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Representable Functors with Values in Arbitrary Categories*

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INTRODUCTION

The representable functors from a category $\mathbf{A}$ to the category of sets $\mathbf{S}$ have the following basic properties:

(1) Let $[A, -] : \mathbf{A} \to \mathbf{S}$, $A \in \mathbf{A}$, be a representable functor and $t : \mathbf{A} \to \mathbf{S}$ be an arbitrary functor. Then there exists a bijection

$$[[A, -], t] \cong tA$$

which is natural in A and t.

(2) Each functor $t : \mathbf{A} \to \mathbf{S}$ is canonically a direct limit of representable functors. (In general the index category is not small.) In other words, the Yoneda embedding $Y^{\mathbf{A}} : \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{S})$, $A \rightsquigarrow [A, -]$, is dense (cf. [25] 1.3 and [25] 1.10).

In this paper we shall define a concept of "representable functor" in an arbitrary functor category $(\mathbf{A}, \mathbf{B})$ in such a way that properties similar to (1) and (2) hold.

For this purpose we first consider the case where $\mathbf{B}$ is right complete and has a small dense ([25] 1.3) subcategory $\bar{\mathbf{B}}$. Let $I : \bar{\mathbf{B}} \to \mathbf{B}$ denote the inclusion. Then by [25] 1.15 $\mathbf{B}$ is a left retract of $(\bar{\mathbf{B}}^{opp}, \mathbf{S})$, i.e., the canonical embedding $S : \mathbf{B} \to (\bar{\mathbf{B}}^{opp}, \mathbf{S}), B \rightsquigarrow [I-, B]$, has a left adjoint $T : (\bar{\mathbf{B}}^{opp}, \mathbf{S}) \to \mathbf{B}$ and the end adjunction $TS \to \mathrm{id}_{\mathbf{B}}$ is an equivalence. Therefore the induced functor $(\mathbf{A}, T) : (\mathbf{A}, (\bar{\mathbf{B}}^{opp}, \mathbf{S})) \to (\mathbf{A}, \mathbf{B})$ is also a left retraction.

Let $\bar{B}$ be an object of $\bar{\mathbf{B}}$. Denote by $\bar{B}\otimes : \mathbf{S} \to \mathbf{B}$ the functor $M \rightsquigarrow \oplus_{m \in M} \bar{B}_m$, where $\bar{B}_m = \bar{B}$. One readily verifies that $\bar{B}\otimes$ is left adjoint to $[I\bar{B}, -] : \mathbf{B} \to \mathbf{S}$.

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(3) In 1.33 we show that the composition

$$(\mathbf{A} \times \bar{\mathbf{B}}^{opp}, \mathbf{S}) \cong (\mathbf{A}, (\bar{\mathbf{B}}^{opp}, \mathbf{S})) \xrightarrow{(\mathbf{A},T)} (\mathbf{A}, \mathbf{B})$$

assigns to a representable functor $[A \times \bar{B}, -] : \mathbf{A} \times \bar{\mathbf{B}}^{opp} \to \mathbf{S}$, $A \in \mathbf{A}$, $\bar{B} \in \bar{\mathbf{B}}$, the composed functor[1]

(4) $$\mathbf{A} \xrightarrow{[A,-]} \mathbf{S} \xrightarrow{\bar{B}\otimes} \mathbf{B}.$$

(5) Since the representable functors are dense in $(\mathbf{A} \times \bar{\mathbf{B}}^{opp}, \mathbf{S})$, the same holds for their images in $(\mathbf{A}, \mathbf{B})$ (cf. [*25*] 1.13), i.e. each functor $t : \mathbf{A} \to \mathbf{B}$ is canonically a direct limit of functors of the form $\bar{B} \otimes [A, -] : \mathbf{A} \to \mathbf{B}$, $\bar{B} \in \bar{\mathbf{B}} \subset \mathbf{B}$, $A \in \mathbf{A}$.[2] As there exists moreover a bijection[3]

(6) $$[\bar{B} \otimes [A, -], t] \cong [\bar{B}, tA]$$

which is natural in $\bar{B}$, A and t, we call the functors $\bar{B} \otimes [A, -]$, $A \in \mathbf{A}$, $\bar{B} \in \bar{\mathbf{B}}$, the (generalized) representable functors. They are in fact a generalization of the set valued representable functors. For let $\mathbf{B}$ be the category of sets $\mathbf{S}$, and let $\{1\}$ denote the dense subcategory of $\mathbf{S}$ whose only object $(=1)$ is a one point set. From the preceeding definition of the tensorproduct, it follows easily that $1 \otimes : \mathbf{S} \to \mathbf{S}$ coincides with the identity of $\mathbf{S}$. Therefore the (generalized) representable functors agree in this case with the usual ones and (6) specializes to the Yoneda lemma (1).

(7) The restrictions made before on $\mathbf{B}$ are not essential. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and let A and B be objects of $\mathbf{A}$ and $\mathbf{B}$ respectively such that the functor $B \otimes [A, -] : \mathbf{A} \to \mathbf{B}$, $X \rightsquigarrow \oplus_{f:A\to X} B_f$, $B_f = B$, exists. Then for each functor $t : \mathbf{A} \to \mathbf{B}$ the generalized Yoneda lemma

$$[B \otimes [A, -], t] \cong [B, tA]$$

holds.

(8) If, moreover, the functors $B \otimes [A, -]$ exist for all pairs $A \in \mathbf{A}$, $B \in \mathbf{B}$, then each functor $t : \mathbf{A} \to \mathbf{B}$ is canonically a direct limit of functors $B \otimes [A, -]$. However, the canonical index category $\mathbf{D}_Y(t)$ in 2.12 ($J = \mathrm{id}$) is very large

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[1] These functors have also been used by André [*1*] p. 6. He showed that they are a generating family in $(\mathbf{A}, \mathbf{B})$, provided $\mathbf{A}$ is small.

[2] In other words the Yoneda functor $Y : \mathbf{A} \times \bar{\mathbf{B}}^{opp} \to (\mathbf{A}, \mathbf{B})$, $A \times \bar{B} \rightsquigarrow \bar{B} \otimes [A, -]$, is dense. If the inclusion $\bar{\mathbf{B}} \subset \mathbf{B}$ is not dense but each object $B \in \mathbf{B}$ is in functorially a direct limit of objects of $\bar{\mathbf{B}}$ (cf. 2.20), then each functor $t : \mathbf{A} \to \mathbf{B}$ is functorially a direct limit of functors $\bar{B} \otimes [A, -] : \mathbf{A} \to \mathbf{B}$, where $A \in \mathbf{A}$, $\bar{B} \in \bar{\mathbf{B}}$(2.21). However, the Yoneda functor $Y : \mathbf{A} \times \bar{\mathbf{B}}^{opp} \to (\mathbf{A}, \mathbf{B})$ need not be dense.

[3] For $\mathbf{A}$ small, this was first observed by Mitchell [*20*].

and depends on t. The following is a smaller one and independent of t. Let the objects of $\mathbf{Mor}(\mathbf{A})$ be the morphisms of $\mathbf{A}$. Let the morphism sets $[\alpha, \beta]$ of $\mathbf{Mor}(\mathbf{A})$ consist of a single element if $\beta = \mathrm{id}_{\mathrm{d}\alpha}$ or $\beta = \mathrm{id}_{\mathrm{r}\alpha}$ (d = domain, r = range), otherwise let them be empty (cf. [25] 2.19). Then[4]

$$(9) \qquad t = \varinjlim_{\alpha\in\mathbf{M}(\mathbf{A})} t\mathrm{d}\alpha \otimes [\mathrm{r}\alpha, -]$$

holds.

(10) Unlike the usual representable functors (1), the generalized ones can be dualized. By composing a functor $B \otimes [A, -] : \mathbf{A} \to \mathbf{B}$ with the dualization functors $\mathbf{A}^{opp} \to \mathbf{A}$ and $\mathbf{B} \to \mathbf{B}^{opp}$, one obtains a corepresentable functor in $(\mathbf{A}^{opp}, \mathbf{B}^{opp})$ which is denoted by[5]

$$(11) \qquad \mathbf{A}^{opp} \xrightarrow{[-,A]} \mathbf{S} \xrightarrow{\boldsymbol{[}-,B\boldsymbol{]}} \mathbf{B}^{opp}$$

In (11) A and B are regarded as objects of $\mathbf{A}^{opp}$ and $\mathbf{B}^{opp}$ respectively, therefore $\boldsymbol{[}[X, A], B\boldsymbol{]} = \prod_{f:X\to A} B_f$, $B_f = B$ holds. By dualizing (7) and (9), one obtains

$$(12) \qquad [t, \boldsymbol{[}-, B\boldsymbol{]} \cdot [-, A]] \cong [tA, B]$$

and

$$(13) \qquad t = \varprojlim_{\alpha\in\mathbf{M}(\mathbf{A})^*} \boldsymbol{[}-, t\mathrm{r}\alpha\boldsymbol{]} \cdot [-, \mathrm{d}\alpha] \text{ [4]}$$

If $\mathbf{B} = \mathbf{S}$, then the symbolic hom-functor $\boldsymbol{[}-, B\boldsymbol{]} : \mathbf{S} \to \mathbf{S}$ coincides with the hom-functor $[-, B] : \mathbf{S} \to \mathbf{S}$. Hence by (13) each functor $t : \mathbf{A} \to \mathbf{S}$ is an inverse limit of "double" hom-functors $[-, B] \cdot [-, A]$, $A \in \mathbf{A}$, $B \in \mathbf{B}$. If there are no strongly inaccessible cardinals (in $\mathbf{S}$), then any infinite set B_0 is codense in $\mathbf{S}$, i.e. each set $M \in \mathbf{S}$ is canonically an inverse limit of sets equal to B_0 (cf. Ulam). Hence by dualizing (3) — (5), it follows that each functor $t : \mathbf{A} \to \mathbf{S}$ is canonically an inverse limit of functors $[-, B_0] \cdot [-, A] : \mathbf{A} \to \mathbf{S}$, $A \in \mathbf{A}$; i.e. the functors $[-, B_0] \cdot [-, A]$, $A \in \mathbf{A}$, are codense in $(\mathbf{A}, \mathbf{S})$.[6]

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[4] Throughout the paper, when $\mathbf{Mor}(\mathbf{A})$ and $\mathbf{Mor}(\mathbf{A})^{opp}$ appear in subscripts, we abbreviate them to $\mathbf{M}(\mathbf{A})$ and $\mathbf{M}(\mathbf{A})^*$ respectively.

[5] $\boldsymbol{[}-, B\boldsymbol{]}$ is the symbolic hom-functor of Freyd [5] p. 87.

[6] F. W. Lawvere has remarked that this implies part of a recent result of Isbell [11]: If $\mathbf{A}$ is small, then $(\mathbf{A}, \mathbf{S})$ has a small codense subcategory. In particular, a primitive category of algebras with only unary operations has a small codense (= right adequate, cf. [25] 1.8) subcategory.

The functors $B \otimes [A, -] : \mathbf{A} \to \mathbf{B}$, $A \in \mathbf{A}$, $B \in \mathbf{B}$, are not additive if **A** and **B** are additive categories. This suggests changing the definition in the additive case. We assume for the moment that **B** is right complete. Then for each $B \in \mathbf{B}$ the hom-functor $[B, -]$ from **B** to the category **Ab.Gr.** of abelian groups has a left adjoint which is denoted by $B \otimes_{\mathbf{Z}} : \mathbf{Ab.Gr.} \to \mathbf{B}$. The additive representable functors are defined as compositions

$$(14) \qquad \mathbf{A} \xrightarrow{[A,-]} \mathbf{Ab.Gr.} \xrightarrow{B\otimes_{\mathbf{Z}}} \mathbf{B}$$

where $A \in \mathbf{A}$ and $B \in \mathbf{B}$.[7] Then all aforementioned properties ((3) — (13)) of the representable functors carry over to the additive ones (14), provided all functors $t : \mathbf{A} \to \mathbf{B}$ under consideration are additive.

Now let **B** be an additive category which has cokernels but not necessarily arbitrary direct limits. Then the additive representable functors can still be defined (and they keep the same properties), provided that in the domain category **A**, all morphism groups $[A, A']$, A, $A' \in \mathbf{A}$, are finitely generated.

(15) Assume furthermore that all the groups $[A, A']$, A', $A \in \mathbf{A}$ can be coherently equipped with a left Λ-module structure, in such a way that they are finitely presentable as Λ-modules.[8] (If Λ is notherian, "finitely presentable" is equivalent to "finitely generated") Then by modifying the definition of representable functors accordingly, their existence can be proved and the basic properties established only assuming cokernels in **B**. The details run as follows.

Let (B, ρ) be a right Λ-object of **B**, i.e. an object $B \in \mathbf{B}$ together with a ring homomorphism $\Lambda^{opp} \to [B, B]$. Then the functor $[B, -] : \mathbf{B} \to \mathbf{Ab.Gr.}$ has a canonical lifting ${}_\Lambda[B, -] : \mathbf{B} \to {}_\Lambda\mathbf{M}$ (${}_\Lambda\mathbf{M}$ = left Λ-modules). Since only cokernels are assumed in **B**, the left adjoint of ${}_\Lambda[B, -] : \mathbf{B} \to {}_\Lambda\mathbf{M}$ can be defined in general only on the subcategory $\mathbf{FP}({}_\Lambda\mathbf{M})$ of finitely presentable Λ-modules. This partial left adjoint is denoted by $(B, \rho)\otimes_\Lambda : \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$. (A detailed study of these partial or relative adjoints is given in [*25*] §2.) Representable functors from **A** to **B** are now defined as compositions

$$(16) \qquad (B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$$

where $A \in \mathbf{A}$ and $(B, \rho) \in \mathbf{B}_\Lambda$ (= category of the right Λ-objects of **B**). If the Λ-modules $[A, A']$, A, $A' \in \mathbf{A}$ are not finitely presentable, but there exist

[7] For **A** small, these functors were first introduced by Freyd [*4*] p. 18. He also proved (18) and (21) in the case $\Lambda = \mathbf{Z}$.

[8] **A** is then a Λ-enriched category in the sense of Kelly (unpublished).

arbitrary direct limits in $\mathbf{B}$, then the representable functors are defined as compositions

$$(17)\qquad (B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{B}$$

where $A \in \mathbf{A}$, $(B, \rho) \in \mathbf{B}_\Lambda$. Hereafter we mean $(B, \rho) \otimes_\Lambda [A, -]$ to be a functor (16) or (17). Since $\mathbf{A}$ is a Λ-enriched category, the values tA, $A \in \mathbf{A}$, of each functor $t : \mathbf{A} \to \mathbf{B}$ can be canonically equipped with ring homomorphisms $\tau_A : \Lambda^{opp} \to [tA, tA]$ (cf. 3.11). The Yoneda lemma then generalizes to

$$(18)\qquad [(B, \rho) \otimes_\Lambda [A, -], t] \cong [(B, \rho), (tA, \tau_A)]$$

This isomorphism is natural in (B, ρ), A and t. The representable functors (16) or (17) are dense in $(\mathbf{A}, \mathbf{B})$ (cf. 2.12). Furthermore, for each functor $t : \mathbf{A} \to \mathbf{B}$ the equation

$$(19)\qquad t = \varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} (t\mathrm{d}\alpha, \tau_{\mathrm{d}\alpha}) \otimes_\Lambda [\mathrm{r}\alpha, -]$$

holds.

Note that, as before (10), by passing to the duals of $\mathbf{A}$ and $\mathbf{B}$, the representable functors (16) or (17) of $(\mathbf{A}, \mathbf{B})$ change over to the corepresentable ones of $(\mathbf{A}^{opp}, \mathbf{B}^{opp})$. Viewed as functors from $\mathbf{A}^{opp}$ to $\mathbf{B}^{opp}$, they are denoted by

$$(20)\qquad {}_\Lambda\mathbf{[}-, (\rho, B)\mathbf{]} \cdot {}_\Lambda[-, A] : \mathbf{A}^{opp} \to \mathbf{B}^{opp}$$

where $A \in \mathbf{A}^{opp}$ and $(\rho, B) \in {}_\Lambda(\mathbf{B}^{opp}) = (\mathbf{B}_\Lambda)^{opp}$. By dualizing (18) and (19) one obtains for each functor $t : \mathbf{A}^{opp} \to \mathbf{B}^{opp}$

$$(21)\qquad [t, {}_\Lambda\mathbf{[}-, (\rho, B)\mathbf{]} \cdot {}_\Lambda[-, A]] \cong [(\tau_A, tA), (\rho, B)]$$

and

$$(22)\qquad t = \varprojlim_{\alpha \in \mathbf{M}(\mathbf{A})^*} {}_\Lambda\mathbf{[}-, (\tau_{\mathrm{r}\alpha}, t\mathrm{r}\alpha)\mathbf{]} \cdot {}_\Lambda[-, \mathrm{d}\alpha]$$

The paper is divided into two sections and an appendix. A short summary is given at the beginning of each section. In order not to make the paper too long, we only treat the additive representable functors. The non-additive case, as discussed in the first part of the introduction, is left to the reader. But once he has understood the additive case, it should not be hard for him to carry out the non-additive one, whereas the converse would be much more difficult.

