Pointwise approximation by Bézier variant of integrated MKZ operators

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Received 28 August 2005
Available online 18 April 2007
Submitted by K. Jarosz

Abstract

In this paper the pointwise approximation of Bézier variant of integrated MKZ operators for general bounded functions is studied. Two estimate formulas of this type approximation are obtained. The approximation of functions of bounded variation becomes a special case of the main result of this paper. In the case of functions of bounded variation, Theorem B of the paper corrects the mistake of Theorem 1 of the article [V. Gupta, Degree of approximation to functions of bounded variation by Bézier variant of MKZ operators, J. Math. Anal. Appl. 289 (2004) 292–300].

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Keywords: Approximation; Bounded functions; Durrmeyer type operators; Bounded variation; Meyer-König and Zeller operators

1. Introduction

In 1988 Guo [1] introduced the Durrmeyer type integral modification of the Meyer-König and Zeller operators as

\[
\hat{M}_n(f, x) = \sum_{k=1}^{\infty} p_{n,k+1}(x) (n + k - 2)(n + k - 3) \frac{1}{n - 2} \int_0^1 p_{n-2,k-1}(t) f(t) \, dt,
\]

(1)

The present investigation was supported by NSFC under Grant 10571145.
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where
\[ p_{n,k}(x) = \binom{n + k - 1}{k} x^k (1 - x)^n, \quad x \in [0, 1], \]
are Meyer-König and Zeller basis functions.

In 2000 Zeng and Chen [2] introduced the Bézier variant of Bernstein–Durrmeyer operators \(D_{n,\alpha}\) and studied the approximation properties of \(D_{n,\alpha}\) for functions of bounded variation. Zeng [3] also introduced two kinds Bézier variant of MKZ operators and studied their approximation properties for functions of bounded variation. Recently Gupta [4] introduced another Bézier variant of MKZ operator \(\hat{M}_{n,\alpha}\) defined as
\[
\hat{M}_{n,\alpha}(f, x) = \sum_{k=1}^{\infty} Q_{n,k+1}^{(\alpha)}(x) \frac{(n + k - 2)(n + k - 3)}{n - 2} \int_{0}^{1} p_{n-2,k-1}(t) f(t) \, dt,
\]
where \(f \in L_1[0, 1], n \in \mathbb{N}, \alpha \geq 1, x \in [0, 1]\), \(Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\) and \(J_{n,k}(x) = \sum_{j=k}^{\infty} p_{n,j}(x)\). In the special case \(\alpha = 1\), the operators \(\hat{M}_{n,\alpha}(f, x)\) reduce to the operators \(\hat{M}_{n,1}(f, x)\). Gupta [4] studied the degree of approximation to function of bounded variation by operators \(\hat{M}_{n,\alpha}\) and gave the following result:

**Theorem 1.** Let \(f\) be a function of bounded variation on \([0, 1]\), \(\alpha \geq 1\) and \(\mu > 2\). Then for every \(x \in (0, 1)\) and \(n\) sufficiently large, we have
\[
\begin{align*}
\left| \hat{M}_{n,\alpha}(f, x) - \left[ \frac{1}{\alpha + 1} f(x+) + \frac{\alpha}{\alpha + 1} f(x-) \right] \right| \\
\leq \frac{3\alpha}{\sqrt{n}x^{3/2}} \left| f(x+) - f(x-) \right| + \frac{2\mu\alpha + x}{nx} \sum_{k=1}^{n} \sqrt{\frac{x+(1-x)/\sqrt{k}}{x-x/\sqrt{k}}} (g_x),
\end{align*}
\]
where \(\sqcup_{a}^{b}(g_x)\) is total variation of \(g_x\) on \([a, b]\) and
\[
g_x(t) = \begin{cases} 
  f(t) - f(x-), & 0 \leq t < x, \\
  0, & t = x, \\
  f(t) - f(x+), & x < t \leq 1.
\end{cases}
\]

Unfortunately, Theorem 1 is incorrect. In fact, if we take the function of bounded variation \(f(t) \equiv 1, t \in [0, 1]\), then \(g_x(t) \equiv 0\) and \(f(x+) = f(x-)\) for \(x \in (0, 1)\). Thus the estimate (3) will derive a contradictory inequality
\[
0 < \left| \hat{M}_{n,1}(1, x) - 1 \right| \leq 0.
\]
In fact, there are two mistakes in [4] which make Theorem 1 incorrect:

