) such that

\[
\langle Ax^*, y - x^* \rangle + f(y) - f(x^*) \geq 0, \quad \forall y \in K, \tag{1.1}
\]

where \( K \) is a nonempty closed convex subset of a Banach space \( B \) with dual \( B^* \), \( A : K \to B^* \) and \( f : K \to R \cup \{+\infty\} \). A convergence result for this iterative method is also given in uniformly convex and uniformly smooth Banach spaces.

2. Preliminaries

In this section, we shall recall some definitions and known results.

Let \( B \) be a real Banach space with dual space \( B^* \) and \( R = (-\infty, +\infty) \). The normalized duality mapping \( J : B \to 2^{B^*} \) is defined by

\[
J(x) = \{ j(x) \in B^* : \langle j(x), x \rangle = \| j(x) \| : \| x \| = \| x \|^2 = \| j(x) \|^2 \},
\]

where \( \langle \cdot , \cdot \rangle \) denotes the duality pairing of \( B^* \) and \( B \). Without confusion, one understands that \( \| j(x) \| \) is the \( B^* \)-norm and \( \| x \| \) is the \( B \)-norm. Many properties of the normalized duality mapping \( J \) can be found in Cioranescu [20] or Takahashi [21] or Vainberg [22]. We list some properties below for easy reference:

(i) \( J \) is a monotone and bounded operator in arbitrary Banach spaces;

(ii) \( J \) is a strictly monotone operator in strictly convex Banach spaces;

(iii) If \( B \) is a smooth Banach space, the \( J \) is single-valued and hemi-continuous, that is, \( J \) is continuous from the strong topology of \( B \) to the weak star topology of \( B^* \);

(iv) \( J \) is uniformly continuous on each bounded set in uniformly smooth Banach spaces;

(v) \( J \) is the identity operator in Hilbert spaces, i.e. \( J = I_B \);

(vi) \( J(x) = \partial(\| x \|^2/2) \), where \( \partial(\| x \|^2/2) \) denotes subdifferential of \( \| x \|^2/2 \) at \( x \);

(vii) If \( B \) is a reflexive strictly convex space with a strictly convex conjugate space \( B^* \) and \( J^* : B^* \to B \) is a normalized duality mapping in \( B^* \), then \( J^{-1} = J^* \), \( JJ^* = I_{B^*} \) and \( J^*J = I_B \).

For any fixed \( \rho > 0 \), let \( G : B^* \times K \to R \cup \{+\infty\} \) be a functional defined as follows:

\[
G(\varphi, x) = \| \varphi \|^2 - 2\varphi(x) + \| x \|^2 + 2\rho f(x),
\]

where \( \varphi \in B^*, \ x \in B \) and \( f : K \subset B \to R \cup \{+\infty\} \) is proper, convex and lower semi-continuous. It is easy to see that \( G(\varphi, x) \geq (\| \varphi \| - \| x \|)^2 + 2\rho f(x) \).

From the definitions of \( G \) and \( f \), it is easy to have the following properties:

(vii) \( G(\varphi, x) \) is convex and continuous with respect to \( \varphi \) when \( x \) is fixed;
(viii) $G(\varphi, x)$ is convex and lower-semi-continuous with respect to $x$ when $\varphi$ is fixed;
(ix) $(\|\varphi\| - \|x\|)^2 + 2\rho f(x) \leq G(\varphi, x) \leq (\|\varphi\| + \|x\|)^2 + 2\rho f(x)$.

**Definition 2.1.** We say that a Banach space $B$ has the property (h) if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ implies $x_n \to x$.

**Remark 2.1.** It is well known that any locally uniformly convex space has the property (h).

**Definition 2.2.** An operator $A : B \to 2^{B^*}$ is said to be maximal monotone if it is monotone and for any $(x, u) \in B \times B^*$, the inequality $\langle u - v, x - y \rangle \geq 0$ holds for $(y, v) \in \text{graph}(A)$ implies that $(x, u) \in \text{graph}(A)$.

**Definition 2.3** ([19]). Let $B$ be a Banach space with dual space $B^*$ and $K$ be a nonempty, closed and convex subset of $B$. We say that $\pi^f_K : B^* \to 2^K$ is a generalized $f$-projection operator if

$$\pi^f_K \varphi = \{ u \in K : G(\varphi, u) = \inf_{y \in K} G(\varphi, y) \} \quad \forall \varphi \in B^*.$$

**Remark 2.2.** (i) Compared with Definition 1.1 of [19], we replace $f$ by $2\rho f$. Theorem 3.6 shows this definition is more suitable than one of [19]; (ii) If $f(x) = 0$ for all $x \in K$, then the generalized $f$-projection operator reduces to the generalized projection operator defined by Alber [1] and Li [9].