Participants of the Moscow conference told me that a forthcoming paper of Pokazeeva contains the following:

Let **A** be a **D**-category in the sense of Linton [*17*]. Then each strong functor $t : \mathbf{A} \to \mathbf{A}$ (cf. [*17*]) is a direct limit of composite functors $\Sigma_Y \cdot \Omega_X : \mathbf{A} \to \mathbf{A}$, where $\Omega_X : \mathbf{A} \to \mathbf{A}$ is the lifting of the hom-functor $[X, -] : \mathbf{A} \to \mathbf{S}$ and $\Sigma_Y : \mathbf{A} \to \mathbf{A}$ the left adjoint of Ω_Y.

This suggested the idea of investigating to what extent the results of the present paper can be generalized to the case of enriched categories (or categories based on one of Linton's autonomous categories, cf. [*17*]). It turns out that in the special case $\Lambda = \mathbf{Z}$ most of our results can be so generalized.[9,10] But in the setting of the present paper, where Λ is an arbitrary ring, it seems unlikely that this can be done. Apparently there is no concept of a strong functor, and it is hard to imagine how a Yoneda lemma generalizing 3.17 would look.

The generalized representable functors have many useful properties as will be shown in subsequent papers ([*22*] and [*23*]). For instance, let $K : \mathbf{A} \to \bar{\mathbf{A}}$ be a functor. Then the right Kan K-extension (cf. [*25*] 2.9b) of $B \otimes [A, -] : \mathbf{A} \to \mathbf{B}$ is $B \otimes [KA, -] : \bar{\mathbf{A}} \to \mathbf{B}$. (For this **A** need not be small.) Using (9) and [*25*] 2.13 we will prove that for each functor $t : \mathbf{A} \to \mathbf{B}$ the right K-extension $E_K(t) : \bar{\mathbf{A}} \to \mathbf{B}$ exists iff $\varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} t\mathrm{d}\alpha \otimes [K\mathrm{r}\alpha, -]$ exists. Moreover

$$E_K(t) = \varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} t\mathrm{d}\alpha \otimes [K\mathrm{r}\alpha, -] \tag{23}$$

is valid. The same holds in the additive case.

Representable functors are used in [*23*] to prove that right satellites and derived functors — provided they exist — can be expressed by tensor products and $\mathrm{Ext}^*(-, -)$. For this let **A** and **B** be abelian categories, **A** Λ-enriched (3.7). Assume $\mathrm{Ext}^j(A, \bar{A})$ is a set for all $A, \bar{A} \in \mathbf{A}$. Then the connected sequence of the right satellites of $(B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{B}$ is $(B, \rho) \otimes_\Lambda \mathrm{Ext}^*(A, -)$. For each additive functor $t : \mathbf{A} \to \mathbf{B}$ the right satellites S^*t ([*25*] 2.9c) exist iff $\varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} (\tau_{\mathrm{d}\alpha}, t\mathrm{d}\alpha) \otimes_\Lambda \mathrm{Ext}^*(\mathrm{r}\alpha, -)$ exists. Moreover

$$S^*t = \varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} (\tau_{\mathrm{d}\alpha}, t\mathrm{d}\alpha) \otimes_\Lambda \mathrm{Ext}^*(\mathrm{r}\alpha, -) \tag{24}$$

holds. Dually, the left satellites S_*t agree with

$$\varprojlim_{\alpha \in \mathbf{M}(\mathbf{A})^*} {}_\Lambda[-, (\tau_{\mathrm{r}\alpha}, t\mathrm{r}\alpha)] \cdot \mathrm{Ext}^*(-, \mathrm{d}\alpha). \tag{25}$$

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[9] The set up of our proofs for 1.33 and 2.12 make it obvious how this can be done. We shall return to this in [*22*].

[10] In particular this includes the above mentioned result of Pokazeeva.

We conclude with a few remarks about terminology and foundations. For the latter we refer the reader to the last section of the introduction of [25]. The terms "small category" and "left (right) complete category" are used as in Mitchell [20]. Functor categories are indicated with parentheses (—, —).

We use brackets [—, —] to denote the morphisms between two objects or the natural transformations between functors. We only consider rings Λ, Γ, Σ... which have a unit element. A ring homomorphism is assumed to preserve the unit element. By ${}_\Lambda\mathbf{M}$ we denote the category of unitary left Λ-modules. All categories and functors are assumed additive unless otherwise stated or unless it is clear from the context that they are not additive. All functors are assumed covariant unless otherwise stated or unless it is clear from the context that the functor under consideration is contravariant (as for instance the contravariant hom-functor $[-, A] : \mathbf{A} \to \mathbf{Ab.Gr.}$). The category of contravariant additive functors from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $(\mathbf{A}^{opp}, \mathbf{B})$. But we do not adopt the notation $t : \mathbf{A}^{opp} \to \mathbf{B}$ for a contravariant functor t because it does not apply to a composition $t' \cdot t : \mathbf{A} \to \mathbf{B} \to \mathbf{B}'$ of contravariant functors. The morphism category of a category $\mathbf{B}$ is denoted by $\mathbf{B}^2$.

1. The Tensor Product and Generalized Representable Functors

Let Λ be a ring and $\mathbf{B}_\Lambda$ the category of right Λ-objects of an additive category $\mathbf{B}$. We define the tensor product as a relative adjoint and construct the bifunctor

$$\otimes_\Lambda : \mathbf{B}_\Lambda \times \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$$

only assuming cokernels in $\mathbf{B}$, where $\mathbf{FP}({}_\Lambda\mathbf{M})$ is the category of finitely presentable Λ-modules. The basic properties are established, in particular the right continuity in both variables. Our presentation is slightly more general than Epstein's [3] p. 5-17 or Mitchell's [20] p. 143. We only give an outline and refer the reader for the details to [3] or [20].

If $\mathbf{B}$ is right complete and has a small dense subcategory $\bar{\mathbf{B}}$, we will show that the functors

$$\bar{B} \otimes_{\mathbf{Z}} [A, -] : \mathbf{A} \to \mathbf{Ab.Gr.} \to \mathbf{B},$$

where $\bar{B} \in \bar{\mathbf{B}}$ and $A \in \mathbf{A}$, are dense in $(\mathbf{A}, \mathbf{B})$ (cf. [25] 1.3, 1.17). If $U \in \mathbf{B}$ is a dense generator (cf. [25] 1.19), then even the functors

$$U \otimes_{\mathbf{Z}} [A, -] : \mathbf{A} \to \mathbf{Ab.Gr.} \to \mathbf{B},$$

are dense in $(\mathbf{A}, \mathbf{B})$, where A runs through $\mathbf{A}$.

(1.1) Let Λ be a ring and ${}_\Lambda\mathbf{M}$ the category of left Λ-modules. By $\mathbf{FP}({}_\Lambda\mathbf{M})$ we denote the full subcategory of finitely presentable Λ-modules (cf. Bourbaki

[*2*] p. 35). As we remarked in [*25*] 1.05c, the inclusion $J : \mathbf{FP}({}_A\mathbf{M}) \to {}_A\mathbf{M}$ is dense. Let $\mathbf{L}(\mathbf{FP}({}_A\mathbf{M})^{opp}, \mathbf{Ab.Gr.})$ denote the full subcategory of $(\mathbf{FP}({}_A\mathbf{M})^{opp}, \mathbf{Ab.Gr.})$ consisting of those functors which take right exact sequences into left exact ones. Then it is not too hard to verify that the functor ${}_A\mathbf{M} \to \mathbf{L}(\mathbf{FP}({}_A\mathbf{M})^{opp}, \mathbf{Ab.Gr.})$, $M \rightsquigarrow [J-, M]$, is an equivalence. For A noetherian this was first proved by P. Gabriel in his thesis (cf. Bull. Soc. Math. France 90, p. 356, 1962).

1.2 Definition. Let A be a ring and $\mathbf{B}$ an additive category. The objects of the category $\mathbf{B}_A$ are pairs (B, ρ_B) consisting of an object $B \in \mathbf{B}$ and a ring homomorphism $\rho_B : A^{opp} \to [B, B]$. A morphism $(B, \rho_B) \to (B', \rho_{B'})$ in $\mathbf{B}_A$ is a morphism $\beta : B \to B'$ with the property $\beta \cdot \rho_B(\lambda) = \rho_{B'}(\lambda) \cdot \beta$, $\lambda \in A$. (The index B is dropped in ρ_B if this does not give rise to misunderstanding).

The category $\mathbf{B}_A$ is called the category of right A-objects of $\mathbf{B}$.

(1.3) The category ${}_A\mathbf{B}$ of the left A-objects of $\mathbf{B}$ is defined dually. (The objects are pairs (ρ_B, B), where B is an object of $\mathbf{B}$ and $\rho_B : A \to [B, B]$ a ring homomorphism).

If $A = \mathbf{Z}$ then both $\mathbf{B}_\mathbf{Z}$ and ${}_\mathbf{Z}\mathbf{B}$ are canonically isomorphic to $\mathbf{B}$. Therefore we denote any of these categories simply by $\mathbf{B}$.

(1.4) The duality functor $\mathbf{B} \to \mathbf{B}^{opp}$ induces an equivalence $\mathbf{B}_A \to {}_A(\mathbf{B}^{opp})$, $(B, \rho : A^{opp} \to [B, B]) \rightsquigarrow (B, \rho : A \to [B, B]^{opp})$. Hence $\mathbf{B}_A$ is dual to ${}_A(\mathbf{B}^{opp})$.

1.5 Lemma. *Let $H : \mathbf{D} \to \mathbf{B}_A$ be a functor and $V : \mathbf{B}_A \to \mathbf{B}$ the forgetful functor $(B, \rho) \rightsquigarrow B$. Assume that H admits a universal* ([*25*] 2.9a) *natural transformation $\Phi' : VH \to \mathrm{const}_B$, where $B \in \mathbf{B}$. Then there exists a ring homomorphism $\rho : A^{opp} \to [B, B]$ and a universal natural transformation $\Phi : H \to \mathrm{const}_{(B,\rho)}$ with the property $V\Phi = \Phi'$. There is a similar statement if H admits a co-universal transformation $\mathrm{const}_{B'} \to H$, $B' \in \mathbf{B}$.*

1.6 *Remark.* If $\mathbf{B}$ is right (left) complete, this lemma implies that $\mathbf{B}_A$ is right (left) complete and that $V : \mathbf{B}_A \to \mathbf{B}$ preserves small direct (inverse) limits.

However we do not know whether V is right (left) continuous (cf. [*25*] 2.15), unless we further assume that $\mathbf{B}$ is complete. Then by 1.29 V has both a right and a left adjoint. Hence by [*25*] 2.13 both right and left limits are preserved.

Proof of 1.5. Since the values of H are right A-objects, each $\lambda \in A^{opp}$ induces a natural transformation $\psi(\lambda) : V \cdot H \to V \cdot H$ and hence a morphism $\varinjlim \psi(\lambda) : B \to B$. Define $\rho(\lambda) = \varinjlim \psi(\lambda)$. Then one readily checks that $\rho : A^{opp} \to [B, B]$ is a ring homomorphism and that the morphisms

$\Phi'(D) : V \cdot H(D) \to B$, $D \in \mathbf{D}$, are compatible with the Λ-structures on $V \cdot H(D)$ and B. Viewed as morphisms $H(D) \to (B, \rho)$ they are denoted by $\Phi(D)$. Thus $V\Phi = \Phi'$ holds. One easily verifies that the universality of Φ' carries over to Φ. Q.E.D.

(1.7) Let (B, ρ) be a right Λ-object of $\mathbf{B}$. Then for each $B' \in \mathbf{B}$ the abelian group $[B, B']$ can be equipped a left Λ-module structure. An element $\lambda \in \Lambda$ acts on a morphism $f : B \to B'$ by composing it with $\rho(\lambda) : B \to B$. Thus the functor $[B, -] : \mathbf{B} \to \mathbf{Ab.Gr.}$ factors through the forgetful functor $U : {}_{\Lambda}\mathbf{M} \to \mathbf{Ab.Gr.}$ Similarly, if (ρ, B) is a left Λ-object of $\mathbf{B}$, the functor $[-, B] : \mathbf{B} \to \mathbf{Ab.Gr.}$ factors through U.

(1.8) We denote the lifting of $[B, -]$ by ${}_{\Lambda}[B, -] : \mathbf{B} \to {}_{\Lambda}\mathbf{M}$ and define the tensor product $(B, \rho) \otimes_{\Lambda} : \mathbf{FP}({}_{\Lambda}\mathbf{M}) \to \mathbf{B}$ as its left adjoint relative to the inclusion $J : \mathbf{FP}({}_{\Lambda}\mathbf{M}) \to {}_{\Lambda}\mathbf{M}$ (cf. [*25*] 2.2).

1.9 Theorem. *Let Λ be a ring and $\mathbf{B}$ an additive category with cokernels. Then there exists a covariant bifunctor (the generalized tensor product)*

$$\otimes_{\Lambda} : \mathbf{B}_{\Lambda} \times \mathbf{FP}({}_{\Lambda}\mathbf{M}) \to \mathbf{B},$$

determined up to an equivalence, which has the following properties:

(1.10) *For each object $(B, \rho) \in \mathbf{B}_{\Lambda}$*

$$(B, \rho) \otimes_{\Lambda} \Lambda \cong B$$

holds.

(1.11) *For each $(B, \rho) \in \mathbf{B}_{\Lambda}$ the functor*

$$(B, \rho)\otimes_{\Lambda} : \mathbf{FP}({}_{\Lambda}\mathbf{M}) \to \mathbf{B}$$

is J-left adjoint to ${}_{\Lambda}[B, -] : {}_{\Lambda}\mathbf{M} \to \mathbf{B}$ (cf. 1.8), *i.e. for each pair $B' \in \mathbf{B}$, $M \in \mathbf{FP}({}_{\Lambda}\mathbf{M})$, there exists an isomorphism*

$$(1.12) \qquad \Theta_{(B,\rho)}(M, B') : [(B, \rho) \otimes_{\Lambda} M, B'] \xrightarrow{\cong} [JM, {}_{\Lambda}[B, B']]$$

natural in B', M and (B, ρ).

(1.13) *For each $M \in \mathbf{FP}({}_{\Lambda}\mathbf{M})$ the functor $\otimes_{\Lambda} M : \mathbf{B}_{\Lambda} \to \mathbf{B}$ preserves those direct limits which are preserved by $V : \mathbf{B}_{\Lambda} \to \mathbf{B}$* (cf. 1.5 *and* 1.6). *Similarly for each $(B, \rho) \in \mathbf{B}_{\Lambda}$ the functor $(B, \rho)\otimes_{\Lambda} : \mathbf{FP}({}_{\Lambda}\mathbf{M}) \to \mathbf{B}$ commutes with those direct limits which are preserved by $J : \mathbf{FP}({}_{\Lambda}M) \to {}_{\Lambda}\mathbf{M}$.*

1.14 *Remark.* *If* **B** *has cokernels and arbitrary sums, then there exists a bifunctor*

$$\otimes_\Lambda : \mathbf{B}_\Lambda \times {}_\Lambda\mathbf{M} \to \mathbf{B}$$

with the same properties. More precisely, the statements 1.10 — 1.13 *remain valid if* $\mathbf{FP}({}_\Lambda\mathbf{M})$ *and* J *are replaced by* ${}_\Lambda\mathbf{M}$ *and the identity of* ${}_\Lambda\mathbf{M}$ *respectively.*

Our proof of 1.9 carries through directly to 1.14.