1. The author of [4, p. 293] mistook that \(\hat{M}_{n,1}(1, x) = 1\). However, Guo [1, p. 13] has showed that \(\hat{M}_{n}(1, x) = 1 + o(n^{-2})\). In fact, note that \(\sum_{k=0}^{\infty} p_{n,k}(x) = 1\) (cf. [5, p. 165]), by an easy computation, we have
\[
\hat{M}_{n}(1, x) = \sum_{k=1}^{\infty} p_{n,k+1}(x) \frac{(n + k - 2)(n + k - 3)}{n - 2} \int_{0}^{1} p_{n-2,k-1}(t) f(t) \, dt
\]
In approximation theory, Durrmeyer type operators are extremely suitable for approximation of integrable functions, they are also suitable for approximation of continuous functions, bounded variation functions and absolutely continuous functions. In view of the importance of Durrmeyer of integrable functions, they are also suitable for approximation of continuous functions, bounded variation functions and absolutely continuous functions. In view of the importance of Durrmeyer type operators in approximation theory, in this paper we will re-discuss the approximation properties of operators $M_{n,\alpha}$ for a class of function $I_B$ defined as follows

$$I_B = \{ f : f \text{ is bounded on } [0, 1] \text{ and } f(x \pm) \text{ exist at } x \in (0, 1) \}.$$ 

It is clear that class $I_B$ is more wide-ranging than the class of functions of bounded variation which was considered in [4]. We introduce the following three quantities

$$\Omega_{x-}(f, \delta_1) = \sup_{t \in [x-\delta_1, x]} |f(t) - f(x)|, \quad \Omega_{x+}(f, \delta_2) = \sup_{t \in [x, x+\delta_2]} |f(t) - f(x)|, \quad \Omega(x, f, \lambda) = \sup_{t \in [x-x/\lambda, x+(1-x)/\lambda]} |f(t) - f(x)|,$$

where $f \in I_B$, $x \in [0, 1]$ is fixed, $0 \leq \delta_1 \leq x$, $0 \leq \delta_2 \leq 1 - x$ and $\lambda \geq 1$. For the properties of $\Omega_{x-}(f, \delta_1)$, $\Omega_{x-}(f, \delta_1)$, $\Omega(x, f, \lambda)$, refer to Zeng and Cheng [6, p. 244]. By means of the technique lines of [2,6], we obtain the main results of this paper as follows.

**Theorem A.** Let $f \in I_B$, $\alpha \geq 1$. Then for every $x \in (0, 1)$ and $n$ sufficiently large, we have

$$\left| \hat{M}_{n,\alpha}(f, x) - \left[ \frac{1}{\alpha + 1} f(x+) + \frac{\alpha}{\alpha + 1} f(x-) \right] \right| \leq \frac{6\alpha + x}{nx} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}) + \frac{6\alpha |f(x+) - f(x-)|}{\sqrt{nx}} + \frac{|f(x+) - f(x-)|}{2} o(n^{-2}), \quad (5)$$

where $g_x(t)$ is defined in (4).

**Theorem B.** Let $f$ be a function of bounded variation on $[0, 1]$, $\alpha \geq 1$. Then for every $x \in (0, 1)$ and $n$ sufficiently large, we have

$$\left| \hat{M}_{n,\alpha}(f, x) - \left[ \frac{1}{\alpha + 1} f(x+) + \frac{\alpha}{\alpha + 1} f(x-) \right] \right| \leq \frac{6\alpha + x}{nx} \sum_{k=1}^{n} \sqrt{x+(1-x)/\sqrt{k}} g_x(t) + \frac{6\alpha |f(x+) - f(x-)|}{\sqrt{nx}} + \frac{|f(x+) - f(x-)|}{2} o(n^{-2}).$$

(6)

Theorem B corrects the mistake of Theorem 1 of [4].
2. Preliminary results

In this section we give some preliminary results, which are necessary to prove Theorems A and B.