**Lemma 2.1** ([20]). Let $B$ be a Banach space and $f : B \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semi-continuous. Then $f$ is subdifferentiable on $\text{int} D(f)$ and the subdifferential $\partial f$ is maximal monotone, where $\text{int} D(f)$ denotes the interior of the effective domain.

**Lemma 2.2** ([20]). Let $B$ be a reflexive Banach space and $A : B \to 2^{B^*}$ be a maximal monotone operator with $D(A) = B$. Then, for every $\lambda > 0$, $A + \lambda J$ is maximal monotone and surjective, and $(A + \lambda J)^{-1} : B^* \to B$ is norm-weak continuous and maximal monotone. Moreover, if $B$ has the property (h), then $(A + \lambda J)^{-1}$ is continuous.

**Lemma 2.3** ([20]). Let $B$ be a real reflexive Banach space and $A : B \to 2^{B^*}$ be a monotone mapping. Then $A$ is maximal monotone if and only if $A + J$ is surjective.

**Lemma 2.4** ([23]). Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space $B$ is uniformly convex if and only if there is a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|),$$

for all $x, y \in B$, and $\lambda \in [0, 1]$, where $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

**Lemma 2.5** ([2]). Let $B$ be a smooth Banach space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2(y, J(x + y)), \quad \forall x, y \in B.$$

### 3. Properties of the generalized $f$-projection operator

In this section, we shall show some properties of the generalized $f$-projection operator and an interesting relation between the generalized $f$-projection operator and the resolvent operator. Moreover, under the suitable conditions, we shall prove that the generalized $f$-projection operator is maximal monotone.

The following theorems provide some properties of the generalized $f$-projection operator $\pi^f_K$ in reflexive Banach spaces.

**Theorem 3.1** ([19,24]). If $B$ is a reflexive Banach space with dual space $B^*$ and $K$ is a nonempty closed convex subset of $B$, then $\pi^f_K \varphi \neq \emptyset$ for all $\varphi \in B^*$.
Remark 3.1. In [24], the assumption that $f$ is bounded from below in Theorem 2.1 of [19] was removed. Moreover, from the proof of Theorem 3.1 in [24], it is easy to see that Theorem 3.1 presented in this paper is true when $f$ is replaced by $2\rho f$.

From the proof of Theorem 2.2 in [19], we can easily see the following Theorem 3.2 holds.

Theorem 3.2 ([19]). If $B$ is a reflexive Banach space with dual space $B^*$ and $K$ is a nonempty, closed and convex subset of $B$, then the following conclusions hold:

(f1) For any given $\varphi \in B^*$, $\pi_k^f \varphi$ is a nonempty, closed and convex subset of $K$;

(f2) $\pi_k^f$ is monotone, i.e., for any $\varphi_1, \varphi_2 \in B^*$, $x_1 \in \pi_k^f \varphi_1$ and $x_2 \in \pi_k^f \varphi_2$,

$$(x_1 - x_2, \varphi_1 - \varphi_2) \geq 0;$$

(f3) If $B$ is smooth, then for any given $\varphi \in B^*$, $x \in \pi_k^f \varphi$ if and only if

$$\langle \varphi - J(x), x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in K;$$

(f4) If $f : K \to R \cup \{+\infty\}$ is positively homogeneous, i.e., $f(tx) = tf(x)$ for all $t > 0$ and $x \in K$ with $tx \in K$, then for any $\varphi \in B^*$ and $x_1, x_2 \in \pi_k^f \varphi$ with $x_1 \neq 0$ and $x_2 \neq 0$, we have $x_1 \neq \mu x_2$ for all $\mu \in (0, +\infty)$ with $\mu \neq 1$;

(f5) If $f : K \to R \cup \{+\infty\}$ is positively homogeneous and $B$ is strictly convex, then the operator $\pi_k^f : B^* \to K$ is single-valued.

By Theorem 3.2 (f3), it is easy to obtain the following result.

