Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 343 (2008) 525–530

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Master partial differential equations for a Type II hidden symmetry

Barbara Abraham-Shrauner [a,*], Keshlan S. Govinder [b]

[a] Department of Electrical and Systems Engineering, Washington University, St. Louis, MO 63130, USA
[b] Astrophysics and Cosmology Research Unit, School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, South Africa

Received 29 June 2007

Available online 2 February 2008

Submitted by P. Broadbridge

Abstract

An approach for determining a class of master partial differential equations from which Type II hidden point symmetries are inherited is presented. As an example a model nonlinear partial differential equation (PDE) reduced to a target PDE by a Lie symmetry gains a Lie point symmetry that is not inherited (hidden) from the original PDE. On the other hand this Type II hidden symmetry is inherited from one or more of the class of master PDEs. The class of master PDEs is determined by the hidden symmetry reverse method. The reverse method is extended to determine symmetries of the master PDEs that are not inherited. We indicate why such methods are necessary to determine the genesis of Type II symmetries of PDEs as opposed to those that arise in ordinary differential equations (ODEs).
© 2008 Elsevier Inc. All rights reserved.

Keywords: Lie group symmetries; Partial differential equations; Ordinary differential equations; Hidden symmetries

1. Introduction

Partial differential equations are widely used in science and engineering. Analytic solutions can be determined for nonlinear PDEs that support solitons by inverse scattering or are solved by the method of characteristics but this leaves a vast number of nonlinear PDEs that cannot be solved by these methods. However, PDEs and as well ODEs have been solved or simplified with the aide of Lie point group symmetries. The traditional approach has been to analyze the Lie algebra associated with the Lie group symmetry of the original differential equation and then to devise a reduction path (reduction in order for ODEs and reduction in the number of variables for PDEs). For both ODEs and PDEs the Lie algebra of the original differential equation does not give all the information when Lie point group symmetries are used in the reduction path. Here we present analysis of Type II hidden symmetries of PDEs that further illustrate this phenomenon. Type II hidden point symmetries of a PDE are Lie point symmetries that appear when the number of independent and dependent variables is reduced that are in addition to the Lie point symmetries inherited from the original PDE [1–3] for the reported cases. Additional Lie symmetries of reduced (target) PDEs had been reported [4–6] but their origin had not been explained until recently. These Type II hidden symmetries of PDEs should not be

* Corresponding author. Fax: +1 (314) 935 7500.
E-mail address: bas@wustl.edu (B. Abraham-Shrauner).

0022-247X/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2008.01.074

confused with other hidden symmetries of PDEs [7,8]. The provenance of these Type II hidden symmetries has been shown [1–3]. They are inherited symmetries of other PDEs that reduce to the target PDE. Type II hidden symmetries of ODEs have a different provenance where they arise from the reduction of the order of an ODE that has contact or nonlocal symmetries [9–11]. Type II hidden symmetries of PDEs may arise in the reduction by Lie point symmetries of the number of variables of a PDE from contact or potential symmetries. That possibility is not discussed here.

The aim here is to determine a wide class of PDEs or master PDEs that reduce to the target PDE and from which is inherited the Type II hidden symmetry of the original PDE. The reduction is restricted to reduction by Lie point symmetries where the Type II point hidden symmetry is a Lie point symmetry of another PDE. This is a complicated problem so that we start with a model nonlinear PDE with three Lie point symmetries that upon reduction is shown to have one Type II hidden symmetry. Our approach will be limited to those master PDEs which result when we deal with a fixed reduction operator (and so the reduction variables are fixed). In addition, we require the combination of symmetries for the master PDE to form a Lie algebra.

2. Type II point hidden symmetries of a model PDE

We start with

$$u_{xxx} + u(u_t + cu_x) + u_x u_{xx} = 0, \tag{1}$$

where the subscripts denote differentiation with respect to that variable. The Lie point symmetries found by the symbolic computer program LIE [12] are

$$U_1 = \frac{\partial}{\partial t}, \qquad U_2 = \frac{\partial}{\partial x}, \qquad U_3 = 3t\frac{\partial}{\partial t} + (x + 2ct)\frac{\partial}{\partial x}. \tag{2}$$

If we let $y = x - ct$, $w = u$ obtained with the variable transformation found from the Lie symmetry represented by $X_c = U_1 + cU_2$, then the target PDE is

$$w_{yyy} + w_y w_{yy} = 0. \tag{3}$$

The Lie point symmetries of (3) are represented by the group generators

$$V_1 = \frac{\partial}{\partial y}, \qquad V_2 = y\frac{\partial}{\partial y}, \qquad V_3 = \frac{\partial}{\partial w}. \tag{4}$$

The group generators V_1 and V_2 are inherited from the Lie symmetries of (1) as represented by (2) but V_3 represents a Type II hidden symmetry, i.e. a symmetry not inherited from (1).