Lemma 1. (See [7] or [2, Lemma 2].) Let \( \{\xi_k\}_{k=1}^\infty \) be a sequence of independent and identically distributed random variables with the expectation \( E(\xi_1) = a_1 \), the variance \( E(\xi_1 - a_1)^2 = \sigma^2 > 0 \), \( E|\xi_1 - a_1|^3 = \rho < \infty \), and let \( F_n \) stand for the distribution function of \( \sum_{k=1}^n (\xi_k - a_1)/\sigma \sqrt{n} \). Then there exists an absolute constant \( C, 1/\sqrt{2\pi} \leq C < 0.8 \), such that for all \( t \) and \( n \)
\[
|F_n(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du| < \frac{C\rho}{\sigma^3 \sqrt{n}}. \tag{7}
\]

Lemma 2. (See [3].) If \( \{\xi_i\} \ (i = 1, 2, \ldots) \) are independent random variables with the same geometric distribution \( P(\xi_i = k) = x^k (1-x) \), \( x \in (0, 1) \), then
\[
E(\xi_i) = \frac{x}{1-x}, \quad E(\xi_i - E(\xi_i))^2 = \frac{x}{(1-x)^2}, \quad E(\xi_i - E(\xi_i))^4 = \frac{x^3 + 7x^2 + x}{(1-x)^4} < \infty,
\]
\[
E|\xi_1 - a_1|^3 = \rho \leq \frac{3x}{(1-x)^3},
\]
and \( \eta_n = \sum_{i=1}^n \xi_i \) is a random variable with distribution
\[
P(\eta_n = k) = \binom{n+k-1}{k} x^k (1-x)^n.
\]

Lemma 3. For \( k = 1, 2, \ldots \), we have
\[
\frac{1}{\sqrt{2\pi}} \int_{I_{n,k}} e^{-t^2/2} dt \leq \frac{1.2}{\sqrt{nx}}, \tag{8}
\]
where
\[
I_{n,k} = \left[ \frac{k - nx/(1-x)}{\sqrt{nx}/(1-x)}, \frac{k - (n-1)x/(1-x)}{\sqrt{(n-1)x}/(1-x)} \right].
\]

Proof. Let
\[
A_1 = \frac{k - nx/(1-x)}{\sqrt{nx}/(1-x)}, \quad A_2 = \frac{k - (n-1)x/(1-x)}{\sqrt{(n-1)x}/(1-x)}.
\]
Then
\[
A_2 - A_1 = \frac{k(1-x) + x\sqrt{n}\sqrt{n-1}}{\sqrt{x}\sqrt{n}\sqrt{n-1}(\sqrt{n} + \sqrt{n-1})} \geq 0.
\]
If \( k \leq \frac{3nx}{1-x} \), then
\[
A_2 - A_1 = \frac{k(1-x) + x\sqrt{n}\sqrt{n-1}}{\sqrt{x}\sqrt{n}\sqrt{n-1}(\sqrt{n} + \sqrt{n-1})} \leq \frac{3\sqrt{nx} + (n-1)x}{\sqrt{n-1}(\sqrt{n} + \sqrt{n-1})} \leq \frac{3}{\sqrt{nx}} \quad (n \geq 3).
\]
Thus
\[ \frac{1}{\sqrt{2\pi}} \int_{I_{n,k}} e^{-t^2/2} dt \leq \frac{1}{\sqrt{2\pi}} |A_2 - A_1| \leq \frac{1.2}{\sqrt{n}}. \]

If \( k > \frac{3nx}{1-x} \), then any upper bound does not exist for quantity \( A_2 - A_1 \). Note that \( A_1 > 0 \), and
\[ \frac{k(1-x) + x\sqrt{n}\sqrt{n-1}}{k(1-x) - nx} < 2. \]

Thus
\[ \frac{1}{\sqrt{2\pi}} \int_{A_1}^{A_2} e^{-t^2/2} dt \leq \frac{e^{-A_1^2/2}}{\sqrt{2\pi}} (A_2 - A_1) \leq \frac{A_2 - A_1}{A_1^2/2} \]
\[ = \frac{2(k(1-x) + x\sqrt{n}\sqrt{n-1})\sqrt{n}}{(k(1-x) - nx)^2\sqrt{n-1}((\sqrt{n} + \sqrt{n-1})} \]
\[ \leq \frac{4\sqrt{n}}{(k(1-x) - nx)\sqrt{n-1}((\sqrt{n} + \sqrt{n-1})} < \frac{1}{\sqrt{n}} (n \geq 2). \]

The proof of Lemma 3 is complete. \( \square \)

**Lemma 4.** For all \( x \in (0, 1) \) and \( k \in N \) we have
\[ |J_{nk}^\alpha(x) - J_{n-1,k}^\alpha(x)| \leq \frac{7\alpha}{\sqrt{n}} \]  \hfill (9)
and
\[ |J_{nk}^\alpha(x) - J_{n-1,k+1}^\alpha(x)| \leq \frac{7\alpha}{\sqrt{n}}. \]  \hfill (10)

**Proof.** Let
\[ A_1 = \frac{k - nx/(1-x)}{\sqrt{n}}, \quad A_2 = \frac{k - (n-1)x/(1-x)}{\sqrt{(n-1)x/(1-x)}}, \]
and \( I_{n,k} = [A_1, A_2] \).