Proof of 1.9. Let (B, ρ) be an object of $\mathbf{B}_\Lambda$. We define

$$(B, \rho) \otimes_\Lambda \Lambda = B \tag{1.15}$$

and similarly for a finite sum $\bigoplus_{i\in I} \Lambda_i$

$$(B, \rho) \otimes_\Lambda (\bigoplus_{i\in I} \Lambda_i) = \bigoplus_{i\in I} B_i \tag{1.16}$$

where $\Lambda_i = \Lambda$ and $B_i = B$.

A Λ-homomorphism $f : \Lambda \to \Lambda$ is determined by $f(1) \in \Lambda$. Therefore we define

$$(B, \rho)\otimes_\Lambda f = \rho(f(1)) \tag{1.17}$$

Similarly a map $g : \bigoplus_{i\in I} \Lambda_i \to \bigoplus_{k\in K} \Lambda_k$ (I, K finite) can be described by the matrix $(g_{ik}(1))$, where g_{ik} is the composition

$$\Lambda_i \xrightarrow{q_i} \bigoplus_{i\in I} \Lambda_i \xrightarrow{g} \bigoplus_{k\in K} \Lambda_k \xrightarrow{p_k} \Lambda_k$$

(p_k = canonical projection, q_i = canonical injection).

Thus we define

$$(B, \rho)\otimes_\Lambda g = (B, \rho)\otimes_\Lambda (g_{ik}(1)) \tag{1.18}$$

For each $M \in \mathbf{FP}({}_\Lambda\mathbf{M})$ there exists by definition an exact sequence

$$\bigoplus_{i\in I} \Lambda_i \xrightarrow{g} \bigoplus_{k\in K} \Lambda_k \to M \to 0 \tag{1.19}$$

where I and K are finite sets. We define

$$(B, \rho)\otimes_\Lambda M = \operatorname{cok}\{(B, \rho)\otimes_\Lambda g\}. \tag{1.20}$$

By standard homological algebra this is well defined, (i.e. up to an equivalence) and in particular compatible with the definitions 1.15 and 1.16. Furthermore

$(B, \rho)\otimes_\Lambda : \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$ is a functor and a morphism $(B, \rho) \to (B', \rho')$ induces a natural transformation $(B, \rho)\otimes_\Lambda \to (B', \rho')\otimes_\Lambda$. Hence $\otimes_\Lambda : \mathbf{B}_\Lambda \times \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$ is a bifunctor.

We now prove 1.12. Let $M \in \mathbf{FP}({}_\Lambda\mathbf{M})$, $B' \in \mathbf{B}$ and $(B, \rho) \in \mathbf{B}_\Lambda$. For each $m \in \mathbf{M}$ denote by $f_m : \Lambda \to M$ the Λ-homomorphism $1 \rightsquigarrow m$. Then the map

$$\Theta_{(B,\rho)}(M, B') : [(B, \rho) \otimes_\Lambda M, B'] \to [M, {}_\Lambda[B, B']], \tag{1.21}$$

which assigns to a morphism $\beta : (B, \rho) \otimes_\Lambda M \to B'$ the Λ-homomorphism $m \rightsquigarrow B = (B, \rho) \otimes_\Lambda \Lambda \xrightarrow{1\mathrm{d}\otimes_\Lambda f_m} (B, \rho) \otimes_\Lambda M \xrightarrow{\beta} B'$, is obviously an isomorphism for $M = \Lambda$. Since the functors

$$[(B, \rho) \otimes_\Lambda (-), B'] : \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{Ab.Gr.} \tag{1.22}$$

and

$$[-, {}_\Lambda[B, B']] : \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{Ab.Gr.} \tag{1.23}$$

are additive, we can define in view of 1.16

$$\Theta(\bigoplus_{i\in I} \Lambda_i, B') = \bigoplus_{i\in I} \Theta(\Lambda_i, B') \tag{1.24}$$

for a finite sum $\bigoplus_{i\in I} \Lambda_i$, $\Lambda_i = \Lambda$. (We drop the index (B, ρ) if this does not give rise to misunderstanding). Hence $\Theta(\bigoplus_{i\in I} \Lambda_i, B')$ is also an isomorphism and one readily checks that it is natural in B', (B, ρ) and compatible with morphisms $\bigoplus_{i\in I} \Lambda_i \to \bigoplus_{k\in K} \Lambda_k$. We abbreviate the functors 1.22 and 1.23 by F and $\bar{F}$ respectively. Since they are left exact, a resolution 1.19 of $M \in \mathbf{FP}({}_\Lambda\mathbf{M})$ gives rise to a commutative diagram

$$\begin{array}{ccccccc}
F(\bigoplus_{i\in I} \Lambda_i) & \xleftarrow{Fg} & F(\bigoplus_{k\in K} \Lambda_k) & \longleftarrow & FM & \longleftarrow & 0 \\
\downarrow{\scriptstyle \Theta(\bigoplus_{i\in I} \Lambda_i, B')} & & \downarrow{\scriptstyle \Theta(\bigoplus_{k\in K} \Lambda_k, B')} & & & & \\
\bar{F}(\bigoplus_{i\in I} \Lambda_i) & \xleftarrow{\bar{F}g} & \bar{F}(\bigoplus_{k\in K} \Lambda_k) & \longleftarrow & \bar{F}M & \longleftarrow & 0
\end{array} \tag{1.25}$$

in which the vertical morphisms are equivalences. We define $\Theta(M, B')$ to be the unique morphism $FM \to \bar{F}M$ which makes the diagram 1.25 commutative. It is obviously an equivalence. By standard homological algebra $\Theta(M, B')$ is well defined, i.e. independent of the chosen resolution 1.19, and natural in (B, ρ), M and B'.

The second part of 1.13 follows immediately from [25] 2.13 and 1.11. To prove the first part of 1.13, we need only show by [25] 2.1 and [25] 2.17 that for each $B' \in \mathbf{B}$ the functor $[(-) \otimes_\Lambda M, B'] : \mathbf{B}_\Lambda \to \mathbf{Ab.Gr.}$ takes those direct limits preserved by $V : \mathbf{B}_\Lambda \to \mathbf{B}$ into inverse limits. By 1.12 the functor $[(-) \otimes_\Lambda M, B']$ is equivalent to the composition

$$[M, -] \cdot {}_\Lambda[V-, B'] : \mathbf{B}_\Lambda \to {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$$

Since $[M, -]$ is left continuous ([25] 2.15), it is enough to show that ${}_\Lambda[V-, B'] : \mathbf{B}_\Lambda \to {}_\Lambda\mathbf{M}$ is left continuous with respect to the above indicated direct limits. This is obviously equivalent to saying that the same property holds for the composition $U \cdot {}_\Lambda[V-, B'] : \mathbf{B}_\Lambda \to {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$, where U is the forgetful functor. The functors $U \cdot {}_\Lambda[V-, B']$ and $[-, B'] \cdot V : \mathbf{B}_\Lambda \to \mathbf{B} \to \mathbf{Ab.Gr.}$ agree. Therefore this property holds because $[-, B']$ takes direct limits into inverse limits.

Q.E.D.

By dualizing 1.9 and 1.14 with the help of 1.4 we obtain

1.26 Theorem. *Let* $\mathbf{B}$ *be an additive category with kernels. Then there exists a bifunctor* (symbolic hom, cf. Freyd [5] p. 87), *determined up to an equivalence,*

$${}_\Lambda[\![-, -]\!] : \mathbf{FP}({}_\Lambda\mathbf{M})^{opp} \times {}_\Lambda\mathbf{B} \to \mathbf{B}$$

with the following properties:

(1) *It is contravariant in the first and covariant in the second variable.*

(2) *For each* $(\rho, B) \in {}_\Lambda\mathbf{B}$

$${}_\Lambda[\![\Lambda, (\rho, B)]\!] \simeq B$$

holds.

(3) *For each* (ρ, B) *the functors* ${}_\Lambda[\![-, (\rho, B)]\!] : \mathbf{FP}({}_\Lambda\mathbf{M})^{opp} \to \mathbf{B}$ *and* ${}_\Lambda[-, B] : \mathbf{B}^{opp} \to {}_\Lambda\mathbf{M}$ (cf. 1.8) *are adjoint on the right*[11] *relative to the inclusion* $J : \mathbf{FP}({}_\Lambda\mathbf{M}) \to {}_\Lambda\mathbf{M}$, *i.e. for each pair* $M \in \mathbf{FP}({}_\Lambda\mathbf{M})$, $B' \in \mathbf{B}$, *there exists an isomorphism*

$$[B', {}_\Lambda[\![M, (\rho, B)]\!]] \simeq [M, {}_\Lambda[B', B]]$$

which is natural in B', M *and* (ρ, B).

(4) *For each* $M \in \mathbf{FP}({}_\Lambda\mathbf{M})$ *the functor* ${}_\Lambda[\![M, -]\!] : {}_\Lambda\mathbf{B} \to \mathbf{B}$ *preserves those inverse limits preserved by* $V : {}_\Lambda\mathbf{B} \to \mathbf{B}$ (cf. 1.5 and 1.6). *Similarly for each* $(\rho, B) \in {}_\Lambda\mathbf{B}$ *the functor* ${}_\Lambda[\![-, (\rho, B)]\!]$ *takes those direct limits preserved by* $J : \mathbf{FP}({}_\Lambda\mathbf{M}) \to {}_\Lambda\mathbf{M}$ *into inverse limits.*

[11] This terminology is due to Freyd (cf. [5], p. 81).

1.27 *Remark. Let* **B** *be an additive category with kernels and arbitrary products. Then there exists a bifunctor*

$${}_{\Lambda}[-, -] : {}_{\Lambda}\mathbf{M} \times {}_{\Lambda}\mathbf{B} \to \mathbf{B}$$

with the same properties. More precisely, the statements 1.26 (1)–(4) *remain valid if* $\mathbf{FP}({}_{\Lambda}\mathbf{M})$ *and* J *are replaced by* ${}_{\Lambda}\mathbf{M}$ *and the identity of* ${}_{\Lambda}\mathbf{M}$ *respectively.*

(1.28) Replace in 1.9, 1.14, 1.26, and 1.27 the ring Λ by Λ^{opp}. Then one obtains bifunctors

$$\otimes_{\Lambda} : {}_{\Lambda}\mathbf{B} \times \mathbf{FP}(\mathbf{M}_{\Lambda}) \to \mathbf{B}, \qquad \otimes_{\Lambda} : {}_{\Lambda}\mathbf{B} \times \mathbf{M}_{\Lambda} \to \mathbf{B},$$
$${}_{\Lambda}[-, -] : \mathbf{FP}(\mathbf{M}_{\Lambda}) \times \mathbf{B}_{\Lambda} \to \mathbf{B} \quad \text{and} \quad {}_{\Lambda}[-, -] : \mathbf{M}_{\Lambda} \times \mathbf{B}_{\Lambda} \to \mathbf{B}$$

with properties as in 1.9, 1.14, 1.26 and 1.27, respectively.

1.29 Theorem. *Let* Λ *be a ring and* **B** *an additive category with cokernels. Assume either that* **B** *has arbitrary sums or that* Λ *is finitely generated as an abelian group. Then there exists a functor* $\mathbf{B} \to \mathbf{B}_{\Lambda}$, $B \rightsquigarrow B \otimes_{\mathbf{Z}} \Lambda$ (cf. 1.9 or 1.14 respectively) *which is left adjoint to* $V : \mathbf{B}_{\Lambda} \to \mathbf{B}$. *Dually, let* **B** *be an additive category with kernels. Assume either that* **B** *has arbitrary products or that* Λ *is finitely generated as an abelian group. Then there exists a functor* $\mathbf{B} \to \mathbf{B}_{\Lambda}$, $B \rightsquigarrow {}_{\mathbf{Z}}[\Lambda, B]$, (cf. 1.26 or 1.27 respectively) *which is right adjoint to* $V : \mathbf{B}_{\Lambda} \to \mathbf{B}$.

We shall not prove 1.29, since we only make incidental use of it later. But we give the reader a hint how 1.29 can be established:

For each pair $(B', \rho') \in \mathbf{B}_{\Lambda}$, $B \in \mathbf{B}$ there exists a commutative diagram

$$\begin{array}{ccc} {}_{\mathbf{B}}[B \otimes_{\mathbf{Z}} \Lambda, V(B', \rho')] & \xrightarrow[\cong]{\Theta(\Lambda, B')} & {}_{\mathbf{Ab.Gr.}}[\Lambda, [B, V(B', \rho')]] \\ \cup \Big\uparrow & & \Big\uparrow \cup \\ {}_{\mathbf{B}_{\Lambda}}[B \otimes_{\mathbf{Z}} \Lambda, (B', \rho')] & \xrightarrow{\cong} & {}_{\mathbf{M}_{\Lambda}}[\Lambda, [B, V(B', \rho')]_{\Lambda}] \cong [B, V(B', \rho')]_{\Lambda} \end{array}$$

where $[B, V(B', \rho')]_{\Lambda}$ is the abelian group $[B, V(B', \rho')]$ equipped with the right Λ-module structure induced by (B', ρ').

(1.30) Let **A** and **B** be additive categories, **B** right complete. This is equivalent to saying that **B** has arbitrary sums and cokernels (cf. Grothendieck [9] p. 133). Hence there exists by 1.14 ($\Lambda = \mathbf{Z}$) for each pair $A \in \mathbf{A}$, $B \in \mathbf{B}$ the composed functor

$$B \otimes_{\mathbf{Z}} [A, -] : \mathbf{A} \to \mathbf{Ab.Gr.} \to \mathbf{B}$$

which is said to be a generalized representable functor. Dually, if $\mathbf{B}$ is left complete, then by 1.27 ($\Lambda = \mathbf{Z}$) there exists for each pair $B \in \mathbf{B}$, $A \in \mathbf{A}$ the composite functor

$$_{\mathbf{Z}}[-, B] \cdot [-, A] : \mathbf{A} \to \mathbf{Ab.Gr.} \to \mathbf{B}$$

which is said to be a corepresentable functor.

(1.31) Let $\mathbf{A}$ and $\bar{\mathbf{B}}$ be additive categories. The tensor product $\bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}$ of $\bar{\mathbf{B}}$ and $\mathbf{A}$ over $\mathbf{Z}$ is an additive category, whose objects are symbols $\bar{B} \otimes_{\mathbf{Z}} A$, where $\bar{B} \in \bar{\mathbf{B}}$ and $A \in \mathbf{A}$. The abelian group of morphisms from $\bar{B} \otimes_{\mathbf{Z}} A$ to $\bar{B}' \otimes_{\mathbf{Z}} A'$ is the tensor product $[\bar{B}, \bar{B}'] \otimes [A, A']$. If $\bar{\mathbf{B}}$ consists of a single objects U, we use the notation $\{U\} \otimes_{\mathbf{Z}} \mathbf{A}$ instead of $\bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}$.

Let $\mathbf{B}$ be a right complete category and $\bar{\mathbf{B}}$ a subcategory of $\mathbf{B}$. The functor

$$Y : \bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), \bar{B} \otimes_{\mathbf{Z}} A \rightsquigarrow \bar{B} \otimes_{\mathbf{Z}} [A, -],$$

which assigns to a morphism $\bar{\beta} \otimes_{\mathbf{Z}} \alpha$ the natural tranformation $\bar{\beta} \otimes_{\mathbf{Z}} [\alpha, -]$ is called the generalized Yoneda functor.