3. Reverse method for PDEs with Type II hidden symmetries

We aim to determine other PDEs that reduce to the target PDE (3) but this time we shall require those PDEs to possess the Type II hidden symmetry of the original PDE (1), i.e. V_3 in (4). There are several possible approaches. First, one can guess the other PDEs but this ad hoc method may miss many possibilities. Second, one may find Lie point group generators by the reverse method for determining PDEs from which only the symmetries in (4) are inherited (this will ensure that the Type II hidden symmetry is also inherited). From these Lie point generators the differential invariants may be computed and then the general form of the PDE given. Third, one can apply the reverse method to determine additional (non-inherited) symmetries of the PDEs found in the second approach. We start with the second approach and illustrate the procedure by an example. In the reverse method we assume that the reduction of the number of variables of the PDE is by a Lie point symmetry. Then we assume that the Type II hidden symmetry originates from a point symmetry of another PDE as has been found for the examples referenced.

We assume a group generator X_a of a PDE that is a function of the independent variables (x, t) and the dependent variable $u(x, t)$ and its derivatives up to third order. We could consider higher-order derivatives but this should suffice for the demonstration. We introduce the group generator X_a that represents the symmetry that reduces to the Type II hidden symmetry represented by V_3 in (4) as

$$X_a = \xi_{ta}(x, t, u)\frac{\partial}{\partial t} + \xi_{xa}(x, t, u)\frac{\partial}{\partial x} + \eta_a(x, t, u)\frac{\partial}{\partial u}, \tag{5}$$

where subscripts here and in what follows are indices and do not denote differentiation. Then we require that X_a reduces to V_3 when $y = x - ct$, $w = u$ where this variable transformation follows from the invariants of X_c. The Lie symmetry that reduces the number of variables of the other PDEs is restricted here to the same Lie symmetry that reduced the model nonlinear PDE (1) to the target PDE (3). This leads to the conditions

$$X_a(x - ct) = 0, \qquad X_a(u) = C_a \tag{6}$$

for C_a a constant that is often set equal to one. Next, we require that the Lie point symmetry used to reduce (1) to (3) obeys

$$[X_c, X_a] = A_a X_c \tag{7}$$

for A_a a constant. This commutation relation usually ensures that the symmetry represented by X_a is inherited (see later for a qualifier of this statement). The solution of (6) and (7) leads to the infinite or pseudo Lie algebra

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 3t\frac{\partial}{\partial t} + (x + 2ct)\frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial u}, \qquad X_5 = t\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right), \\ &X_6 = f(x - ct)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right). \end{aligned} \tag{8}$$

(Note that f is an arbitrary function of its argument except that the linear part is excluded as it can be obtained by linear combinations of the other symmetries and we have assumed that ξ_{ta} is independent of u for ease of calculation.) The group generator X_a splits into three group generators X_4 to X_6. To understand this result we note that (6) yields

$$c\xi_{ta} - \xi_{xa} = 0, \qquad \eta_a = C_a. \tag{9}$$

Then (7) gives

$$\frac{\partial \xi_{ta}}{\partial t} + c\frac{\partial \xi_{ta}}{\partial x} = A_a. \tag{10}$$

From (10) we have that

$$\xi_{ta} = (A_a - bc)t + bx + f(x - ct). \tag{11}$$

The constant b will be taken as zero here for ease of calculation. (We note that other values of b will result in different forms for X_5, for example, when $b = A_a/c$ we find a form of ξ_{ta} that reduces X_a to xX_c.) We assume that the forms of the arbitrary function $f(x - ct)$ exclude the linear part of its argument in order to prevent an 'overcounting' of symmetries.