Using Lemmas 1, 2 and a simple computation we have
\[ \left| \sum_{j=0}^{k} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_i} e^{-t^2/2} dt \right| \leq \frac{2.4}{\sqrt{n}}, \quad i = 1, 2. \]

Thus by Lemma 3 and using the inequality \(|a^\alpha - b^\alpha| \leq \alpha|a - b| (0 \leq a, b \leq 1, \alpha \geq 1)\) (cf. [2, p. 4]), we obtain
\[ |J_{nk}^\alpha(x) - J_{n-1,k}^\alpha(x)| \]
\[ \leq \alpha \left| \sum_{j=0}^{k} p_{n,j}(x) - \sum_{j=0}^{k} p_{n-1,j}(x) \right| \]
\[
\leq \alpha \left| \sum_{j=0}^{k} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1} e^{-t^2/2} dt \right| + \alpha \left| \sum_{j=0}^{k} p_{n-1,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_2} e^{-t^2/2} dt \right|
+ \frac{\alpha}{\sqrt{2\pi}} \int_{l_{n,k}} e^{-t^2/2} dt
\leq 2.4\alpha \frac{\sqrt{n}}{\sqrt{n}} + 1.2\alpha \frac{\sqrt{n}}{\sqrt{n}} = 6\alpha.
\]

The proof of inequality (10) is similar. \(\square\)

**Lemma 5.** (See [1, Lemma 4].) For \(k \geq 1, x \in (0, 1)\) and \(n > 2\), we have

\[
\sum_{j=0}^{k-1} p_{n-1,j}(x) = \frac{(n+k-2)(n+k-3)}{n-2} \int_{x}^{1} p_{n-2,k-1}(t) dt.
\]  

(11)

**Lemma 6.** Let

\[
K_{n,\alpha}(x, t) = \sum_{k=1}^{\infty} \frac{(n+k-2)(n+k-3)}{n-2} Q_{n,k+1}^{(\alpha)}(x) \int_{0}^{t} p_{n-2,k-1}(u) du.
\]  

(12)

Then for \(n\) sufficiently large, we have

\[
K_{n,\alpha}(x, y) \leq \frac{3\alpha x(1-x)^2}{n(x-y)^2}, \quad 0 \leq y < x.
\]  

(13)

**Proof.** By direct computation

\[
K_{n,\alpha}(x, y) \leq \sum_{k=1}^{\infty} \frac{(n+k-2)(n+k-3)}{n-2} Q_{n,k+1}^{(\alpha)}(x) \int_{0}^{y} \frac{(x-u)^2}{(x-y)^2} p_{n-2,k-1}(u) du
\]

\[
\leq \hat{M}_{n,\alpha}((t-x)^2, x) \frac{\alpha}{(x-y)^2}
\]

and by a result of Guo [1, Lemma 6]

\[
\hat{M}_{n,1}((t-x)^2, x) = \frac{2x(1-x)^2}{(n-2)} + o(n^{-2}).
\]  

(15)

Inequality (13) holds from (14) and (15). \(\square\)

3. Proofs of Theorems A and B

Let \(f \in I_B\). Then \(f(t)\) can be expressed as

\[
f(t) = \frac{1}{\alpha + 1} f(x+) + \frac{\alpha}{\alpha + 1} f(x-) + \frac{f(x+)+ f(x-)}{2} \left[ \text{sgn}(t-x) + \frac{\alpha-1}{\alpha+1} \right]
+ g_{\alpha}(t) + \delta_{\alpha}(t) \left[ f(x) - \frac{f(x+)+ f(x-)}{2} \right],
\]  

(16)
where $g_x(t)$ is defined in (4), $\text{sgn}(t)$ is sign function and
\[
\delta_x(t) = \begin{cases} 
1, & t = x, \\
0, & t \neq x.
\end{cases}
\]

