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A short proof of representability of fork algebras

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Abstract

In this paper a strong relationship is demonstrated between fork algebras and quasi-projective relation algebras. With the help of Tarski's classical representation theorem for quasi-projective relation algebras, a short proof is given for the representation theorem of fork algebras. As a by-product, we will discuss the difference between relative and absolute representation theorems.

Fork algebras, due to their expressive power and applicability in computer science, have been intensively studied in the last four years. Their literature is alive and productive. See e.g. Baum et al. [33, 34.3, 6–10] and Sain–Simon [27].

As described in the textbook [15] 2.7.46, in algebra, there are two kinds of representation theorems: absolute and relative representation. For fork algebras absolute representation was proved almost impossible in [18, 27, 26]. [1] However, relative representation is still possible and is quite useful (see e.g. [2]).

The distinction between the two kinds of representation is not absolutely necessary for understanding the main contribution of this paper; therefore we postpone the description of this distinction to Remark 11 at the end of the paper.

Several papers concentrate on giving a proof or an outline of proof for relative representability of fork algebras (e.g. [2, 3, 8, 10]). The contribution of the present note is threefold.

(i) We provide a very short and easy proof for this (relative) representation theorem.

(ii) We elaborate the connection between the recent theory of fork algebras and Tarski's theory of pairing relation algebras which is a classical branch of algebraic logic going back to 1941 ([32, p. 168]). For historical reasons we will use the expression quasi-projective relation algebra instead of pairing relation algebra. [2]

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[1] We will return to the issue of making this 'almost impossible' task possible at the end of this paper.

[2] The expression used in [32] is "relation algebra with conjugated quasi-projections" or "QRA".

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(iii) In Remark 11 we try to summarize some methodological considerations on what representation problems are about and what kinds of solutions (representation theorems) they admit. These considerations are based on Henkin et al. Tarski [15] and Németi [22].

Fork algebras, as defined in Definition 1 below, form an equational class that contains all the 'fork algebras' found in the literature.[3] In the earlier version [12] of the present paper we introduced the notion *pre-fork algebra* to denote the class we call here *fork algebra*. This notation is no longer necessary since in their latest paper Baum, Frias, Haeberer and López also decided to adopt the wider class, we call here fork algebras, as the central topic of study (cf. [2]), and there they refer to it as *fork algebras*. (In the earlier papers they used that term in a strictly narrower sense.)

We cite first the basic definitions relevant to our topic.

Definition 1. (a) An algebraic structure $\mathscr{A}=\langle A,+,\cdot,-,0,1,;,\breve{\ },Id\rangle$ is a *relation algebra* if it satisfies the axioms listed in [32, p. 235 Ra I–X]. The class of all relation algebras is denoted by **RA**. We note that the definition of **RA** in [16, 5.3] and also in [11] is equivalent with the one above.

(b) A relation algebra $\mathscr{A}$ is *proper*[4] if there is a set U such that A is a set of binary relations over U and the operations $+,\cdot,-,;,\breve{\ },Id$ coincide respectively with the set theoretic operations of union, intersection, complementation relative to the greatest element, relative product, conversion, and identity $Id_U=\{\langle u,u\rangle \mid u\in U\}$.

Considering the fact that in a proper relation algebra, U is the domain of Id, U is uniquely determined. We call U the *base* of the proper relation algebra.

We note that the greatest element of a proper relation algebra is an equivalence relation over the base set.

(c) A relation algebra is *representable* if it is isomorphic to a proper relation algebra. The class of all representable relation algebras is denoted by **RRA**.

(d) A relation algebra is *quasi-projective* if its universe contains a pair of special elements p,q, that satisfy $p\breve{\ };p\leqslant Id$, $q\breve{\ };q\leqslant Id$, $p\breve{\ };q=1$. Here p,q are called *quasi-projections*.

(e) An algebra $\mathscr{A}=\langle A,+,\cdot,-,0,1,;,\breve{\ },Id,\triangledown\rangle$ is a *fork algebra* if its fork-free reduct $\langle A,+,\cdot,-,0,1,;,\breve{\ },Id\rangle$ is a relation algebra and Eqs. (1)–(3) below are valid in $\mathscr{A}$.