Theorem 3.3 ([19]). Let $A$ be an arbitrary operator acting from the reflexive smooth Banach space $B$ to $B^*$, $\xi \in B^*$ and $\rho > 0$. Then the point $x^* \in K \subset B$ is a solution of the variational inequality

$$(Ax - \xi, y - x) + f(y) - f(x) \geq 0, \quad \forall y \in K,$$

if and only if $x^*$ is a solution of the following inclusion

$$x \in \pi_k^f (J(x) - \rho (Ax - \xi)).$$

Theorem 3.4. Let $B$ be a reflexive and strictly convex Banach space with dual space $B^*$ and $K$ be a nonempty compact convex subset of $B$. If $f : K \to R \cup \{+\infty\}$ is proper, convex, lower semi-continuous and positively homogeneous, then $\pi_k^f : B^* \to K$ is continuous.

Proof. Since $B$ is a reflexive and strictly convex Banach space, and $f : K \to R \cup \{+\infty\}$ is positively homogeneous, it follows from Theorems 3.1 and 3.2 that $\pi_k^f$ is well defined and single-valued. Suppose that $\varphi_n \to \varphi$ as $n \to \infty$, $\pi_k^f (\varphi_n) = x_n$ for $n = 1, 2, \ldots$, and $\pi_k^f (\varphi) = x$. The compactness of $K$ implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $x_0$ such that $\{x_{n_i}\}$ converges to $x_0$. Since $G(\varphi_{n_i}, x_{n_i}) \leq G(\varphi_{n_i}, y)$ for every $y \in K$, we have

$$G(\varphi, x_0) = \lim \inf_{i \to \infty} G(\varphi_{n_i}, x_{n_i}) \leq \lim \inf_{i \to \infty} G(\varphi_{n_i}, y) = G(\varphi, y), \quad \forall y \in K$$

and so $x_0 = \pi_k^f (\varphi) = x$. Similar to the above arguments, we know that $x$ is the unique limit point of $\{x_n\}$. Therefore, $x_n \to x$ as $n \to \infty$. It follows that $\pi_k^f$ is continuous. This completes the proof.

Theorem 3.5. Let $B$ be a reflexive and strictly convex Banach space with dual space $B^*$ and $K$ be a nonempty closed convex subset of $B$. Suppose that $f : K \to R \cup \{+\infty\}$ is proper, convex, lower semi-continuous, positively homogeneous and bounded from below. Then

(i) $\pi_k^f : B^* \to K$ is norm-weak continuous;

(ii) Moreover, if $B$ has the property (h), then $\pi_k^f : B^* \to K$ is continuous.
Proof. (i) Since $B$ is a reflexive and strictly convex Banach space, and $f : K \rightarrow R \cup \{+\infty\}$ is positively homogeneous, it follows from Theorems 3.1 and 3.2 that $\pi^f_K$ is well defined and single-valued. Suppose that $\varphi_n \rightharpoonup \varphi$ as $n \rightarrow \infty$, $\pi^f_K(\varphi_n) = x_n$ for $n = 1, 2, \ldots$, and $\pi^f_K(\varphi) = x$. For any $y \in K$ with $f(y) < +\infty$, we have

$$(\|\varphi_n\| - \|x_n\|)^2 + 2\rho f(x_n) \leq G(\varphi_n, x_n) \leq G(\varphi, y) \leq (\|\varphi_n\| + \|y\|)^2 + 2\rho f(y).$$

Since $f$ is bounded from below, it follows that $\{x_n\}$ is bounded. The reflexivity of $B$ implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $x_0 \in K$ such that $\{x_{n_i}\}$ converges to $x_0$ weakly. Since $G(\varphi_{n_i}, x_{n_i}) \leq G(\varphi_{n_i}, y)$ for every $y \in K$, we have

$$G(\varphi, x_0) = \liminf_{i \rightarrow \infty} G(\varphi_{n_i}, x_{n_i}) \leq \liminf_{i \rightarrow \infty} G(\varphi_{n_i}, y) = G(\varphi, y), \quad \forall y \in K$$

and so $x_0 = \pi^f_K(\varphi) = x$. Similar to the above arguments, we know that $x$ is the unique weak limit point of $\{x_n\}$. Therefore, $x_n \rightharpoonup x$ weakly as $n \rightarrow \infty$. It follows that $\pi^f_K$ is norm-weak continuous.