The possible PDEs can be constructed from the Lie algebra in (8) by computing the invariants. We compute the differential invariants for each group generator separately. The usual procedure computes the differential invariants for one group generator, constrains its differential invariants by the symmetries of a second group generator and then the others in succession. We do not do that since we are interested in the subalgebras of symmetries (and indeed, in the context of our problem, a rather specific subalgebra). The differential invariants for each of the symmetries in (8) are, to third order

$$\begin{aligned} X_1:\quad & (x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}), \\ X_2:\quad & (t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}), \\ X_3:\quad & \left(\frac{x - ct}{t^{1/3}}, u, \frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \frac{u_{xtt} + 2cu_{xxt} + c^2 u_{xxx}}{u_{xxx}^{7/3}},\right. \\ & \left.\frac{u_{ttt} + 3cu_{xtt} + 3c^2 u_{xxt} + c^3 u_{xxx}}{u_{xxx}^3}\right), \\ X_4:\quad & (t, x, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}), \\ X_5:\quad & \big[x - ct, u, u_x, t(u_t + cu_x), u_{xx}, t(u_{xt} + cu_{xx}), t^2\big(u_{tt} + 2cu_{xt} + c^2 u_{xx}\big), u_{xxx}, t(u_{xxt} + cu_{xxx}), \\ & t^2\big(u_{xtt} + 2cu_{xxt} + c^2 u_{xxx}\big), t^3\big(u_{ttt} + 3cu_{xtt} + 4c^2 u_{xxt} + 2c^3 u_{xxx}\big)\big], \\ X_6:\quad & \big(x - ct, u, u_t + cu_x, u_{tt} + 2cu_{xt} + c^2 u_{xx}, u_{ttt} + 3cu_{xtt} + 3c^2 u_{xxt} + c^3 u_{xxx}\big). \end{aligned} \tag{12}$$

A set of differential invariants of X_6 above will not reduce to (3) by the transformation $y = x - ct$, $w = u$. (We ignore the symmetries of X_6 where the linear part of the characteristic function ξ_{ta} is covered by the differential invariants of X_5 or an alternative form of X_5.)

The master PDEs that reduce to the target nonlinear PDE (3) from which the Type II hidden symmetry is inherited must include at least the symmetries X_c, X_3 and X_4. Other symmetries may be added but these extra symmetries restrict the general form of a master equation as can be seen from (12). The general form of a PDE which admits X_c, X_3 and X_4 is

$$F\left(\frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \frac{u_{xtt} + 2cu_{xxt} + c^2u_{xxx}}{u_{xxx}^{7/3}}, \frac{u_{ttt} + 3cu_{xtt} + 3c^2u_{xxt} + c^3u_{xxx}}{u_{xxx}^3}\right) = 0. \tag{13}$$

In order to ensure that we obtain (3) via X_c we restrict the form of (13) as follows:

$$\begin{aligned} u_{xxx} + u_x u_{xx} = u_x^3 \overline{F}\Bigg(&\frac{u_t + cu_x}{u_x^3}, \frac{u_{xx}}{u_x^2}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxx}}{u_x^3}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \\ &\frac{u_{xtt} + 2cu_{xxt} + c^2u_{xxx}}{u_{xxx}^{7/3}}, \frac{u_{ttt} + 3cu_{xtt} + 3c^2u_{xxt} + c^3u_{xxx}}{u_{xxx}^3}\Bigg) \\ \times G\Bigg(&\frac{u_t + cu_x}{u_x^3}, \frac{u_{tt} + cu_{xt}}{u_{xx}^3}, \frac{u_{xt} + cu_{xx}}{u_{xx}^2}, \frac{u_{xxt} + cu_{xxx}}{u_{xxx}^{5/3}}, \frac{u_{xtt} + 2cu_{xxt} + c^2u_{xxx}}{u_{xxx}^{7/3}}, \\ &\frac{u_{ttt} + 3cu_{xtt} + 3c^2u_{xxt} + c^3u_{xxx}}{u_{xxx}^3}\Bigg) \end{aligned} \tag{14}$$

such that $G(0, 0, 0, 0, 0, 0) = 0$. In addition singularities of G must also be avoided. As a result (14) is the general form of a master PDE that reduces to the target equation (3) via X_c and supplies the inherited Lie symmetry that is the Type II hidden symmetry in (3).