(1.32) Dually, assume that $\bar{\mathbf{B}}$ is a subcategory of a left complete additive category $\mathbf{B}$. Then there exists a functor

$$Y' : \mathbf{B} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), \bar{B} \otimes_{\mathbf{Z}} A \rightsquigarrow {}_{\mathbf{Z}}[-, \bar{B}] \cdot [-, A],$$

which assigns to a morphism $\bar{\beta} \otimes_{\mathbf{Z}} \alpha$ the natural transformation $_{\mathbf{Z}}[-, \bar{\beta}] \cdot [-, \alpha]$

1.33 Theorem. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be additive categories. Assume that* $\mathbf{B}$ *is right complete and has small dense subcategory* $\bar{\mathbf{B}}$.[12] *Then the Yoneda functor*

$$Y : \bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), \bar{B} \otimes_{\mathbf{Z}} A \rightsquigarrow \bar{B} \otimes_{\mathbf{Z}} [A, -],$$

is dense ([25] 1.3), i.e. *each functor* $t : \mathbf{A} \to \mathbf{B}$ *is canonically a direct limit of generalized representable functors* $\bar{B} \otimes_{\mathbf{Z}} [A, -]$. *Furthermore it follows immediately from* [25] 1.11 *and* [25] 1.17 *that any subcategory of* $(\mathbf{A}, \mathbf{B})$ *is dense whose objects are the functors* $\bar{B} \otimes_{\mathbf{Z}} [A, -]$ *and whose morphisms include the natural transformations* $\bar{\beta} \otimes_{\mathbf{Z}} [\alpha, -]$, *where* A, α *and* $\bar{B}$, $\bar{\beta}$ *run through* $\mathbf{A}^{opp}$ *and* $\bar{\mathbf{B}}$ *respectively.*

(1.34) *Dually assume that* $\mathbf{B}$ *is left complete and has a small codense subcategory* $\bar{\mathbf{B}}$. *Then the functor*

$$Y' : \bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), \bar{B} \otimes_{\mathbf{Z}} A \rightsquigarrow {}_{\mathbf{Z}}[-, \bar{B}] \cdot [-, A],$$

[12] The smallness of $\bar{\mathbf{B}}$ is not essential. Using different methods one can show that for any dense functor $J : \bar{\mathbf{B}} \to \mathbf{B}$ the Yoneda functor $\bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B})$, $\bar{B} \otimes_{\mathbf{Z}} A \rightsquigarrow J\bar{B} \otimes_{\mathbf{Z}} [A, -]$, is dense (cf. 2.12).

is codense, i.e. each functor $t : \mathbf{A} \to \mathbf{B}$ *is canonically an inverse limit of corepresentable functors* $_Z[-, \bar{B}] \cdot [-, A]$. *Furthermore, any subcategory of* $(\mathbf{A}, \mathbf{B})$ *is codense whose objects are the functors* $_Z[-, \bar{B}] \cdot [-, A]$ *and whose morphisms include the natural transformations* $_Z[-, \bar{\beta}] \cdot [-, \alpha]$, *where* A, α *and* $\bar{B}$, $\bar{\beta}$ *run through* $\mathbf{A}^{opp}$ *and* $\mathbf{B}$ *respectively.*

Proof. We give a proof for 1.33; the statement 1.34 then follows by duality. Let $J : \bar{\mathbf{B}} \to \mathbf{B}$ denote the dense inclusion. According to [*25*] 1.15 and [*25*] 1.17 the functor $S : \mathbf{B} \to (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.})$, $B \rightsquigarrow [J-, B]$, has a left adjoint $T : (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}) \to \mathbf{B}$ which is a retraction. By [*25*] 2.22 and [*25*] 2.1 the induced functors $(\mathbf{A}, S) : (\mathbf{A}, \mathbf{B}) \to (\mathbf{A}, (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}))$, $t \rightsquigarrow S \cdot t$, and $(\mathbf{A}, T) : (\mathbf{A}, (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.})) \to (\mathbf{A}, \mathbf{B})$, $r \rightsquigarrow T \cdot r$, are also adjoint and $(\mathbf{A}, T)$ is a left retraction. Since the functor

$$R : (\bar{\mathbf{B}}^{opp} \otimes_Z \mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.})),$$

$\{R(F)(A)\}(\bar{B}) = F(\bar{B} \otimes_Z A)$, is an equivalence, $(\mathbf{A}, T) \cdot R$ is left adjoint to $R^{-1} \cdot (\mathbf{A}, S)$ and $(\mathbf{A}, T) \cdot R$ is a retraction. By [*25*] 1.10 and [*25*] 1.17 the Yoneda embedding $(\bar{\mathbf{B}}^{opp} \otimes_Z \mathbf{A})^{opp} \to (\bar{\mathbf{B}}^{opp} \otimes_Z \mathbf{A}, \mathbf{Ab.Gr.})$, $\bar{B} \otimes_Z A \rightsquigarrow [\bar{B} \otimes_Z A, -]$, is dense. Therefore by [*25*] 1.13 and [*25*] 1.17 the same holds for the composite of $\bar{\mathbf{B}} \otimes_Z \mathbf{A}^{opp} = (\bar{\mathbf{B}}^{opp} \otimes_Z \mathbf{A})^{opp} \to (\bar{\mathbf{B}}^{opp} \otimes_Z \mathbf{A}, \mathbf{Ab.Gr.})$ and $(\mathbf{A}, T) \cdot R : (\bar{\mathbf{B}}^{opp} \otimes_Z \mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, \mathbf{B})$. Hence to prove 1.33 it is enough to show that this composite is equivalent to $Y : \bar{\mathbf{B}} \otimes_Z \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B})$, $\bar{B} \otimes_Z A \rightsquigarrow \bar{B} \otimes_Z [A, -]$. To establish this we need some preparation (1.35 — 1.38).

(1.35) For each abelian group G and object $B \in \mathbf{B}$ the functors $[G, [J-, B]] : \bar{\mathbf{B}}^{opp} \to \mathbf{Ab.Gr.}$ and $[J-, {}_Z[G, B]] : \bar{\mathbf{B}}^{opp} \to \mathbf{Ab.Gr.}$ are equivalent by 1.27 (cf. 1.26 3)). Hence the diagram

(1.36)
$$\begin{array}{ccc}
(\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}) & \xrightarrow{(\bar{\mathbf{B}}^{opp}, [G,-])} & (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}) \\
\uparrow{\scriptstyle S} & & \uparrow{\scriptstyle S} \\
\mathbf{B} & \xrightarrow{\quad {}_Z[G,-] \quad} & \mathbf{B}
\end{array}$$

is commutative up to an equivalence. The left adjoints of the functors S, $(\bar{\mathbf{B}}^{opp}, [G, -])$ and $_Z[G, -]$ are T, $(\bar{\mathbf{B}}^{opp}, G\otimes)$ (cf. [*25*] 2.22) and $\otimes_Z G : \mathbf{B} \to \mathbf{B}$ respectively [for the latter combine 1.14 (1.12) and 1.27 (1.26 2)]. Since the diagram 1.36 is commutative up to an equivalence, it follows that

(1.37)
$$T \cdot (\bar{\mathbf{B}}^{opp}, G\otimes) \cong \otimes_Z G \cdot T.$$

Since $T : (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}) \to \mathbf{B}$ is left adjoint to S, the groups $[T[-, \bar{B}], B]$ and $[[-, \bar{B}], SB] = [[-, \bar{B}], [J-, B]] \cong [J\bar{B}, B]$ are equivalent for all $B \in \mathbf{B}$, $\bar{B} \in \bar{\mathbf{B}}$. Therefore it follows from the Yoneda lemma that there is an equivalence between $\bar{B}$ and $T[-, \bar{B}]$ which is natural in $\bar{B}$. By 1.37 we obtain

$$(1.38) \qquad \{T \cdot (\bar{\mathbf{B}}^{opp}, G\otimes)\}[-, \bar{B}] = T\{G\otimes \cdot [-, \bar{B}]\} \cong (\otimes_{\mathbf{Z}} G \cdot T)[-, \bar{B}] \cong \bar{B} \otimes_{\mathbf{Z}} G.$$

The value of $R : (\bar{\mathbf{B}}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}))$ at $[\bar{B} \otimes_{\mathbf{Z}} A, -]$ is the functor $\mathbf{A} \to (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.})$, $A' \rightsquigarrow [A, A'] \otimes [-, \bar{B}]$. By 1.38 (with $G = [A, A']$) the functor $T : (\bar{\mathbf{B}}^{opp}, \mathbf{Ab.Gr.}) \to \mathbf{B}$ assigns to $[A, A'] \otimes [-, \bar{B}]$ an object which is equivalent to $\bar{B} \otimes_{\mathbf{Z}} [A, A']$. Hence we obtain

$$(1.39) \qquad \{(\mathbf{A}, T) \cdot R[\bar{B} \otimes_{\mathbf{Z}} A, -]\} (A') \cong \bar{B} \otimes_{\mathbf{Z}} [A, A']$$

Let $\alpha : A' \to A''$ be a morphism. Then one readily checks that the morphisms $\{(\mathbf{A}, T) \cdot R[\bar{B} \otimes_{\mathbf{Z}} A, -]\} (\alpha)$ and

$$\bar{B} \otimes_{\mathbf{Z}} [A, \alpha] : \bar{B} \otimes_{\mathbf{Z}} [A, A'] \to \bar{B} \otimes_{\mathbf{Z}} [A, A'']$$

and the two equivalences (1.39) for A' and A'' respectively form a commutative diagram in an obvious way. This proves that the image of $[\bar{B} \otimes_{\mathbf{Z}} A, -]$ under $(\mathbf{A}, T) \cdot R : \{\bar{\mathbf{B}}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.}\} \to (\mathbf{A}, \mathbf{B})$ is equivalent to the functor $\bar{B} \otimes_{\mathbf{Z}} [A, -] : \mathbf{A} \to \mathbf{B}$. We leave it to the reader to check that this equivalence is natural in $\bar{B} \otimes_{\mathbf{Z}} A$. Thus the composite of the Yoneda embedding

$$\bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} = (\bar{\mathbf{B}}^{opp} \otimes_{\mathbf{Z}} \mathbf{A})^{opp} \to (\bar{\mathbf{B}}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.})$$

and $(\mathbf{A}, T) \cdot R : (\bar{\mathbf{B}}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, \mathbf{B})$ is equivalent to

$$Y : \bar{\mathbf{B}} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), \ \bar{B} \otimes_{\mathbf{Z}} A \rightsquigarrow \bar{B} \otimes_{\mathbf{Z}} [A, -].$$

This proves that Y is dense.

Q.E.D.

1.40 THEOREM. *Let* **A** *and* **B** *be additive categories. Assume that* **B** *is right complete and has a dense generator* U (cf. [25] 1.19). *Then the functor* (cf. 1.31)

$$Y : \{U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), \ U \otimes_{\mathbf{Z}} A \rightsquigarrow U \otimes_{\mathbf{Z}} [A, -],$$

is dense; *i.e. each functor* $t : \mathbf{A} \to \mathbf{B}$ *is canonically a direct limit of generalized representable functors* $U \otimes_{\mathbf{Z}} [A, -]$, $A \in \mathbf{A}^{opp}$. *Furthermore, it follows immediately from* [25] 1.11 *and* [25] 1.17 *that any subcategory of* $(\mathbf{A}, \mathbf{B})$ *is dense whose objects are the functors* $U \otimes_{\mathbf{Z}} [A, -]$ *and whose morphisms include the* *natural transformations* $\gamma \otimes_{\mathbf{Z}} [\alpha, -]$, *where* A, α *and* γ *run through* $\mathbf{A}^{opp}$ *and* $\{U\}$ *respectively.*

(1.41) Dually, assume that $\mathbf{B}$ is left complete and has a codense cogenerator Q (cf. [*25*] 1.19). Then the functor

$$Y' : \{Q\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}), Q \otimes_{\mathbf{Z}} A \rightsquigarrow {}_{\mathbf{Z}}[-, Q] \cdot [-, A],$$

is codense, i.e., each functor $t : \mathbf{A} \to \mathbf{B}$ is canonically an inverse limit of corepresentable functors ${}_{\mathbf{Z}}[-, Q] \cdot [-, A]$. Furthermore, any subcategory of $(\mathbf{A}, \mathbf{B})$ is codense whose objects are the functors ${}_{\mathbf{Z}}[-, Q] \cdot [-, A]$ and whose morphisms include the natural transformations ${}_{\mathbf{Z}}[-, \gamma][-, \alpha]$, where A, α and γ run through $\mathbf{A}^{opp}$ and $\{Q\}$ respectively.

Proof. The statements 1.40 and 1.41 are actually special cases of 1.33 and 1.34 respectively. We show this for 1.40 and 1.33. For 1.41 and 1.34 this follows by duality.

Let $\{\oplus_F U\}$ denote the full subcategory of $\mathbf{B}$ whose objects are U, $U \oplus U$, $U \oplus U \oplus U$, ... In the proof of [*25*] 1.23 (ii) $\to$ (i) it was shown that the inclusion $\{\oplus_F U\} \to \mathbf{B}$ is dense. Hence by 1.33 the Yoneda functor $Y_0 : \{\oplus_F U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B})$, $(\oplus_{i=1}^n U_i) \otimes_{\mathbf{Z}} A \rightsquigarrow (\oplus_{i=1}^n U_i) \otimes_{\mathbf{Z}} [A, -]$, is dense ($U_i = U$). Let $I : \{U\} \to \{\oplus_F U\}$ be the inclusion. Then it follows readily from the definition of the tensor product of categories (1.31) that the functors $I \otimes_{\mathbf{Z}} \mathbf{A}^{opp} : \{U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to \{\oplus_F U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp}$ and

$$\{\oplus_F U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to \{U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp}, (\oplus_{i=1}^n U_i) \otimes_{\mathbf{Z}} A \rightsquigarrow U \otimes_{\mathbf{Z}} (\oplus_{i=1}^n A_i),$$

where $A_i = A$, establish an equivalence between $\{U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp}$ and $\{\oplus_F U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp}$. The Yoneda functor $Y : \{U\} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B})$ is the composite of Y_0 with the equivalence $I \otimes_{\mathbf{Z}} \mathbf{A}^{opp}$. Since Y_0 is dense, the same holds for Y.

Q.E.D.

2. The Basic Properties of Generalized Representable Functors

The essential theorems of this section have been stated in the introduction (16) — (22).

2.1 Assumptions. In this section, we adopt some notations and make certain assumptions which are collected here for convenience.

(2.2) A category denoted by $\mathbf{A}$ will be assumed to be Λ-enriched (3.7), where Λ is a ring.

(2.3) A category denoted by **B** is assumed to have cokernels.

(2.4) A pair of categories denoted by **A** and **B** will be assumed to satisfy one of the following conditions:

(i) The category **B** has infinite sums. Then the symbol Λ denotes any object of **A**.

(ii) For each pair $X, Y \in \mathbf{A}$ the Λ-module ${}_\Lambda[X, Y]$ is finitely presentable. Then the symbol A denotes any object of **A**. The only exception is 2.8 where A is restricted because (ii) is replaced by the weaker condition:

(ii)′ There exists an object $A \in \mathbf{A}$ for which the functor ${}_\Lambda[A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M}$ factors through the inclusion $J : \mathbf{FP}({}_\Lambda\mathbf{M}) \to {}_\Lambda\mathbf{M}$ (cf. 1.1)

The duals of the assumptions 2.2 – 2.4 are denoted by 2.2* – 2.4*.

(2.5) Let **A** and **B** be additive categories satisfying 2.2 – 2.4. Then by 1.9 and 1.14 there exists for each $(B, \rho) \in \mathbf{B}_\Lambda$ a generalized representable functor

$$(B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$$

or

$$(B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{B}$$

(2.6) Dually, if **A** and **B** satisfy 2.2* – 2.4*, then by 1.26 and 1.27 there exists for each $(\rho, B) \in {}_\Lambda\mathbf{B}$ a generalized corepresentable functor

$${}_\Lambda[-, (\rho, B)] \cdot {}_\Lambda[-, A] : \mathbf{A} \to \mathbf{B}$$

(2.7) Since **A** is Λ-enriched, there exists by (3.8e) for each $A \in \mathbf{A}$ ring homomorphisms $\sigma_A : \Lambda^{opp} \to [A, A]$ and $\pi_A : \Lambda \to [A, A]$ with the property $\pi_A(\lambda) = \sigma_A(\lambda)$, where $\Lambda \ni \lambda \in \Lambda^{opp}$. Thus the values tA of a functor $t : \mathbf{A} \to \mathbf{B}$ together with the ring homomorphisms $\Lambda^{opp} \to [A, A] \to [tA, tA]$ and $\Lambda \to [A, A] \to [tA, tA]$ are both right and left Λ-objects. Since the values of these composed ring homomorphisms agree at each λ, where $\Lambda \ni \lambda \in \Lambda^{opp}$, we denote them both by τ_A.