It is obvious that $\hat{M}_{n,\alpha}(\delta_x, x) = 0$. Note that $\hat{M}_n(1, x) = 1 + o(n^{-2})$ [1, p. 13], a simple computation shows that $\hat{M}_{n,\alpha}(1, x) = (\hat{M}_n(1, x))^\alpha = 1 + o(n^{-2})$. Thus from (16) we have
\[
\left| \hat{M}_{n,\alpha}(f, x) - \left[ \frac{1}{\alpha + 1} f(x+) + \frac{\alpha}{\alpha + 1} f(x-) \right] \right|
\leq \left| \hat{M}_{n,\alpha}(g_x, x) \right| + \frac{|f(x+) - f(x-)|}{2} \left| \hat{M}_{n,\alpha}(\text{sgn}(t - x), x) + \frac{\alpha - 1}{\alpha + 1} \right|
+ \frac{|f(x+) - f(x-)|}{2} o(n^{-2}).
\] (17)

First, we apply the technique lines of [2] to estimate $|\hat{M}_{n,\alpha}(\text{sgn}(t - x), x)|$. By definitions and using Lemma 5, we have
\[
\left| \hat{M}_{n,\alpha}(\text{sgn}(t - x), x) \right|
= \sum_{k=1}^{\infty} Q_{n,k+1}^{(\alpha)}(x) \frac{(n - 2)(n + k - 3)}{n - 2} \left( - \int_0^x p_{n-2,k-1}(t) \, dt + \int_x^1 p_{n-2,k-1}(t) \, dt \right)
\leq -1 + 2 \sum_{k=1}^{\infty} Q_{n,k+1}^{(\alpha)}(x) \frac{(n - 2)(n + k - 3)}{n - 2} \int_x^1 p_{n-2,k-1}(t) \, dt
= -1 + 2 \sum_{k=1}^{\infty} Q_{n,k+1}^{(\alpha)}(x) \sum_{j=0}^{k-1} p_{n-1,j}(x)
= -1 + 2 \sum_{j=0}^{\infty} p_{n-1,j}(x) \sum_{k=j}^{\infty} Q_{n,k+1}^{(\alpha)}(x)
= -1 + 2 \sum_{j=0}^{\infty} p_{n-1,j}(x) J_{n,j+1}^{\alpha}(x).
\]

Thus
\[
\hat{M}_{n,\alpha}(\text{sgn}(t - x), x) + \frac{\alpha - 1}{\alpha + 1} = 2 \sum_{j=0}^{\infty} p_{n-1,j}(x) J_{n,j+1}^{\alpha}(x) - \frac{2}{\alpha + 1}
= 2 \sum_{j=0}^{\infty} p_{n-1,j}(x) J_{n,j+1}^{\alpha}(x) - \frac{2}{\alpha + 1} \sum_{j=0}^{\infty} Q_{n-1,j}^{(\alpha+1)}(x).
\]

By mean value theorem
\[
Q_{n-1,j}^{(\alpha+1)}(x) = J_{n-1,j}^{\alpha+1}(x) - J_{n-1,j+1}^{\alpha+1}(x) = (\alpha + 1) p_{n-1,j}(x) \gamma_{n,j}^{(\alpha)}(x),
\] (18)
where $J_{n-1,j+1}(x) < \gamma_{n,j}(x) < J_{n-1,j}(x)$.

Hence it follows from Lemma 4 and (18) that
Next, we estimate $|\hat{M}_{n,\alpha}(g_x,x)|$. By Lebesgue–Stieltjes integral representation, we have

$$ \hat{M}_{n,\alpha}(g_x,x) = \int_0^1 g_x(t) \, d_t K_{n,\alpha}(x,t) = \int_{I_1} + \int_{I_2} + \int_{I_3} g_x(t) \, d_t K_{n,\alpha}(x,t) $$

where

$$ I_1 = [0, x - x/\sqrt{n}], \quad I_2 = [x - x/\sqrt{n}, x + (1 - x)/\sqrt{n}], \quad I_3 = [x + (1 - x)/\sqrt{n}, 1]. $$

We shall evaluate $\Delta_1,n, \Delta_2,n, \Delta_3,n$. Firstly, for $\Delta_2,n$ note that $g_x(x) = 0$ we have

$$ |\Delta_2,n| \leq \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(t) \, d_t K_{n,\alpha}(x,t) \leq \Omega(x, g_x, \sqrt{n}) \leq \frac{1}{n} \sum_{k=1}^n \Omega(x, g_x, \sqrt{k}). $$