4. Reverse method for non-inherited Lie symmetries

We require that C_a equals zero in (6) and that the commutator below holds

$$[X_c, X_a] = X_b, \tag{15}$$

where X_b cannot be equal to $A_a X_c$. Then we have that

$$X_a = \xi_{ta} X_c, \qquad X_b = (X_c \xi_{ta}) X_c. \tag{16}$$

These conditions indicate that X_b cannot be equal to $X_d = U_1 - cU_2$, X_3, X_4. However, we can have

$$X_b = X_a. \tag{17}$$

This leads to

$$\xi_{ta} = \exp(t) g(x - ct, u). \tag{18}$$

If we let the arbitrary function g be a constant, then the master PDE is

$$\begin{aligned} u_{xxx} + u_x u_{xx} = F\big\{&x - ct, u, u_x, u_{xx}u_{xxx}, e^t(u_t + cu_x), e^t(u_{tx} + cu_{xx}), \\ &e^{2t}\big[\big(u_{tt} + 2cu_{xt} + c^2u_{xx}\big) + (u_t + cu_x)\big]\big\}, \end{aligned} \tag{19}$$

where we do not include all third-order terms here. It is evident that if we add X_b to the two in (13) and (14), then all the factors with exponentials in t must be combined to eliminate these factors. This restricts (14) further. If $g(x - ct, u)$ is a function of its arguments, more complicated invariants are found that restrict (14) also. The most general form of X_b is not obvious but $\xi_{ta} = h(t)g(x - ct, u)$ is possible where $h(t)$ is an arbitrary function. The examples indicated here further restrict the master equation (14). We have not found an X_a that increases the possible forms of (13).

5. Discussion

In the above we have illustrated how the reverse method could be used to obtain master PDEs that reduce to the target PDE in which Type II hidden symmetries were observed. One could argue that a more "systematic" approach should be adopted. The obvious direct route (a more general approach would be to look at *all* appropriate dimensioned Lie algebras) is as follows: Eq. (3) possesses the three symmetries given in (4). The sole non-zero Lie bracket relation for these symmetries is

$$[V_1, V_2] = V_1 \tag{20}$$

so that these symmetries form the Lie algebra $A_{2,1} \oplus A_1$, where we use the notation of Mubarakzyanov [13] as quoted in [14]. In order to determine the master PDE which reduced to Eq. (3), all we need to do is look for a Lie algebra which contains four elements, three of which are given in (4). Simply determining the PDE invariant under these four symmetries will yield the equation from which Eq. (3) has inherited symmetries not contained in Eq. (1). However, as we show below, this approach has some rather large pitfalls for the problem we consider.

For simplicity, we insist that the (new) fourth element of our required four-dimensional Lie algebra commutes with the other three. In other words, the Lie algebra we require is $A_{2,1} \oplus 2A_1$. Looking at Table 3 of [14], we find four options. The possibilities for the fourth symmetry are

$$V_4^1 = \frac{\partial}{\partial t}, \qquad V_4^2 = t\frac{\partial}{\partial y} \quad \left(\text{with } V_2 = y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right), \qquad V_4^3 = t\frac{\partial}{\partial w} \quad \left(\text{with } V_2 = y\frac{\partial}{\partial y} + \phi(t)\frac{\partial}{\partial w}\right) \quad \text{and}$$
$$V_4^4 = t\frac{\partial}{\partial y} \quad \left(\text{with } V_2 = y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w} + t\frac{\partial}{\partial t},\ V_3 = w\frac{\partial}{\partial y}\right).$$

The first case works, but corresponds to the rather trivial case of merely forcing $w = w(y)$ to become $w = w(y,t)$ which is obvious by inspection—there is thus little need for this approach as this case is more easily obtained. Using any of the symmetries of the other four-dimensional Lie algebras will not reduce that Lie algebra to the one in (4)! This is due to the fact that the Lie algebra of symmetries for PDEs does not always provide us with the relevant information to determine which symmetries remain after a reduction of variables (see also [15] in this regard). Specifically, if we look at the Lie algebra $A_{2,1} \oplus 2A_1$, we would expect that, if we "use up" one of the symmetries forming part of the $2A_1$ subalgebra, we would be left with $A_{2,1} \oplus A_1$. This certainly holds if the symmetries are symmetries of ODEs. However, in the case of PDEs, this is not the case. In fact except for the case mentioned above, all the other cases reduce to 2-dimensional or 1-dimensional Lie algebras.