2.8 Lemma (*Yoneda*).[13] *Let* **A** *and* **B** *be additive categories satisfying* 2.2, 2.3 *and either* 2.4(i) *or* 2.4(ii)′. *Let* $t : \mathbf{A} \to \mathbf{B}$ *be a functor. Then for each* $(B, \rho) \in \mathbf{B}_\Lambda$ *the homomorphism*

(2.9) $$\Omega((B, \rho), A, t) : [(B, \rho) \otimes_\Lambda [A, -], t] \to [B, tA]$$

[13] For small categories **A** and $\Lambda = \mathbf{Z}$ this was first observed by Freyd [4], p. 18. Using different methods he proved that there exists an isomorphism between the groups $[B \otimes_{\mathbf{Z}} [A, -], t]$ and $[B, tA]$.

which assigns to $\psi : (B, \rho) \otimes_\Lambda [A, -] \to t$ *the composed morphism*

$$B = (B, \rho) \otimes_\Lambda \Lambda \xrightarrow{\mathrm{id} \otimes_\Lambda f_A} (B, \rho) \otimes_\Lambda [A, A] \xrightarrow{\psi(A)} tA,$$

where $f_A(1) = \mathrm{id}_A$, *establishes a 1–1 correspondence between the natural transformations from* $(B, \rho) \otimes_\Lambda [A, -]$ *to* t *and the* Λ*-morphisms from* (B, ρ) *to* (tA, τ_A).[14]

(2.10)[13] *Dually, assume that* **A** *and* **B** *satisfy* 2.2*, 2.3* *and either* 2.4(i)* *or* 2.4(ii)′*. *Then for each functor* $t : \mathbf{A} \to \mathbf{B}$ *and* $(\rho, B) \in {}_\Lambda\mathbf{B}$ *there exists an isomorphism*

$$[t, {}_\Lambda[-, (\rho, B)] \cdot {}_\Lambda[-, A]] \cong [(\tau_A, tA), (\rho, B)]$$

similar to 2.9.

Proof. We prove 2.8 if **A** and **B** satisfy 2.2, 2.3 and 2.4(ii)′. Then by 2.5 $(B, \rho) \otimes_\Lambda [A, -]$ is the composition $\mathbf{A} \to \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$. The proof for the other case is similar but simpler. The statement 2.10 then follows by duality. By 1.12 the functor $(B, \rho) \otimes_\Lambda : \mathbf{FP}({}_\Lambda\mathbf{M}) \to \mathbf{B}$ is left adjoint to ${}_\Lambda[B, -] : \mathbf{B} \to {}_\Lambda\mathbf{M}$ (cf. 1.7) relative to the inclusion $J : \mathbf{FP}({}_\Lambda\mathbf{M}) \to {}_\Lambda\mathbf{M}$ (cf. [25] 2.2). Then by [25] 2.22 and [25] 2.1 the induced functor $(\mathbf{A}, (B, \rho) \otimes_\Lambda) : (\mathbf{A}, \mathbf{FP}({}_\Lambda\mathbf{M})) \to (\mathbf{A}, \mathbf{B})$ is left adjoint to $(\mathbf{A}, {}_\Lambda[\mathbf{B}, -]) : (\mathbf{A}, \mathbf{B}) \to (\mathbf{A}, {}_\Lambda\mathbf{M})$ relative to $(\mathbf{A}, J) : (\mathbf{A}, \mathbf{FP}({}_\Lambda\mathbf{M})) \to (\mathbf{A}, {}_\Lambda\mathbf{M})$. Hence for ${}_\Lambda[A, -] \in (\mathbf{A}, \mathbf{FP}({}_\Lambda\mathbf{M}))$ and $t \in (\mathbf{A}, \mathbf{B})$ there exists an isomorphism

$$(2.11)\quad [(B, \rho) \otimes_\Lambda [A, -], t] \xrightarrow{\cong} [{}_\Lambda[A, -], {}_\Lambda[B, -] \cdot t] = [{}_\Lambda[A, -], {}_\Lambda[B, t-]].$$

Let $\xi \in {}_\Lambda[A, X]$ and $f_\xi : \Lambda \to {}_\Lambda[A, X]$ be the Λ-homomorphism $1 \leadsto \xi$. From 1.21 and the proof of [25] 2.22 one can easily deduce that the isomorphism 2.11 assigns to $\psi : (B, \rho) \otimes_\Lambda [A, -] \to t$ the natural transformation $\psi' : {}_\Lambda[A, -] \to {}_\Lambda[B, t-]$, where $\psi'(X)(\xi)$ is the composed morphism $B = (B, \rho) \otimes_\Lambda \Lambda \xrightarrow{\mathrm{id} \otimes_\Lambda f_\xi} (B, \rho) \otimes_\Lambda [A, X] \xrightarrow{\psi(X)} tX$. Thus the Yoneda map 3.18 (for the functor ${}_\Lambda[B, t-]$) $Y(A) : [{}_\Lambda[A, -], {}_\Lambda[B, t-]] \to {}_\Lambda[B, tA]$ assigns to $\psi' : {}_\Lambda[A, -] \to {}_\Lambda[B, t-]$ the composite morphism

$$B = (B, \rho) \otimes_\Lambda \Lambda \xrightarrow{\mathrm{id} \otimes_\Lambda f_A} (B, \rho) \otimes_\Lambda [A, A] \xrightarrow{\psi(A)} tA.$$

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[14] Actually the Yoneda Lemma 2.8 is also valid, if **A** is not Λ-enriched. It suffices that for the object A of **A** the functor $[A, -]$ can be decomposed into $U \cdot {}_\Lambda[A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$ (cf. the remark after the proof of 3.17).

The composition of $Y(A)$ with the isomorphism 2.11 therefore yields $\Omega((B, \rho), A, t)$ (cf. 2.9). From the Yoneda lemma 3.17 it follows that $\Omega((B, \rho), A, t)$ establishes a 1–1 correspondence between the natural transformation from $(B, \rho) \otimes_\Lambda [A, -]$ to t and the morphisms $f: B \to tA$ with the property $\lambda f = f\lambda$, $\lambda \in \Lambda$. According to 1.7 λf is the composition $f \cdot \rho(\lambda): B \to B \to tA$. By 3.21 $f\lambda$ is equal to ${}_\Lambda[B, t\sigma_A(\lambda)](f)$ and since $t\sigma_A(\lambda) = \tau_A(\lambda)$ (cf. 2.7), we obtain $f\lambda = \tau_A(\lambda) \cdot f$. Thus the morphisms $f: B \to tA$ with the property $\lambda f = f\lambda$, $\lambda \in \Lambda$, are the Λ-morphisms from (B, ρ) to (tA, τ_A). This proves the Yoneda lemma 2.8 for generalized representable functors. Q.E.D.

2.12 THEOREM. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be additive categories satisfying* 2.2 – 2.4 *and* $J: \mathbf{C} \to \mathbf{B}_\Lambda$ *an additive functor which is dense* (cf. [25] 1.3, *for instance* $J = \mathrm{id}_{\mathbf{B}_\Lambda}$). *Then the generalized Yoneda functor*

$$Y: \mathbf{C} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}),\ C \otimes_{\mathbf{Z}} A \rightsquigarrow JC \otimes_\Lambda [A, -],$$

is dense (for $\mathbf{C} \otimes_{\mathbf{Z}} \mathbf{A}^{opp}$ cf. 1.31), i.e. *each functor* $t: \mathbf{A} \to \mathbf{B}$ *is canonically a direct limit of representable functors* $JC \otimes_\Lambda [A, -]$, *where* $C \in \mathbf{C}$ *and* $A \in \mathbf{A}$. *Furthermore it follows from* [25] 1.11 *and* [25] 1.17 *that a subcategory of* $(\mathbf{A}, \mathbf{B})$ *is dense, provided its objects are the functors* $JC \otimes_\Lambda [A, -]$ *and its morphisms include the natural transformations* $J\gamma \otimes_\Lambda [\alpha, -]$, *where* A, α *and* C, γ *run through* $\mathbf{A}^{opp}$ *and* $\mathbf{C}$ *respectively. In particular if* $\mathbf{A}$ *and* $\mathbf{C}$ *are small, then* $(\mathbf{A}, \mathbf{B})$ *has a small dense subcategory.*

(2.13) *Dually, assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy* 2.2* – 2.4* *and that* $J: \mathbf{C} \to {}_\Lambda\mathbf{B}$ *is a codense functor* (e.g. $J = \mathrm{id}_{\Lambda\mathbf{B}}$). *Then the functor*

$$Y': \mathbf{C} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}),\ C \otimes_{\mathbf{Z}} A \rightsquigarrow {}_\Lambda[\![-, JC]\!] \cdot {}_\Lambda[-, A],$$

is codense, i.e. each functor is canonically an inverse limit of corepresentable functors ${}_\Lambda[\![-, JC]\!] \cdot {}_\Lambda[-, A]$. *Furthermore any subcategory of* $(\mathbf{A}, \mathbf{B})$ *is codense whose objects are the functors* ${}_\Lambda[\![-, JC]\!] \cdot {}_\Lambda[-, A]$ *and whose morphisms include the natural transformations* ${}_\Lambda[\![-, J\gamma]\!] \cdot {}_\Lambda[-, \alpha]$, *where* A, α *and* C, γ *run through* $\mathbf{A}^{opp}$ *and* $\mathbf{C}$ *respectively. In particular if* $\mathbf{A}$ *and* $\mathbf{C}$ *are small, then* $(\mathbf{A}, \mathbf{B})$ *has a small codense subcategory.*

Proof. We prove 2.12. The assertion 2.13 then follows by duality. To establish that Y is dense we need some preparation.

Since $\mathbf{A}$ is Λ-enriched (3.7), there is a functor $F: \mathbf{A} \to \mathbf{A}_\Lambda$ (cf. 3.8). By 3.11 and 3.12 every functor $t: \mathbf{A} \to \mathbf{B}$ can be canonically decomposed into $\mathbf{A} \xrightarrow{F} \mathbf{A}_\Lambda \xrightarrow{t_\Lambda} \mathbf{B}_\Lambda \xrightarrow{V} \mathbf{B}$. (For the notation, see 3.11, 3.12. The Λ-structure on tA is given by $\Lambda^{opp} \xrightarrow{\sigma} [A, A] \xrightarrow{t} [tA, tA]$, where $(A, \sigma) = FA$.) The composition $t_\Lambda \cdot F : \mathbf{A} \to \mathbf{A}_\Lambda \to \mathbf{B}_\Lambda$ is called the canonical lifting of t. Call a functor $s : \mathbf{A} \to \mathbf{B}_\Lambda$ strong if it is the canonical lifting of a functor $\mathbf{A} \to \mathbf{B}$. The full subcategory of $(\mathbf{A}, \mathbf{B}_\Lambda)$ generated by the strong functors is denoted by $Str(\mathbf{A}, \mathbf{B}_\Lambda)$. From 3.11 and 3.14 it follows that the forgetful functor $V : \mathbf{B}_\Lambda \to \mathbf{B}$ induces an isomorphism $Str(\mathbf{A}, \mathbf{B}_\Lambda) \xrightarrow{\cong} (\mathbf{A}, \mathbf{B})$, $s \rightsquigarrow V \cdot s$. Thus the Yoneda functor $Y : \mathbf{C} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B})$, $C \otimes_{\mathbf{Z}} A \rightsquigarrow JC \otimes_\Lambda [A, -]$ gives rise to a composite

$$(2.14) \quad Str(\mathbf{A}, \mathbf{B}_\Lambda) \xrightarrow{\cong} (\mathbf{A}, \mathbf{B}) \to (\mathbf{C}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.}),\ t_\Lambda F \rightsquigarrow t \rightsquigarrow [Y-, t].$$

To prove the density of Y, it suffices by [25] 1.7, 1.17 to show that the composite 2.14 is full and faithful.

Since $J : \mathbf{C} \to \mathbf{B}_\Lambda$ is dense, the functor $\mathbf{B}_\Lambda \to (\mathbf{C}^{opp}, \mathbf{Ab.Gr.})$, $(B, \rho) \rightsquigarrow [J-, (B, \rho)]$ is full and faithful. Thus the induced functor

$$Q : Str(\mathbf{A}, \mathbf{B}_\Lambda) \to (\mathbf{A}, (\mathbf{C}^{opp}, \mathbf{Ab.Gr.})),\ s \rightsquigarrow (A \rightsquigarrow [J-, sA]),$$

is also full and faithful. Recall that the functor $R : (\mathbf{C}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, (\mathbf{C}^{opp}, \mathbf{Ab.Gr.}))$, where $\{R(E)(A)\}(C) = E(C \otimes_{\mathbf{Z}} A)$, is an equivalence (cf. proof of 1.33). Hence the composite

$$R^{-1} \cdot Q : Str(\mathbf{A}, \mathbf{B}_\Lambda) \to (\mathbf{A}, (\mathbf{C}^{opp}, \mathbf{Ab.Gr.})) \to (\mathbf{C}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.})$$

is full and faithful.

We now are in a position to show that the composite functors 2.14 and $R^{-1} \cdot Q$ are equivalent. This clearly implies that 2.14 is full and faithful. Hence Y is dense.

Let $t_\Lambda \cdot F : \mathbf{A} \to \mathbf{B}_\Lambda$ be the canonical lifting of a functor t. By definition of R and Q the value of the functor $R^{-1} \cdot Q(t_\Lambda \cdot F) : \mathbf{C}^{opp} \otimes_{\mathbf{Z}} \mathbf{A} \to \mathbf{Ab.Gr.}$ at $C \otimes_{\mathbf{Z}} A$ is $[JC, (tA, \tau_A)]$, where $(tA, \tau_A) = (t_\Lambda \cdot F)\, A$ cf. 2.7. On the other hand the value of the functor $[Y-, t] \in (\mathbf{C}^{opp} \otimes_{\mathbf{Z}} \mathbf{A}, \mathbf{Ab.Gr.})$ at $C \otimes_{\mathbf{Z}} A$ is $[JC \otimes_\Lambda [A, -], t]$ (cf. 2.14). Since there is an isomorphism $[JC \otimes_\Lambda [A, -], t] \cong [JC, (tA, \tau_A)]$, which is natural in C and A and t, the composite functors 2.14 and $R^{-1} \cdot Q$ are equivalent (cf. 2.9).

Q.E.D.

If $\mathbf{B}_\Lambda$ does not admit a dense functor $J : \mathbf{C} \to \mathbf{B}_\Lambda$ with a small domain, then the canonical index category in the representation of a functor $t : \mathbf{A} \to \mathbf{B}$ as direct limit of generalized representable functors is very large. The following theorem shows that in this case there is a smaller index category.

2.15 THEOREM.[15] *Let* **A** *and* **B** *be additive categories satisfying* 2.2 – 2.4. *Then for each functor* $t : \mathbf{A} \to \mathbf{B}$, *each object* $X \in \mathbf{A}$ *and each morphism* $\xi : X \to X'$

$$tX = \varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} (td\alpha, \tau_{d_\alpha}) \otimes_\Lambda [r\alpha, X] \tag{2.16}$$

and

$$t\xi = \varinjlim_{\alpha \in \mathbf{M}(\mathbf{A})} (td\alpha, \tau_{d_\alpha}) \otimes_\Lambda [r\alpha, \xi]$$

hold (for $\mathbf{M}(\mathbf{A})$ and $\mathbf{M}(\mathbf{A})^*$ cf. [*25*] 2.19).

Dually, assume that **A** *and* **B** *satisfy* 2.2* – 2.4*. *Then for each functor* $t : \mathbf{A} \to \mathbf{B}$, *each object* $X \in \mathbf{A}$ *and each morphism* $\xi : X \to X'$

$$tX = \varprojlim_{\alpha \in \mathbf{M}(\mathbf{A})^*} {}_\Lambda[{}_\Lambda[X, d\alpha], (\tau_{r_\alpha}, tr\alpha)] \tag{2.17}$$

and

$$t\xi = \varprojlim_{\alpha \in \mathbf{M}(\mathbf{A})^*} {}_\Lambda[{}_\Lambda[\xi, d\alpha], (\tau_{r_\alpha}, tr\alpha)]$$

are valid.

Proof. We give an outline for the first half of 2.16. From this the second part can be readily deduced. The assertion (2.17) then follows by duality.