(ii) Suppose that $\varphi_n \rightharpoonup \varphi$ as $n \rightarrow \infty$, $\pi^f_K(\varphi_n) = x_n$ for $n = 1, 2, \ldots$, and $\pi^f_K(\varphi) = x$. Then $G(\varphi_n, x_n) \leq G(\varphi_n, x)$ and $G(\varphi, x) \leq G(\varphi, x_n)$. It follows that

$$2(\varphi_n - \varphi, x - x_n) + 2\varphi_n(x_n) + 2\rho f(x_n) - 2\rho f(x) \leq \|x\|^2 - \|x_n\|^2. \tag{3.1}$$

From (i), we know that $x_n \rightharpoonup x$ weakly as $n \rightarrow \infty$. By the weak lower semi-continuity of the norm $\|\cdot\|$, we have $\|x\| = \liminf\|x_n\|$. Since $f$ is lower semi-continuous and convex, it follows from (3.1) that $\limsup\|x_n\| \leq \|x\|$ and so

$$\|x_n\| \rightarrow \|x\| \quad \text{as} \quad n \rightarrow \infty.$$

Now the property (h) implies that $x_n \rightharpoonup x$ as $n \rightarrow \infty$. This completes the proof. \(\square\)

When $K = B$, we have the following interesting result.

**Theorem 3.6.** Let $B$ be a reflexive and smooth Banach space. If $f : B \rightarrow R \cup \{+\infty\}$ is proper, convex and lower semi-continuous. Then for any $\varphi \in B^*$, $\pi^f_B\varphi = (J + \rho \partial f)^{-1}\varphi$.

Proof. It follows from Lemma 2.1 that the subdifferential $\partial f$ of a proper, convex and lower semi-continuous functional $f$ is maximal monotone. It follows from Lemma 2.3 that $J + \rho \partial f$ is surjective. Thus $(J + \rho \partial f)^{-1}\varphi$ is well-defined. Since $B$ is a reflexive and smooth Banach space, $J$ is single-valued. First we prove that $\pi^f_B\varphi \subset (J + \rho \partial f)^{-1}\varphi$. In fact, from Theorem 3.2 ($f_3$), for any $x \in \pi^f_B\varphi$, we have

$$\langle \varphi - J(x), x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in B.$$

This implies that $\varphi - J(x) \in \rho \partial f(x)$ and so $\varphi \in (J + \rho \partial f)(x)$. Therefore, $x \in (J + \rho \partial f)^{-1}\varphi$.

Next we prove that $(J + \rho \partial f)^{-1}\varphi \subset \pi^f_B\varphi$. For any $x \in (J + \rho \partial f)^{-1}\varphi$, we have

$$\rho f(y) - \rho f(x) \geq \langle \varphi - J(x), y - x \rangle, \quad \forall y \in B.$$

That is,

$$\langle \varphi - J(x), x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in B.$$

It follows from Theorem 3.2 ($f_3$) that $x \in \pi^f_B\varphi$. This completes the proof. \(\square\)

**Remark 3.2.** From Theorem 3.6, we know that the generalized $f$-projection operator is a generalization of the resolvent operator for subdifferential $\partial f$ of a proper, convex and lower semi-continuous functional $f$ from Hilbert spaces to Banach spaces.

**Theorem 3.7.** Let $B$ be a reflexive and smooth Banach space with dual space $B^*$ and $f : B \rightarrow R$ be a proper, convex and lower semi-continuous functional. Then

(i) $\pi^f_K : B^* \rightarrow B$ is a maximal monotone and norm-weak continuous operator;
(ii) If $B$ has the property (h), then $\pi^B_K : B^* \rightarrow B$ is continuous.

**Proof.** Since $f : B \rightarrow \mathbb{R}$ is convex and lower semi-continuous, it follows from Lemma 2.1 that $f$ is subdifferentiable on $B$ and the subdifferential $\partial f$ is maximal monotone. By Lemma 2.2 and Theorem 3.6, conclusions (i) and (ii) of Theorem 3.7 hold. This completes the proof. \(\square\)

4. Approximating solutions of the generalized variational inequality

Due to the presence of nonlinear terms, the projection methods presented in [2,11,12,18] cannot be applied to suggest any iterative scheme for generalized variational inequality (1.1) in Banach spaces. In this section, by employing the generalized $f$-projection operator, we shall propose an iterative method for finding approximating solutions of generalized variational inequality (1.1). Throughout this section, we suppose that $\rho$ is a given positive number.