In the case of ODEs, if two point symmetries commute, then reduction via either one of the symmetries would result in the other (suitably transformed) being a point symmetry of the reduced equation. That this is not always the case for PDEs was observed in [15] and can be illustrated via a simple example. Take a PDE in which $u = u(x,t)$ and which admits the symmetries

$$G_1 = \frac{\partial}{\partial x}, \qquad G_2 = \theta(t,u)\frac{\partial}{\partial x}. \tag{21}$$

These symmetries clearly commute. The symmetry G_1 defines $p = t$ and $q = u$ as reduction variables for the PDE. However, these variables mean that the remaining symmetry, G_2, does not have any relevance for the reduced equation (which is now in the new variables p and q). As a result, the only methods thus far available must entail the rather intricate approach of determining the possible "original" symmetries via reversal of transformations and a calculation of the necessary invariants.

6. Conclusion

A procedure for determining a wide class of master partial differential equations has been presented. The master partial differential equations possess Lie point symmetries that are inherited by a target PDE but are Type II hidden symmetries of the original PDE. The hidden symmetry reverse method has been used to determine the master PDE for a model nonlinear PDE. This original PDE has three Lie point symmetries. A Type II hidden symmetry is found when this PDE is reduced to the target PDE. The wide class of master PDEs is found by assuming these PDEs possess the Lie symmetry used to reduce the original PDE and the Lie symmetry that is inherited as the Type II hidden symmetry. Two approximations are made in the form of the characteristic function ξ_{ta} but these are not expected to alter the conclusions. The differential invariants of the two symmetries give the general form for the PDEs. A further restriction on the form is that many invariants vanish in the reduction. The class of possible Lie point symmetries of the master PDEs has been expanded to include Lie symmetries not inherited. These symmetries appear to restrict the class of the master PDEs.

Acknowledgments

The research of B. Abraham-Shrauner was partially supported by the Professional Development Fund of the School of Engineering and Applied Science of Washington University. K.S. Govinder thanks the University of KwaZulu-Natal and the National Research Foundation of South Africa for continuing support.

References

[1] B. Abraham-Shrauner, K.S. Govinder, D.J. Arrigo, Type-II hidden symmetries of the linear 2D and 3D wave equations, J. Phys. A: Math. Gen. 39 (2006) 5739–5747.
[2] B. Abraham-Shrauner, K.S. Govinder, Provenance of Type II hidden symmetries from nonlinear partial differential equations, J. Nonlinear Math. Phys. 13 (2006) 612–622.
[3] B. Abraham-Shrauner, Type II hidden symmetries of the second heavenly equation, Phys. Lett. A 369 (4) (2007) 299–302.
[4] M.P. Edwards, P. Broadbridge, Exceptional symmetry reductions of Burgers' equation in two and three spatial dimensions, Z. Angew. Math. Phys. 46 (1995) 595–622.
[5] H. Stephani, Differential Equations, Cambridge Univ. Press, Cambridge, 1989.
[6] B. Abraham-Shrauner, Type II hidden symmetries of some partial differential equations, 1005th AMS Meeting, Newark, Delaware, 2005, pp. 22, 37.
[7] R.J. Moitsheki, P. Broadbridge, M.P. Edwards, Systematic construction of hidden nonlocal symmetries for the inhomogeneous nonlinear diffusion equation, J. Phys. A 37 (2004) 8279–8286.
[8] I. Yehorchenko, Group classification with respect to hidden symmetry, Proc. Inst. Math. NAS Ukraine 50 (2004) 290–297.
[9] K.S. Govinder, P.G.L. Leach, On the determination of nonlocal symmetries, J. Phys. A 28 (1995) 5349–5359.
[10] B. Abraham-Shrauner, P.G.L. Leach, K.S. Govinder, G. Ratcliff, Hidden symmetries and contact symmetries of ordinary differential equations, J. Phys. A 28 (1995) 6707–6716.
[11] B. Abraham-Shrauner, Hidden symmetries and nonlocal group generators for ordinary differential equations, IMA J. Appl. Math. 56 (1996) 235–252.
[12] A. Head, LIE, a PC program for Lie analysis of differential equations, Comput. Phys. Comm. 71 (1993) 241–248; Comput. Phys. Comm. 96 (1996) 311–313.
[13] G.M. Mubarakzyanov, On solvable Lie algebras, Izv. Vyssh. Uchebn. Zaved. Mat. 32 (1) (1963) 114–123.
[14] R.O. Popovych, V.M. Boyko, M.O. Nesterenko, M.W. Lutfullin, Realization of real low-dimensional Lie algebras, J. Phys. A: Math. Gen. 36 (2003) 7337–7360.
[15] K.S. Govinder, Lie subalgebras, reduction of order and group invariant solutions, J. Math. Anal. Appl. 258 (2001) 720–732.