From 1.12 it follows for each $Y \in \mathbf{A}$ that the functors $G : \mathbf{Mor}(\mathbf{A}) \to \mathbf{Ab.Gr.}$, $\alpha \rightsquigarrow [(td\alpha, \tau_{d_\alpha}) \otimes_\Lambda [r\alpha, X], Y]$, and $G' : \mathbf{Mor}(\mathbf{A}) \to \mathbf{Ab.Gr.}$, $\alpha \rightsquigarrow [{}_\Lambda[r\alpha, X], {}_\Lambda[td\alpha, Y]]$, are equivalent. There is a lemma similar to [*25*] 2.20 for contravariant functors. (One only has to replace $\mathbf{N}$ in [*25*] 2.20 by $\mathbf{N}^{opp}$.) Therefore $\varprojlim G' = [{}_\Lambda[-, X], {}_\Lambda[t-, Y]]$ is valid. Since ${}_\Lambda[t-, Y] : \mathbf{A}^{opp} \to {}_\Lambda\mathbf{M}$ is the canonical lifting of $[t-, Y] : \mathbf{A}^{opp} \to \mathbf{Ab.Gr.}$ (cf. 3.11 and 3.12), it follows from 3.14 that $[{}_\Lambda[-, X], {}_\Lambda[t-, Y]] \cong [[-, X], [t-, Y]] \cong [tX, Y]$. Hence

$$\varprojlim G \cong \varprojlim G' \cong [tX, Y] \tag{2.18}$$

holds. By [*25*] 2.10 and [*25*] 2.1 $\varprojlim G$ is isomorphic to the group of natural transformations from $\mathbf{Mor}(\mathbf{A}) \to \mathbf{B}$, $\alpha \rightsquigarrow (td\alpha, \tau_{d_\alpha}) \otimes_\Lambda [r\alpha, X]$, to the constant

[15] If $\Lambda = \mathbf{Z}$, $\mathbf{B} = \mathbf{Ab.Gr.}$ and **A** is small, then it can be shown that 2.16 is equivalent to the following assertion of Yoneda (cf. [*26*], 4.31*)

$$\int_{Y \in \mathbf{A}} tY \otimes \mathrm{Hom}(Y, X) = tX$$

functor const(Y) : $\mathbf{Mor}(\mathbf{A}) \to \mathbf{B}$. Since this isomorphism and those in 2.18 are natural in Y, the object tX is the direct limit of $\mathbf{Mor}(\mathbf{A}) \to \mathbf{B}$, $\alpha \rightsquigarrow (td\alpha, \tau_{d\alpha}) \otimes_{\Lambda} [r\alpha, X]$. Q.E.D.

(2.19) We now sketch a useful generalization of 2.12 and 1.33. Let $\mathbf{A}$ and $\mathbf{B}$ be as in 2.1 and $J : \mathbf{C} \to \mathbf{B}_{\Lambda}$ an additive functor such that each $(B, \rho) \in \mathbf{B}_{\Lambda}$ is functorially a direct limit of objects JC, where $C \in \mathbf{C}$. This is made precise in 2.20. Examples are dense functors J. However density is a much stronger condition than 2.20; from this weaker condition we will show that each functor $t : \mathbf{A} \to \mathbf{B}$ is functorially a direct limit of functors of the form $JC \otimes_{\Lambda} [A, -] : \mathbf{A} \to \mathbf{B}$, where $C \in \mathbf{C}$.

(2.20) Let $J : \mathbf{C} \to \mathbf{B}_{\Lambda}$ be a functor with the following properties:

(*a*) For each $(B, \rho) \in \mathbf{B}_{\Lambda}$ there is a category $\mathbf{D}(B, \rho)$ and a functor $F(B, \rho) : \mathbf{D}(B, \rho) \to \mathbf{C}$ such that $(B, \rho) = \varinjlim J \cdot F(B, \rho)$.

(*b*) For each morphism $f : (B, \rho) \to (B', \rho')$ there is a functor

$$H(f) : \mathbf{D}(B, \rho) \to \mathbf{D}(B', \rho')$$

and a natural transformation $\psi(f) : F(B, \rho) \to F(B', \rho') \cdot H(F)$, such that $f = \varinjlim J\psi(f)$. If $f = \mathrm{id}$, then $H(f) = \mathrm{id}$ and $\psi(f) = \mathrm{id}$.

2.21 THEOREM.[16] *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be additive categories as in* 2.1 *and* $J : \mathbf{C} \to \mathbf{B}_{\Lambda}$ *an additive functor satisfying* 2.20. *Then the property* 2.20 *carries over to the functor*

$$Z : \mathbf{C} \otimes_{\mathbf{Z}} \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{B}),\ C \otimes_{\mathbf{Z}} A \rightsquigarrow JC \otimes_{\Lambda} [A, -].$$

The dual statement is left to the reader.

Proof. (Sketch). In 2.15 we showed that each functor $t : \mathbf{A} \to \mathbf{B}$ can be represented as a direct limit of functors $(td\alpha, \tau_{d\alpha}) \otimes_{\Lambda} [r\alpha, -] : \mathbf{A} \to \mathbf{B}$. Roughly speaking it therefore suffices to prove that such a functor is a direct limit of functors $JC \otimes_{\Lambda} [r\alpha, -] : \mathbf{A} \to \mathbf{B}$, where $C \in \mathbf{C}$.

Let $(B, \rho) \otimes_{\Lambda} [A, -] : [\mathbf{A} \to \mathbf{B}$ be a generalized representable functor 2.5. By 2.20 $(B, \rho) = \varinjlim JC_{\iota}$ is valid, where $\iota \in \mathbf{D}(B, \rho)$. (We prefer this notation to $(B, \rho) = \varinjlim J \cdot F(B, \rho)$.) The following calculation based on 2.8 and [25] 2.10 shows that $(B, \rho) \otimes_{\Lambda} [A, -]$ is a direct limit of functors $JC_{\iota} \otimes_{\Lambda} [A, -]$, where ι runs through $\mathbf{D}(B, \rho)$.

[16] The proof that 2.21 is actually a generalization of 2.12 requires a rather elaborate cofinality argument.

$$(2.22)\quad [[B,\rho]\otimes_\Lambda[A,-],t]\cong[(B,\rho),(tA,\tau_A)]\cong[\varinjlim JC_\iota,(tA,\tau_A)]$$
$$\cong\varprojlim[JC_\iota,(tA,\tau_A)]\cong\varprojlim[JC_\iota\otimes_\Lambda[A,-],t]$$
$$(\cong[\varinjlim JC_\iota\otimes_\Lambda[A,-],t])$$

(2.23) For a functor $t:\mathbf{A}\to\mathbf{B}$ define a category $\mathbf{D}(t)$ as follows: The objects are pairs (ι,α), where α is an object of $\mathbf{Mor}(\mathbf{A})$ (i.e., a morphism of $\mathbf{A}$, cf. [25], 2.19) and ι an object of $\mathbf{D}(t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha})$. Let $\alpha:\mathrm{id}_{\mathrm{d}\alpha}\to\mathrm{id}_{\mathrm{r}\alpha}$ be a morphism in $\mathbf{A}$. A morphism $(\iota,\alpha)\to(\kappa,\mathrm{id}_{\mathrm{d}\alpha})$ in $\mathbf{D}(t)$ is a morphism $f:C_\iota\to C_\kappa$ in $\mathbf{C}$ such that the diagram

$$\begin{array}{ccccc} JC_\iota & \xrightarrow{u_\iota} & \varinjlim JC_\iota & = & (t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha}) \\ {\scriptstyle Jf}\downarrow & & & & \Vert \\ JC_\kappa & \xrightarrow{u_\kappa} & \varinjlim JC_\kappa & = & (t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha}) \end{array}$$

is commutative, where u_ι and u_κ are the canonical morphisms into the direct limit. Similarly a morphism $(\iota,\alpha)\to(\nu,\mathrm{id}_{\mathrm{r}\alpha})$ is a morphism $g:C_\iota\to C_\nu$ such that $v_\nu\cdot Jg=t\alpha\cdot u_\iota$, where $v_\nu:JC_\nu\to(t\mathrm{r}\alpha,\tau_{\mathrm{r}\alpha})$ is the canonical morphism into the direct limit. These are the only morphisms in $\mathbf{D}(t)$. Composition is defined in the obvious way. The category $\mathbf{D}(t)$ has all properties of a cofibre category except for one. [The base would be $\mathbf{Mor}(\mathbf{A})$, cf. [25] 2.19, and the cofibre at $\alpha\in\mathbf{Mor}(\mathbf{A})$ would be $\mathbf{D}(t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha})$.]

Define $F(t):\mathbf{D}(t)\to\mathbf{C}\otimes_{\mathbf{Z}}\mathbf{A}^{opp}$ to be the functor $(\iota,\alpha)\rightsquigarrow C_\iota\otimes_{\mathbf{Z}}\mathrm{r}\alpha$. With the notation as before (2.23) the value of $F(t)$ at a morphism $(\iota,\alpha)\to(\kappa,\mathrm{id}_{\mathrm{d}\alpha})$ or $(\iota,\alpha)\to(\nu,\mathrm{id}_{\mathrm{r}\alpha})$ is $f\otimes_{\mathbf{Z}}\alpha$ or $g\otimes_{\mathbf{Z}}\mathrm{id}$ respectively. For each $(\iota,\alpha)\in\mathbf{D}(t)$ there is a composite

$$(2.24)\qquad JC_\iota\otimes_\Lambda[\mathrm{r}\alpha,-]\xrightarrow{u_\iota\otimes_\Lambda\mathrm{id}}(t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha})\otimes_\Lambda[\mathrm{r}\alpha,-]\xrightarrow{w_\alpha}t$$

where u_ι is the canonical morphism into the direct limit and w_α is the natural tranformation which corresponds to $t\alpha$ under the Yoneda isomorphism $[(t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha})\otimes_\Lambda[\mathrm{r}\alpha,-],t]\cong[(t\mathrm{d}\alpha,\tau_{\mathrm{d}\alpha}),(t\mathrm{r}\alpha,\tau_{\mathrm{r}\alpha})]$, cf. 2.8. Using 2.22 and 2.16 it is not difficult to check that t together with the natural transformations 2.24 is the direct limit of $Z\cdot F(t):\mathbf{D}(t)\to(\mathbf{A},\mathbf{B})$, $(\iota,\alpha)\rightsquigarrow JC_\iota\otimes_\Lambda[\mathrm{r}\alpha,-]$. We leave it to the reader to show that this representation of t as a direct limit is functorial in t, i.e., that 2.20(b) is valid. Q.E.D.

2.25 *Remark.* Let Λ' be the ring Λ made commutative. As mentioned in 3.10, the canonical projection $p:\Lambda\to\Lambda'$ induces a 1-1 correspondence between the Λ-enrichments of $\mathbf{A}$ and the Λ'-enrichments of $\mathbf{A}$. Furthermore the functors $(t\mathrm{d}\alpha, \tau_{\mathrm{d}\alpha}) \otimes_\Lambda [\mathrm{r}\alpha, -] : \mathbf{A} \to \mathbf{B}$ and

$$_\Lambda[-, (\tau_{\mathrm{r}\alpha}, t\mathrm{r}\alpha)] \cdot {}_\Lambda[-, \mathrm{d}\alpha] : \mathbf{A} \to \mathbf{B}$$

considered in 2.16 and 2.17 have the property that $\tau_{\mathrm{d}\alpha}$ and $\tau_{\mathrm{r}\alpha}$ factor through $p^{opp} : \Lambda^{opp} \to \Lambda'$ and $p : \Lambda \to \Lambda'$ respectively (cf. 2.7). Therefore it is not a restriction to assume Λ commutative in 2.16 and 2.17. This is in contrast to 2.12, because the representable functors considered there do not have the property mentioned above.

(2.26) *For the following consequence of* 2.16 *assume Λ commutative. From* 1.5 *it easily follows that* 2.16 *remains valid if* $t : \mathbf{A} \to \mathbf{B}$ *and*

$$(t\mathrm{d}\alpha, \tau_{\mathrm{d}\alpha}) \otimes_\Lambda [\mathrm{r}\alpha, -] : \mathbf{A} \to \mathbf{B}$$

are replaced by their canonical liftings $t_\Lambda \cdot F_\Lambda : \mathbf{A} \to \mathbf{B}_\Lambda$ *and*

$$((t\mathrm{d}\alpha, \tau_{\mathrm{d}\alpha}) \otimes_\Lambda [\mathrm{r}\alpha, -])_\Lambda \cdot F_\Lambda : \mathbf{A} \to \mathbf{B}_\Lambda$$

(cf. 3.11). *A similar remark holds for* 2.17. *However this presentation of* $t_\Lambda \cdot F_\Lambda$ *as a direct limit of functors* $((t\mathrm{d}\alpha, \tau_{\mathrm{d}\alpha}) \otimes_\Lambda [\mathrm{r}\alpha, -])_\Lambda \cdot F_\Lambda$, $\alpha \in \mathbf{A}$, *is of little use in applications unless the latter are representable functors in* $(\mathbf{A}, \mathbf{B}_\Lambda)$. *We now prove that this is actually the case.*

(2.27) Let $\mathbf{B}$ be right complete and $(B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{B}$ a representable functor, where $A \in \mathbf{A}$ and $(B, \rho) \in \mathbf{B}_\Lambda$. (The case 2.4(ii) can be treated similarly.) Since Λ is commutative, $\rho : \Lambda \to [B, B]$ factors through the inclusion $I : [(B, \rho), (B, \rho)] \to [B, B]$. We denote the factorization also by ρ. Hence $((B, \rho), \rho)$ is an object of $(\mathbf{B}_\Lambda)_\Lambda$. We will show that

$$((B, \rho), \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{B}_\Lambda$$

and the canonical lifting of $(B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{B}$ coincide (cf. 3.11). From the construction of the tensor product (cf. 1.14 and 1.15 – 1.20) it easily follows that the diagram

$$\begin{array}{ccc} {}_\Lambda\mathbf{M} & \xrightarrow{((B,\rho),\rho)\otimes_\Lambda} & \mathbf{B}_\Lambda \\ \Vert & & \downarrow V \\ {}_\Lambda\mathbf{M} & \xrightarrow{(B,\rho)\otimes_\Lambda} & \mathbf{B} \end{array}$$

is commutative. Furthermore the endomorphism of $V\{((B, \rho), \rho) \otimes_\Lambda M\}$, $M \in {}_\Lambda\mathbf{M}$, corresponding to $\lambda \in \Lambda$ coincides with $\rho(\lambda) \otimes_\Lambda \mathrm{id} : (B, \rho) \otimes_\Lambda M \to$

$(B, \rho) \otimes_\Lambda M$. By 3.11 the canonical lifting of $(B, \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{B}$ is given by the morphisms $\mathrm{id} \otimes_\Lambda \sigma_X(\lambda) : (B, \rho) \otimes_\Lambda [A, X] \to (B, \rho) \otimes_\Lambda [A, X]$, where $X \in \mathbf{A}$ and $\lambda \in \Lambda$. Using 3.1 one readily checks that $\sigma_X(\lambda) : {}_\Lambda[A, X] \to {}_\Lambda[A, X]$ agrees with the action $\lambda(-) : {}_\Lambda[A, X] \to {}_\Lambda[A, X]$, $\xi \rightsquigarrow \lambda(\xi)$, of λ on ${}_\Lambda[A, X]$. Therefore we need only show that the morphisms

$$\rho(\lambda) \otimes_\Lambda \mathrm{id} : (B, \rho) \otimes_\Lambda [A, X] \to (B, \rho) \otimes_\Lambda [A, X]$$

and $\mathrm{id} \otimes_\Lambda \lambda(-) : (B, \rho) \otimes_\Lambda [A, X] \to (B, \rho) \otimes_\Lambda [A, X]$ coincide. From its construction (cf. 1.14 and 1.15 – 1.20), one easily deduces that the tensor product is bilinear. Hence $\rho(\lambda) \otimes_\Lambda \mathrm{id}$ and $\mathrm{id} \otimes_\Lambda \lambda(-)$ are equal, $\lambda \in \Lambda$. This proves that $((B, \rho), \rho) \otimes_\Lambda [A, -] : \mathbf{A} \to \mathbf{B}_\Lambda$ and $((B, \rho) \otimes_\Lambda [A, -])_\Lambda \cdot F_\Lambda : \mathbf{A} \to \mathbf{B}_\Lambda$ agree. Q.E.D.