**Lemma 4.1.** If $f(x) \geq 0$ for all $x \in K$, then

$$G(Jx, y) \leq G(\varphi, y) + 2\rho f(y), \quad \forall \varphi \in B^*, \ y \in K, \ x \in \pi^K(\varphi).$$

**Proof.** From Theorem 3.2 ($f_3$), we know that

$$\langle \varphi - J(x), x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in K$$

and so

$$-2\langle \varphi, y \rangle + 2\rho f(y) \geq 2\|x\|^2 - 2\langle \varphi, x \rangle - 2\langle J(x), y \rangle + 2\rho f(x).$$

It follows that

$$\|\varphi\|^2 - 2\langle \varphi, y \rangle + \|y\|^2 + 2\rho f(y) \geq \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2 + 2\rho f(x) + \|Jx\|^2 - 2\langle J(x), y \rangle + \|y\|^2.$$

This implies that

$$G(\varphi, y) \geq G(Jx, y) + (\|\varphi\| - \|x\|)^2 + 2\rho f(x) - 2\rho f(y) \geq G(Jx, y) - 2\rho f(y)$$

and so

$$G(Jx, y) \leq G(\varphi, y) + 2\rho f(y), \quad \forall \varphi \in B^*, \ y \in K, \ x \in \pi^K(\varphi).$$

The proof is completed. \(\square\)

**Theorem 4.1.** Let $K$ be a nonempty compact convex subset of an uniformly convex and uniformly smooth Banach space $B$ with dual space $B^*$ and $0 \in K$. Let $A : K \rightarrow B^*$ be a continuous mapping and $f : K \rightarrow \mathbb{R}$ be convex, lower semi-continuous and positively homogeneous. Suppose that

1. $f(x) \geq 0$ for all $x \in K$ and $f(0) = 0$;
2. For any $x \in K$,

$$\langle J(x - \pi^K(Jx - \rho Ax)), J^*(Jx - J(x - \pi^K(Jx - \rho Ax))) \rangle \geq 0. \quad (4.1)$$

Let $x_0 \in K$ and the sequence $\{x_n\}$ be generated by the following iteration scheme:

$$x_{n+1} = \pi^K(Jx_n - \alpha_n J(x_n - \pi^K(Jx_n - \rho Ax_n))), \quad n = 0, 1, 2, \ldots, \quad (4.2)$$

where $\{\alpha_n\}$ satisfies the conditions:

(a) $0 \leq \alpha_n \leq 1$, for all $n = 0, 1, 2, \ldots$;
(b) $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$.