(20)

Next, we estimate $|\Delta_1,n|$: 

$$ |\Delta_1,n| = \int_{x-x/\sqrt{n}}^{x-x/\sqrt{n}} g_x(t) \, d_t K_{n,\alpha}(x,t) \leq \int_0^\frac{y}{n} \Omega_x^-(g_x, x - t) \, d_t K_{n,\alpha}(x,t). $$

Integration by parts with $y = x - x/\sqrt{n}$, we obtain

$$ \int_0^\frac{y}{n} \Omega_x^-(g_x, x - t) \, d_t K_{n,\alpha}(x,t) \leq \Omega_x^-(g_x, x - y) K_{n,\alpha}(x,y) + \int_0^y K_{n,\alpha}(x,t) \, d_t (-\Omega_x^-(g_x, x - t)). $$

(21)

Thus using Lemma 6 it follows that

$$ |\Delta_1,n| \leq \Omega_x^-(g_x, x - y) \frac{3\alpha x(1-x)^2}{n(x-y)^2} + \frac{3\alpha x(1-x)^2}{n} \int_0^\frac{y}{n} \frac{1}{(x-t)^2} \, d_t (-\Omega_x^-(g_x, x - t)). $$

(22)
Integrating by parts in the last integral of (22) we get
\[
\int_0^y \frac{1}{(x-t)^2} d_t (-\Omega_x (g_x, x-t)) = -\frac{1}{(x-t)^2} \Omega_x (g_x, x-t) \bigg|_0^y + \int_0^y \frac{2 \Omega_x (g_x, x-t)}{(x-t)^3} dt.
\]  
(23)

So we have from (22), (23):
\[
|\Delta_{1,n}| \leqslant \frac{3\alpha x (1-x)^2}{nx^2} \Omega_x (g_x, x) + \frac{3\alpha x (1-x)^2}{n} \int_0^{x-x/\sqrt{n}} \Omega_x (g_x, x-t) \frac{2}{(x-t)^3} dt.
\]

Putting \( t = x - x/\sqrt{n} \) for the last integral we get
\[
\int_0^{x-x/\sqrt{n}} \Omega_x (g_x, x-t) \frac{2}{(x-t)^3} dt = \frac{1}{x^2} \int_1^n \Omega_x (g_x, x/\sqrt{u}) du
\]
\[
\leqslant \frac{1}{x^2} \sum_{k=1}^n \Omega_x (g_x, x/\sqrt{k}).
\]

Consequently
\[
|\Delta_{1,n}| \leqslant \frac{3\alpha (1-x)^2}{nx} \left( \Omega_x (g_x, x) + \sum_{k=1}^n \Omega_x (g_x, x/\sqrt{k}) \right)
\]
\[
\leqslant \frac{6\alpha (1-x)^2}{nx} \sum_{k=1}^n \Omega_x (g_x, x/\sqrt{k}) \leqslant \frac{6\alpha (1-x)^2}{nx} \sum_{k=1}^n \Omega (x, g_x, \sqrt{k}).
\]  
(24)

Using the similar method to estimate \( |\Delta_{3,n}| \) we get
\[
|\Delta_{3,n}| \leqslant \frac{6\alpha x^2}{nx} \sum_{k=1}^n \Omega (x, g_x, \sqrt{k}).
\]  
(25)

From (20), (24) and (25) we obtain
\[
|\hat{M}_{n,\alpha} (g_x, x)| \leqslant |\Delta_{1,n}| + |\Delta_{2,n}| + |\Delta_{3,n}| \leqslant \frac{6\alpha + x}{nx} \sum_{k=1}^n \Omega (x, g_x, \sqrt{k}).
\]  
(26)

Theorem A now follows from (17), (19) and (26).

If \( f \) is a function of bounded variation on \([0, 1]\), then \( f \in I_B \), and obviously,
\[
\sum_{k=1}^n \Omega (x, g_x, \sqrt{k}) \leqslant \sum_{k=1}^n \sqrt{\frac{n+x(1-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x)}.
\]

Thus we obtain Theorem B from Theorem A immediately.
Acknowledgments

The author thanks the associate editor and the referee for several important comments and suggestions which improved the quality of this article.

References