Appendix. The Yoneda Lemma for Nonstrong Functors $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$

By a Λ-enriched category we mean an additive category $\mathbf{A}$ with the following properties[17]: Each hom-functor $[A, -] : \mathbf{A} \to \mathbf{Ab.Gr.}$, $A \in \mathbf{A}$, factors through the forgetful functor $U : {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$, and each morphism $\alpha : A \to A'$ induces a natural transformation ${}_\Lambda[\alpha, -]$ between the liftings ${}_\Lambda[A', -]$ and ${}_\Lambda[A, -]$ with the property $U_\Lambda [\alpha, -] = [\alpha, -]$. We show in 3.8 that this is equivalent to saying that there exist functors $F : \mathbf{A} \to \mathbf{A}_\Lambda$ and $F' : \mathbf{A} \to {}_\Lambda\mathbf{A}$ which, composed with the forgetful functors $V : \mathbf{A}_\Lambda \to \mathbf{A}$ and $V : {}_\Lambda\mathbf{A} \to \mathbf{A}$, yield the identity of $\mathbf{A}$. Thus a functor $t : \mathbf{A} \to \mathbf{B}$ can be factored into $\mathbf{A} \xrightarrow{F} \mathbf{A}_\Lambda \xrightarrow{t_\Lambda} \mathbf{B}_\Lambda \xrightarrow{V} \mathbf{B}$ and $\mathbf{A} \xrightarrow{F'} {}_\Lambda\mathbf{A} \xrightarrow{{}_\Lambda t} {}_\Lambda\mathbf{B} \xrightarrow{V} \mathbf{B}$, where t_Λ and ${}_\Lambda t$ denote the canonically induced functors (cf. 3.11). A functor $s : \mathbf{A} \to {}_\Lambda\mathbf{M}$ is said to be strong if $s = {}_\Lambda(U \cdot s) \cdot F'$. If Λ is commutative, this is equivalent to requiring the map ${}_\Lambda[A, \bar{A}] \to [sA, s\bar{A}]$, $\alpha \rightsquigarrow s\alpha$, to be a Λ-homomorphism for each pair $A, \bar{A} \in \mathbf{A}$.

Let $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ be an additive functor, not necessarily strong. Since $({}_\Lambda\mathbf{M})_\Lambda = {}_\Lambda\mathbf{M}_\Lambda$ it follows from the above that the values of t can be equipped with a Λ-Λ bimodule structure.

The Yoneda lemma (3.17) for a Λ-enriched category $\mathbf{A}$ then states that the map $Y(A) : [{}_\Lambda[A, -], t] \to tA$, $\psi \rightsquigarrow \psi(A)\,\mathrm{id}_A$, establishes a 1-1 correspondence between the natural transformations ψ from ${}_\Lambda[A, -]$ to t and the elements $a \in tA$ with the property $\lambda a = a\lambda$, $\lambda \in \Lambda$. The map $Y(A)$ is an isomorphism iff t is strong (3.23). Each functor $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ has a maximal strong subfunctor Yt which assigns to $A \in \mathbf{A}$ the submodule $\{a \in tA \mid \lambda a = a\lambda, \lambda \in \Lambda\}$ of tA. The functor $G : (\mathbf{A}, {}_\Lambda\mathbf{M}) \to (\mathbf{A}, \mathbf{Ab.Gr.})$, $t \rightsquigarrow U \cdot Yt$ is right adjoint

[17] This terminology is due to Kelly (unpublished, cf. 3.8b) and 3.8c)).

to (**A**, **Ab.Gr.**) $\rightarrow$ (**A**, ${}_{\Lambda}$**M**), $s \rightsquigarrow {}_{\Lambda}s \cdot F'$. Furthermore G is a right retraction (3.24).

3.1 Lemma (Eilenberg). *Let Λ be a ring and* **A** *an additive category. The right Λ-structures on an object $A \in$* **A** *(i.e. the ring homomorphisms $\Lambda^{opp} \rightarrow [A, A]$* cf. 1.2) *are in* 1-1 *correspondence to the factorizations of $[A, -]$:* **A** $\rightarrow$ **Ab.Gr.** *through U : ${}_{\Lambda}$***M** $\rightarrow$ **Ab.Gr.** *Dually there exists a* 1-1 *correspondence between the left Λ-structures on $A \in$* **A** *and the factorizations of $[-, A]$:* **A** $\rightarrow$ **Ab.Gr.** *through U.*

Proof. Since this lemma is well known, we merely give an outline of the first half. The second half follows by duality.

Let $\rho_A : \Lambda^{opp} \rightarrow [A, A]$, $A \in$ **A**, be a ring homomorphism and let $X \in$ **A**. We define the left Λ-module structure, corresponding to ρ_A, on the group $[A, X]$ by

$$\lambda(\xi) = \xi \cdot \rho_A(\lambda), \tag{3.2}$$

where $\lambda \in \Lambda$[18] and $\xi \in [A, X]$. The group $[A, X]$ equipped with this Λ-module structure is denoted by ${}_{\Lambda}[A, X]$. One easily checks that ${}_{\Lambda}[A, -]$ is a functor from **A** to ${}_{\Lambda}$**M** with the property $[A, -] = U \cdot {}_{\Lambda}[A, -]$.

(3.3) To establish the converse, we define for a lifting ${}_{\Lambda}[A, -]$: **A** $\rightarrow {}_{\Lambda}$**M** of $[A, -]$, $A \in$ **A**,

$$\rho_A(\lambda) = \lambda(\mathrm{id}_A). \tag{3.4}$$

A morphism $\xi : A \rightarrow X$ induces a Λ-homomorphism ${}_{\Lambda}[A, \xi] : {}_{\Lambda}[A, A] \rightarrow {}_{\Lambda}[A, X]$; hence we obtain for each $\lambda \in \Lambda$

$$\lambda(\xi) = \lambda({}_{\Lambda}[A, \xi]\, \mathrm{id}_A) = {}_{\Lambda}[A, \xi]\, \lambda(\mathrm{id}_A) = \xi \cdot \rho_A(\lambda). \tag{3.5}$$

Using this, one readily checks that $\rho_A : \Lambda^{opp} \rightarrow [A, A]$ (cf. 3.4) is a ring homomorphism.

The processes 3.2 and 3.3 obviously yield the 1-1 correspondence stated in 3.1.

3.6 Definition. Let Λ and Γ be rings and **A** an additive category. The objects of the category ${}_{\Gamma}\mathbf{A}_{\Lambda}$ are triples (π, A, ρ) consisting of an object $A \in$ **A** and ring homomorphisms $\pi : \Gamma \rightarrow [A, A]$ and $\rho : \Lambda^{opp} \rightarrow [A, A]$ with the property $\rho(\lambda) \cdot \pi(\gamma) = \pi(\gamma) \cdot \rho(\lambda)$, where $\lambda \in \Lambda$, $\gamma \in \Gamma$. A morphism $(\pi, A, \rho) \rightarrow (\pi', A', \rho')$ in ${}_{\Gamma}\mathbf{A}_{\Lambda}$ is a morphism $\alpha : A \rightarrow A'$ with the properties $\alpha \cdot \rho(\lambda) = \rho'(\lambda) \cdot \alpha$ and $\alpha \cdot \pi(\gamma) = \pi'(\gamma) \cdot \alpha$, where $\lambda \in \Lambda^{opp}$, $\gamma \in \Gamma$.

[18] Throughout this paper we write $\lambda \in \Lambda$ instead of $\Lambda \ni \lambda \in \Lambda^{opp}$ if it does not give rise to misunderstanding.

3.7 DEFINITION. *Let Λ be a ring. An additive category* $\mathbf{A}$ *is said to be Λ-enriched if it has the property* 3.8(b). *Sometimes we also say that* $\mathbf{A}$ *is equipped with a Λ-structure.*

3.8 THEOREM. *Let* $\mathbf{A}$ *be an additive category and Λ a ring. The following are equivalent*:

(a) *There exists a functor $F : \mathbf{A} \to \mathbf{A}_\Lambda$ which, composed with $V : \mathbf{A}_\Lambda \to \mathbf{A}$, yields the identity of* $\mathbf{A}$.

(b) *For each functor $[A, -] : \mathbf{A} \to \mathbf{Ab.Gr.}$, $A \in \mathbf{A}$, there exists a decomposition $U \cdot {}_\Lambda[A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$, and each morphism $\alpha : A \to A'$ induces a natural transformation ${}_\Lambda[\alpha, -] : {}_\Lambda[A', -] \to {}_\Lambda[A, -]$ which, composed with U, yields* $[\alpha, -]$.

(c) *For each functor $[-, A] : \mathbf{A} \to \mathbf{Ab.Gr.}$, $A \in \mathbf{A}$, there exists a decomposition $U \cdot {}_\Lambda[-, A] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$ and each morphism $\alpha : A \to A'$ induces a natural transformation ${}_\Lambda[-, \alpha] : {}_\Lambda[-, A] \to {}_\Lambda[-, A']$ with the property $U \cdot {}_\Lambda[-, \alpha] = [-, \alpha]$.*

(d) *There exists a functor $F' : \mathbf{A} \to {}_\Lambda\mathbf{A}$ which, composed with the forgetful functor $V : {}_\Lambda\mathbf{A} \to \mathbf{A}$, yields the identity of* $\mathbf{A}$.

(e) *There exists a functor $F'' : \mathbf{A} \to {}_\Lambda\mathbf{A}_\Lambda$ with the following properties*:

(1) *The composition $V \cdot F'' : \mathbf{A} \to {}_\Lambda\mathbf{A}_\Lambda \to \mathbf{A}$ is the identity of $\mathbf{A}$ (V denores the forgetful functor ${}_\Lambda\mathbf{A}_\Lambda \to \mathbf{A}$).*

(2) *Let $F''A = (\pi, A, \rho)$. Then the images of $\pi : \Lambda \to [A, A]$ and $\rho : \Lambda^{opp} \to [A, A]$ are commutative rings and they coincide, i.e. $\pi(\lambda) = \rho(\lambda)$ for $\Lambda \ni \lambda \in \Lambda^{opp}$.*

Proof. In view of 1.4 the proof of (a) $\leftrightarrow$ (b) is dual to that of (c) $\leftrightarrow$ (d). The implications (e) $\to$ (a) and (e) $\to$ (d) are trivial. Moreover (b) and (c) are equivalent because they both essentially assert that the hom-functor $[-, -] : \mathbf{A}^{opp} \times \mathbf{A} \to \mathbf{Ab.Gr}.$ factors through $U : {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$ Thus we need only show (a) $\to$ (b) $\to$ (e).

(a) $\to$ (b): Let $A \in \mathbf{A}$. Then $FA = (A, \sigma_A)$ holds, where σ_A is a ring homomorphism $\Lambda^{opp} \to [A, A]$. By 3.1 σ_A gives rise to a decomposition of $[A, -] : \mathbf{A} \to \mathbf{Ab.Gr.}$ into $U \cdot {}_\Lambda[A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$ According to 3.2 an element $\lambda \in \Lambda$ acts on $\xi \in {}_\Lambda[A, X]$ by composing ξ with $\sigma_A(\lambda)$. Since F is a functor, each morphism $\alpha : A \to A'$ is also a Λ-morphism $(A, \sigma_A) \to (A', \sigma_{A'})$. Therefore $\alpha \cdot \sigma_A(\lambda) = \sigma_{A'}(\lambda) \cdot \alpha$ holds for all $\lambda \in \Lambda$. Using this and 3.2, one easily checks that $[\alpha, -]$ is a natural transformation ${}_\Lambda[A', -] \to {}_\Lambda[A, -]$.

(b) $\to$ (e) Let A and X be objects of $\mathbf{A}$. By 3.1 and 3.5 the lifting ${}_\Lambda[A, -]$ of $[A, -]$ gives rise to a ring homomorphism $\rho_A : \Lambda^{opp} \to [A, A]$ and an element $\lambda \in \Lambda$ acts on $\xi \in {}_\Lambda[A, X]$ by composing ξ with $\rho_A(\lambda)$. Since each

$\alpha : A \to A'$ induces a natural transformation ${}_{\Lambda}[\alpha, -] : {}_{\Lambda}[A', -] \to {}_{\Lambda}[A, -]$, we obtain for $\xi \in {}_{\Lambda}[A', X]$ and $\lambda \in \Lambda$

$$\xi \cdot \alpha \cdot \rho_A(\lambda) = \lambda(\xi \cdot \alpha) = \lambda([\alpha, X]\, \xi) = [\alpha, X](\lambda\xi) \\ = (\lambda\xi) \cdot \alpha = \xi \cdot \rho_{A'}(\lambda) \cdot \alpha \tag{3.9}$$

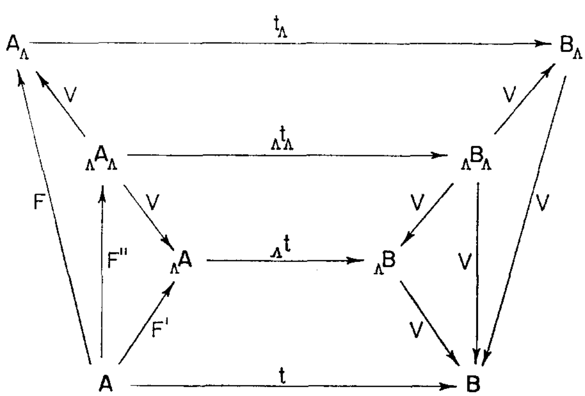
Let $X = A = A'$, $\xi = \mathrm{id}_A$ and $\alpha = \rho_A(\mu)$, where $\mu \in \Lambda^{opp}$. Then by 3.9 the image of $\rho_A : \Lambda^{opp} \to [A, A]$ is a commutative ring. Therefore ρ_A defines also a ring homomorphism $\Lambda \to [A, A]$, $\lambda \rightsquigarrow \rho_A(\lambda)$, which we denote by π_A. Thus we define $F''A = (\pi_A, A, \rho_A)$ and using 3.9 one readily verifies that F'' is a functor from $\mathbf{A}$ to ${}_{\Lambda}\mathbf{A}_{\Lambda}$ with the properties stated in 3.8(e).

3.10 *Remark. The property* 3.8(e) *of a Λ-enriched category shows that it would not be an essential restriction to assume Λ commutative.* For let Λ' be the ring Λ made commutative and $p : \Lambda \to \Lambda'$ the canonical projection. Then p establishes in an obvious way a 1-1 correspondence between the Λ'-structures (3.7) on $\mathbf{A}$ and the Λ-structures on $\mathbf{A}$. But $(\mathbf{A}, {}_{\Lambda'}\mathbf{M})$ is in general only a subcategory of $(\mathbf{A}, {}_{\Lambda}\mathbf{M})$. *Since we can establish the Yoneda Lemma* (3.17) *for all functors $t : \mathbf{A} \to {}_{\Lambda}\mathbf{M}$, we would restrict* 3.17 *if we only consider commutative rings Λ.* In view of this we did not assume Λ commutative in 3.8.

The following is an immediate consequence of 3.8:

3.11 Corollary. *Let $\mathbf{A}$ be a Λ-enriched category and $t : \mathbf{A} \to \mathbf{B}$ a functor. Denote by ${}_{\Lambda}t_{\Lambda} : {}_{\Lambda}\mathbf{A}_{\Lambda} \to {}_{\Lambda}\mathbf{B}_{\Lambda}$, $t_{\Lambda} : \mathbf{A}_{\Lambda} \to \mathbf{B}_{\Lambda}$ and ${}_{\Lambda}t : {}_{\Lambda}\mathbf{A} \to {}_{\Lambda}\mathbf{B}$ the canonically induced functors* (e.g. $t_{\Lambda}(A, \rho) = (A, t(A, A) \cdot \rho)$, *where* $t(A, A) : [A, A] \to [tA, tA]$, $\alpha \rightsquigarrow t\alpha$). *Then there exists a commutative diagram*

(31.2)



where V denotes the forgetful functors and F', F and F'' are as in 3.8.

3.13 Definition. *Let* $\mathbf{A}$ *be a* Λ*-enriched category. A functor* $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ *is called strong if it agrees with canonical lifting* ${}_\Lambda(U \cdot t) \cdot F'$ *of* $U \cdot t : \mathbf{A} \to$ **Ab.Gr**. (cf. 3.11).

The canonical projection $p : \Lambda \to \Lambda'$ (cf. 3.10) induces an embedding $I : {}_{\Lambda'}\mathbf{M} \to {}_\Lambda\mathbf{M}$ which is one-one on objects. *Then it readily follows from* 3.8(e) *and* 3.11 *that a strong functor* $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ *factors through* I. *Furthermore one easily checks that the factorization* $t' : \mathbf{A} \to {}_{\Lambda'}\mathbf{M}$ *is a strong functor in the sense of* Linton [*17*] p. 323, i.e. *for each pair* $A, \bar{A} \in \mathbf{A}$, *the map* ${}_{\Lambda'}[A, \bar{A}] \to [t'A, t'\bar{A}]$, $\alpha \rightsquigarrow t'\alpha$, *is a* Λ'*-homomorphism* (Since Λ' is commutative, $[t'A, t'\bar{A}]$ can be viewed as a Λ'-module). Conversely, any functor $s : \mathbf{A} \to {}_{\Lambda'}\mathbf{M}$ with this property induces a strong functor $I \cdot s : \mathbf{A} \to {}_\Lambda\mathbf{M}$. Instances of strong functors are the hom-functors ${}_\Lambda[A, -] : \mathbf{A} \to {}_\Lambda\mathbf{M}$, $A \in \mathbf{A}$.