Then generalized variational inequality (1.1) has a solution $x^* \in K$ and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightarrow x^*, \quad as \ i \rightarrow \infty.$$
Proof. Since $B$ is an uniformly convex and uniformly smooth Banach space, we know that $J$ is a bijection from $B$ onto $B^*$ and uniformly continuous on any bounded subsets of $B$. Thus the sequence $\{x_n\}$ is well defined by the iteration scheme (4.2). Let $G_2(x, y) = G(Jx, y)$. Then,
\[
G_2(x_n, 0) = G(Jx_n, 0) = \|Jx_n\|^2 - 2\langle Jx_n, 0 \rangle + \|0\|^2 + 2 \rho f(0) = \|Jx_n\|^2.
\]
By Lemma 2.5 and condition (4.1),
\[
\|Jx_n - J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|^2 \\
\leq \|Jx_n\|^2 - 2\langle J(x_n - \pi_K^f (Jx_n - \rho Ax_n)), J^*(Jx_n - J(x_n - \pi_K^f (Jx_n - \rho Ax_n))) \rangle \\
\leq G_2(x_n, 0).
\]
Note that $K$ is a compact subset of $B$. Thus taking $p = 2$ and a suitable positive number $r$, we deduce from Lemma 2.4 that there exists a continuous, strictly increasing and convex function $g : R^+ \rightarrow R^+$ with $g(0) = 0$ such that
\[
\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|),
\]
for all $x, y \in B_r$ and $\lambda \in [0, 1]$, where $K \subseteq B_r$. This together with (4.3) and Lemma 4.1 implies that
\[
G_2(x_{n+1}, 0) = G(J\pi_K^f (Jx_n - \alpha_n J(x_n - \pi_K^f (Jx_n - \rho Ax_n))), 0) \\
\leq G(Jx_n - \alpha_n J(x_n - \pi_K^f (Jx_n - \rho Ax_n)), 0) + 2 \rho f(0) \\
= \|Jx_n - \alpha_n J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|^2 \\
= \|(1 - \alpha_n)Jx_n + \alpha_n (Jx_n - J(x_n - \pi_K^f (Jx_n - \rho Ax_n)))\|^2 \\
\leq (1 - \alpha_n)\|Jx_n\|^2 + \alpha_n \|Jx_n - J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|^2 \\
- \alpha_n (1 - \alpha_n)g(\|J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|) \\
= (1 - \alpha_n)G_2(x_n, 0) + \alpha_n \|Jx_n - J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|^2 \\
- \alpha_n (1 - \alpha_n)g(\|J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|) \\
\leq (1 - \alpha_n)G_2(x_n, 0) + \alpha_n G_2(x_n, 0) - \alpha_n (1 - \alpha_n)g(\|J(x_n - \pi_K^f (Jx_n - \rho Ax_n))\|) \\
= G_2(x_n, 0) - \alpha_n (1 - \alpha_n)g(\|x_n - \pi_K^f (Jx_n - \rho Ax_n)\|).
\]
Taking the sum for $n = 0, 1, 2, \ldots, m$ in the above inequality, we get
\[
\sum_{n=0}^{m} \alpha_n (1 - \alpha_n)g(\|x_n - \pi_K^f (Jx_n - \rho Ax_n)\|) \leq G_2(x_0, 0) - G_2(x_{m+1}, 0).
\]
Since $G_2(x_0, 0) = \|x_0\|^2$ and $G_2(x_{m+1}, 0) = \|x_{m+1}\|^2$,
\[
\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n)g(\|x_n - \pi_K^f (Jx_n - \rho Ax_n)\|) < \infty.
\]
From condition $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$ and inequality (4.4), it is easy to know that there exists a subsequence $\{x_{n_i}\} \subseteq \{x_n\}$ such that
\[
g(\|x_{n_i} - \pi_K^f (Jx_{n_i} - \rho Ax_{n_i})\|) \rightarrow 0, \quad as \ i \rightarrow \infty.
\]
Applying the properties of $g$, we have
\[
\|x_{n_i} - \pi_K^f (Jx_{n_i} - \rho Ax_{n_i})\| \rightarrow 0, \quad as \ i \rightarrow \infty.
\]
From the compactness of $K$, the sequence $\{x_{n_i}\}$ must have a subsequence which converges strongly to a point $x^* \in K$. Without loss of generality, we may assume that the subsequence is $\{x_{n_i}\}$ such that $x_{n_i} \rightarrow x^* \in K$ as $i \rightarrow \infty$. By Theorem 3.4, we conclude that
\[
x^* = \pi_K^f (Jx^* - \rho Ax^*).
\]
Now it follows from Theorem 3.3 that $x^*$ is a solution of generalized variational inequality (1.1). This completes the proof. □

**Corollary 4.1.** Let $H$ be a Hilbert space, $K$ a compact convex subset of $H$ and $0 \in K$. Let $A : K \to B^*$ be a continuous mapping and $f : K \to R$ be convex, lower semi-continuous and positively homogeneous. Suppose that

1. $f(x) \geq 0$ for all $x \in K$ and $f(0) = 0$;
2. For any $x \in K$,
   
   \[
   \langle x - \pi_K^f(x - \rho Ax), \pi_K^f(x - \rho Ax) \rangle \geq 0.
   \]

Let $x_0 \in K$ and the sequence $\{x_n\}$ be generated by the iteration scheme:

\[
x_{n+1} = \pi_K^f(x_n - \alpha_n(x_n - \pi_K^f(x_n - \rho Ax))), \quad n = 0, 1, 2, \ldots,
\]

where $\{\alpha_n\}$ satisfies the conditions:

(a) $0 \leq \alpha_n \leq 1$, for all $n = 0, 1, 2, \ldots$;
(b) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then generalized variational inequality (1.1) has a solution $x^* \in K$ and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

\[
x_{n_i} \to x^*, \quad \text{as } i \to \infty.
\]

**Proof.** Since $H^* = H$ and $J^* = J = I_H$ for a Hilbert space $H$, the conclusion follows from Theorem 4.1. This completes the proof. □

**Acknowledgements**

The authors are grateful to the referees for their valuable comments and suggestions.

**References**