3.14 Theorem. *Let* Λ *be a ring and* $\mathbf{A}$ *a* Λ*-enriched category and* t, $\bar{t} : \mathbf{A} \to \mathbf{B}$ *functors. Then* 3.12 *gives rise to a commutative diagram*

$$
\begin{array}{ccc}
[{}_\Lambda t_\Lambda \cdot F'', {}_\Lambda\bar{t}_\Lambda \cdot F''] & \longrightarrow & [t_\Lambda \cdot F, \bar{t}_\Lambda \cdot F] \\
\downarrow & \searrow & \downarrow{\scriptstyle v} \\
[{}_\Lambda t \cdot F', {}_\Lambda\bar{t} \cdot F'] & \xrightarrow{w} & [t, \bar{t}]
\end{array}
\tag{3.15}
$$

in which all maps are isomorphisms.

Proof. We show that v (3.15) is an isomorphism. The proof for the other maps in 3.15 is similar. Since the forgetful functor $V : \mathbf{B}_\Lambda \to \mathbf{B}$ is faithful, v is injective. To prove that v is surjective, we show that each natural transformation $\psi : t \to \bar{t}$ is also a natural transformation $t_\Lambda \cdot F \to \bar{t}_\Lambda \cdot F$. For this we need only verify that for each $A \in \mathbf{A}$ the morphism $\psi(A) : tA \to \bar{t}A$ is compatible with the canonical Λ-structures on tA and $\bar{t}A$, in other words that $\psi(A)$ is a morphism $t_\Lambda \cdot FA \to \bar{t}_\Lambda \cdot FA$. By 3.8(a) FA is equal to (A, σ_A), where σ_A is a ring homomorphism $\Lambda^{opp} \to [A, A]$. By 3.1, 3.5 and 3.11, an element $\lambda \in \Lambda$ acts on $\psi(A) \in [tA, \bar{t}A]$ by composing $\psi(A)$ with $t\sigma_A(\lambda)$, i.e. $\lambda(\psi(A)) = \psi(A) \cdot t\sigma_A(\lambda)$. Since $\sigma_A(\lambda)$ is a morphism $A \to A$, it follows from the naturality of ψ that

$$\psi(A) \cdot t\sigma_A(\lambda) = \bar{t}\sigma_A(\lambda) \cdot \psi(A)$$

Thus $\psi(A)$ is a morphism $t_\Lambda \cdot FA \to \bar{t}_\Lambda \cdot FA$.

Q.E.D.

3.16 Corollary. (Yoneda *lemma for strong functors*, cf. Linton [*17*] Theorem 3.3) *Let* Λ *be a ring*, $\mathbf{A}$ *a* Λ*-enriched category and* $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ *a* *strong functor. Then for each $A \in \mathbf{A}$ the group $[_{\Lambda}[A, -], t]$ can be equipped with a left Λ-module structure and the Yoneda assignment*

$$Y(A) : [_{\Lambda}[A, -], t] \to tA, \psi \rightsquigarrow \psi(A)\, \mathrm{id}_A ,$$

is an isomorphism of Λ-modules.

Proof. According to the definition of strong (cf. 3.10) $t = {}_{\Lambda}(U \cdot t) \cdot F'$ holds. The map $UY(A)$ is obviously the composition of $[_{\Lambda}[A, -], {}_{\Lambda}(U \cdot t) \cdot F'] \to [[A, -], U \cdot t]$, $\psi \rightsquigarrow U\psi$, with the Yoneda isomorphism $[[A, -], U \cdot t] \cong UtA$. Since ${}_{\Lambda}[A, -] = {}_{\Lambda}(U \cdot {}_{\Lambda}[A, -]) \cdot F'$ it follows from 3.14 (with $\mathbf{B} =$ **Ab.Gr.**) that $\psi \rightsquigarrow U\psi$ is an isomorphism. Thus the left Λ-module structure on $U \cdot tA$ induces a canonical left Λ-module structure on $[_{\Lambda}[A, -], t]$ and $Y(A)$ is a Λ-isomorphism.

In 3.23 we will prove the converse of 3.16, i.e. any functor $t : \mathbf{A} \to {}_{\Lambda}\mathbf{M}$ is strong for which the Yoneda maps $Y(A)$, $A \in \mathbf{A}$, are isomorphisms. For Λ commutative this was first observed by Linton [*17*], theorem 3.12. However, an additive functor $t : \mathbf{A} \to {}_{\Lambda}\mathbf{M}$ is in general not strong, even if Λ is commutative. For instance let $\pi : \Lambda^{opp} \to [A, A]$, $A \in \mathbf{A}$, be a ring homomorphism which is different from the canonical one given by 3.7 and 3.1. Then by 3.1 π induces a lifting ${}_{\Lambda}[A, -]' : \mathbf{A} \to {}_{\Lambda}\mathbf{M}$ of $[A, -] : \mathbf{A} \to$ **Ab.Gr.** which does not agree with ${}_{\Lambda}[A, -] : \mathbf{A} \to {}_{\Lambda}\mathbf{M}$ (cf. 3.7 and 3.8(b)). Since ${}_{\Lambda}[A, -]$ is strong and $U \cdot {}_{\Lambda}[A, -] = U \cdot {}_{\Lambda}[A, -]'$ holds, the functor ${}_{\Lambda}[A, -]'$ is not strong.

3.17 Lemma (*Yoneda*) (*for nonstrong functors*). *Let Λ be a ring and $\mathbf{A}$ a Λ-enriched category* (3.7). By 3.11 (with $\mathbf{B} = {}_{\Lambda}\mathbf{M}$) *there exists for each additive functor $t : \mathbf{A} \to {}_{\Lambda}\mathbf{M}$ a canonical decomposition into $V \cdot t_{\Lambda} \cdot F : \mathbf{A} \to \mathbf{A}_{\Lambda} \to {}_{\Lambda}\mathbf{M}_{\Lambda} \to {}_{\Lambda}\mathbf{M}$. Thus, for each $A \in \mathbf{A}$, tA can be equipped with a Λ-Λ bimodule structure.*

Then $[_{\Lambda}[A, -], t]$ is a left Λ-module and the Yoneda assignment

$$Y(A) : [_{\Lambda}[A, -], t] \to tA, \psi \rightsquigarrow \psi(A)\, \mathrm{id}_A , \tag{3.18}$$

is a Λ-homomorphism which maps $[_{\Lambda}[A, -], t]$ isomorphically on the submodule of tA consisting of the elements $a \in tA$ with the property $\lambda a = a\lambda$, $\lambda \in \Lambda$.

Proof. The group $[_{\Lambda}[A, -], t]$ can be equipped with a left Λ-module structure as follows: Let $\psi : {}_{\Lambda}[A, -] \to t$ be a natural transformation and $\lambda \in \Lambda$. Define $\lambda\psi$ at $X \in \mathbf{A}$ as

$${}_{\Lambda}[A, X] \to tX, \xi \rightsquigarrow \psi(X)(\lambda\xi). \tag{3.19}$$

We have to show that $\mu(\lambda\psi) = (\mu \cdot \lambda)\, \psi$, $\mu \in \Lambda$. According to 3.1 and 3.5 the lifting ${}_{\Lambda}[A, -]$ of $[A, -]$ gives rise to a ring homomorphism $\sigma_A : \Lambda^{opp} \to [A, A]$, and an element $\lambda \in \Lambda$ acts on $\xi \in {}_{\Lambda}[A, X]$ by composing ξ with $\sigma_A(\lambda)$. In 3.9 let $\alpha = \sigma_A(\mu)$, where $\mu \in \Lambda$. Then it follows from 3.9 that

$$(3.20) \qquad \mu(\lambda(\xi)) = \xi \cdot \sigma_A(\lambda) \cdot \sigma_A(\mu) = \xi \cdot \sigma_A(\mu) \cdot \sigma_A(\lambda) = \lambda\mu(\xi).$$

From this one readily deduces that $\mu(\lambda\psi) = (\mu\lambda)\,\psi$. The other properties of a Λ-module structure are evident. Hence by 3.19 a left Λ-module structure is defined on $[{}_{\Lambda}[A, -], t]$. Furthermore it follows easily from 3.19 that the Yoneda map $Y(A)$ (3.18) is a Λ-homomorphism. We now show that the elements $a \in \operatorname{im} Y(A)$ have the property $\lambda a = a\lambda$, $\lambda \in \Lambda$. According to 3.11 the right Λ-module structure on tA is given by the composite $\Lambda^{opp} \to [A, A] \to [tA, tA]$, $\lambda \rightsquigarrow t\sigma_A(\lambda)$. Hence we have

$$(3.21) \qquad a\lambda = t\sigma_A(\lambda)(a)$$

for each $a \in tA$. Since $\psi : {}_{\Lambda}[A, -] \to t$ is natural and $\lambda(\mathrm{id}_A) = \sigma_A(\lambda) \cdot \mathrm{id}_A = {}_{\Lambda}[A, \sigma_A(\lambda)]\,\mathrm{id}_A$, it follows that

$$\begin{aligned}\lambda(Y(A)\,\psi) &= \lambda(\psi(A)\,\mathrm{id}_A) = \psi(A)\,\lambda(\mathrm{id}_A) = \psi(A) \cdot {}_{\Lambda}[A, \sigma_A(\lambda)]\,(\mathrm{id}_A)\\ &= t\sigma_A(\lambda) \cdot \psi(A)(\mathrm{id}_A) = t\sigma_A(\lambda)(Y(A)\,\psi) = (Y(A)\,\psi)\,\lambda.\end{aligned}$$

Conversly, let $a \in tA$ be an element with the property $\lambda a = a\lambda$, $\lambda \in \Lambda$. We prove that

$$\psi_a : {}_{\Lambda}[A, -] \to t, \; \psi_a(X)(\xi) = t\xi(a),$$

where $\xi \in {}_{\Lambda}[A, X]$, is a natural transformation. The compatibility of ψ_a with morphisms $X \to X'$ is standard. Therefore we need only verify that $\psi_a(X) : {}_{\Lambda}[A, X] \to tX$ is a Λ-homomorphism. Since for $\lambda \in \Lambda$, $a \in tA$, and $\xi \in {}_{\Lambda}[A, X]$, $a\lambda = t\sigma_A(\lambda)(a)$ and $\lambda(\xi) = \xi \cdot \sigma_A(\lambda)$ hold, we obtain

$$\begin{aligned}\psi_a(X)(\lambda\xi) &= t(\lambda\xi)(a) = t(\xi \cdot \sigma_A(\lambda))(a) = t\xi \cdot t\sigma_A(\lambda)(a)\\ &= t\xi(a\lambda) = t\xi(\lambda a) = \lambda(t\xi(a)) = \lambda(\psi_a(X)\,\xi).\end{aligned}$$

The partially defined map $tA \to [{}_{\Lambda}[A, -], t]$, $a \rightsquigarrow \psi_a$, is left inverse to $Y(A) : [{}_{\Lambda}[A, -], t] \to tA$. Hence $Y(A)$ is injective, and the elements $a \in tA$ with the property $\lambda a = a\lambda$, $\lambda \in \Lambda$, constitute a submodule of tA which is isomorphic to $[{}_{\Lambda}[A, -], t]$.

Q.E.D.

Remark: *The* Yoneda *lemma* 3.17 *can be generalized as follows*: *The category* $\mathbf{A}$ *need not be* Λ*-enriched. It suffices that for the given object* A *of* $\mathbf{A}$, *the functor* $[A, -]$ *can be decomposed into* $U \cdot {}_{\Lambda}[A, -] : \mathbf{A} \to {}_{\Lambda}\mathbf{M} \to \mathbf{Ab.Gr}$. *The composition* $\Lambda^{opp} \to [A, A] \to [tA, tA]$, $\lambda \rightsquigarrow t\sigma_A(\lambda)$, *makes* tA *into a* Λ-Λ *bimodule, where* $\sigma_A : \Lambda^{opp} \to [A, A]$ *is the ring homomorphism associated with* ${}_{\Lambda}[A, -]$

(cf. 3.1). *Then the map* $Y(A)$ (3.18) *establishes a* 1-1 *correspondence between* $[_\Lambda[A, -], t]$ *and the set of elements* $a \in tA$ *with the property* $\lambda a = a\lambda$, $\lambda \in \Lambda$. *In general the group* $[_\Lambda[A, -], t]$ *can be equipped with a left* Λ*-module structure only if the image of* $t\sigma_A$ *is a commutative ring. In this case the Yoneda map* $Y(A)$ *is also a* Λ*-homomorphism.*

We shall make only incidental use of this generalization later.

From 3.17 and 3.18 it follows that the image of $Y(A)$ (cf. 3.18) is a submodule of tA. This leads to the following

3.22 DEFINITION. *Let* Λ *be a ring,* $\mathbf{A}$ *a* Λ*-enriched category and* $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ *a functor. The* Yoneda *part* $Yt : \mathbf{A} \to {}_\Lambda\mathbf{M}$ *of* t *is the subfunctor which assigns to* $A \in \mathbf{A}$ *the submodule* $\{a \in tA \mid \lambda a = a\lambda, \lambda \in \Lambda\} = \text{im } Y(A)$ of tA.

(3.23) *From* 3.21 *and* 3.11 *it follows easily that* $Yt : \mathbf{A} \to {}_\Lambda\mathbf{M}$ *is a strong functor and that* Yt *contains every strong subfunctor of* t. *Hence* t *is strong iff* $Yt = t$. *This implies that the* Yoneda *map* $Y(A) : [_\Lambda[A, -], t] \to tA$ *is an isomorphism iff* t *is strong.*

3.24 THEOREM. *Let* Λ *be a ring and* $\mathbf{A}$ *a* Λ*-enriched category. Then the functor* $E : (\mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, {}_\Lambda\mathbf{M})$, $s \rightsquigarrow {}_\Lambda s \cdot F'$ (cf. 3.11), *is left adjoint to* $G : (\mathbf{A}, {}_\Lambda\mathbf{M}) \to (\mathbf{A}, \mathbf{Ab.Gr.})$, $t \rightsquigarrow U \cdot Yt$. (Recall that $U : {}_\Lambda\mathbf{M} \to \mathbf{Ab.Gr.}$ is the forgetful functor). *Furthermore for each* $s \in (\mathbf{A}, \mathbf{Ab.Gr.})$ *the functors* $U \cdot Y({}_\Lambda s \cdot F')$ (cf. 3.11) *and* s *coincide and the front adjunction at* s *is the identity of* s. *Hence* G *is a right retraction.*

Proof. Let $t : \mathbf{A} \to {}_\Lambda\mathbf{M}$ be a functor and A an object of $\mathbf{A}$. Then by 3.17 there exists an isomorphism between the underlying group of $[_\Lambda[A, -], t]$ and $U \cdot Yt(A) \cong [[A, -], U \cdot Yt]$ which is natural in A and t. Hence $E \cdot Y^{\mathbf{A}} : \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{Ab.Gr.}) \to (\mathbf{A}, {}_\Lambda\mathbf{M})$, $A \rightsquigarrow {}_\Lambda[A, -]$, is left adjoint to $G : (\mathbf{A}, {}_\Lambda\mathbf{M}) \to (\mathbf{A}, \mathbf{Ab.Gr.})$, $t \rightsquigarrow U \cdot Yt$, relative to the Yoneda embedding $Y^{\mathbf{A}} : \mathbf{A}^{opp} \to (\mathbf{A}, \mathbf{Ab.Gr.})$, $A \rightsquigarrow [A, -]$, (cf. [*25*], 2.2). Since $Y^{\mathbf{A}}$ is dense (cf. [*25*], 1.10, 1.17) and E right continuous, it follows from [*25*], 2.1, 2.24 that E is left adjoint to G. The rest of 3.24 is an immediate consequence of 3.22 and 3.23.

Q.E.D.

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