(1) $x\triangledown y=(x;(Id\triangledown 1))\cdot(y;(1\triangledown Id))$,

(2) $(x\triangledown y);(z\triangledown w)\breve{\ }=(x;z\breve{\ })\cdot(y;w\breve{\ })$,

(3) $(Id\triangledown 1)\breve{\ }\triangledown(1\triangledown Id)\breve{\ }\leqslant Id$.

The class of fork algebras is denoted by FA.

(f) An algebra $\mathscr{A}=\langle A,+,\cdot,-,0,E,;,\breve{\ },Id,\triangledown\rangle$ is called a *proper fork algebra*[5] if $\langle A,+,\cdot,-,0,E,;,\breve{\ },Id\rangle$ is a proper relation algebra with base U (for some set U), $\triangledown$

[3] Fork algebras in the sense of [3, 8, 10] do not form an equational class and their axiomatization involves a quite complicated formula not equivalent to any equation or even quasi-equation.

[4] Strong version of proper is used; see Remark 11.

[5] Weak version of proper is used.

is a binary operation on A, and there is an injective function $*: E \to U$ such that

$$R \triangledown S = \{\langle x, *(y,z)\rangle \mid x,y,z \in U,\ xRy,\ xSz\} \tag{†}$$

whenever $R, S \in A$.

We note that xRy, xSz imply that $\langle y,z\rangle \in R^{\smile}; S \subseteq E$ therefore $*(y,z)$ is defined. If $*$ meets the requirement (†), then it sends the pair $\langle y,z\rangle$ to an element of the E-class of x (that coincides with the E-class of y and that of z).

Claim 2. *Any proper fork algebra is a fork algebra.*

Proof. We need to check that the listed axioms are satisfied in any proper fork algebra. Since this checking is straightforward, we leave it to the reader. □

To prove the relative representation theorem of fork algebras, we recall the following theorem from the literature.

Theorem 3 (Tarski [30]). *Every quasi-projective relation algebra is representable.*

This theorem was already used in Tarski's 1941 paper [30] on the calculus of relations. A proof was outlined in [31]. A purely algebraic proof was given in [17]. A proof of this theorem is also available in the textbook [32, p. 242 item (iii)]. A new kind of simple algebraic proof is in Simon [29].

Corollary 4. *The relation algebraic reduct of a fork algebra is a quasi-projective relation algebra, so it is representable.*

Proof. Put p and q to be $(Id \triangledown 1)^{\smile}$ and $(1 \triangledown Id)^{\smile}$ respectively. From (2) of the definition of a fork algebra we can infer that $p^{\smile}; p \leqslant Id$, $q^{\smile}; q \leqslant Id$, $p^{\smile}; q = 1$, as follows.

We use that in **RA** we have $Id^{\smile} = Id$, $1^{\smile} = 1$, $x^{\smile\smile} = x$, $x; Id = Id; x = x$. If we substitute $(Id, 1, Id, 1)$ into (x,y,z,w) in (2), then we get $p^{\smile}; p = (Id \triangledown 1)^{\smile\smile}; (Id \triangledown 1)^{\smile} = (Id \triangledown 1); (Id \triangledown 1)^{\smile} = (Id; Id^{\smile}) \cdot (1; 1^{\smile}) = Id \cdot (1; 1^{\smile}) \leqslant Id$. In a similar way we get $q^{\smile}; q \leqslant Id$. ($(1, Id, 1, Id)$ should be substituted.) Finally, the substitution of $(Id, 1, 1, Id)$ leads to $p^{\smile}; q = (Id; 1^{\smile}) \cdot (1; Id^{\smile}) = 1$. □

Theorem 5 (Relative representation theorem of fork algebras). *Every fork algebra is isomorphic to a proper fork algebra.*

Proof. Let $\mathscr{A}$ be a fork algebra. By the previous theorem, the relation algebraic reduct of $\mathscr{A}$ is isomorphic to a proper relation algebra, say $\mathscr{B}$ with base U and greatest element E, and there is a bijective function f between the universes of $\mathscr{A}$ and $\mathscr{B}$ that proves the isomorphism. With the help of f, $\mathscr{B}$ can be expanded to a fork algebra

$\mathscr{B}^* = \langle \mathscr{B}, \triangledown \rangle$ by setting

$$R \triangledown S \stackrel{\text{def}}{=} f(f^{-1}(R) \triangledown f^{-1}(S)).$$

Clearly, $\mathscr{A}$ and $\mathscr{B}^*$ are isomorphic algebras since f is an isomorphism between $\mathscr{A}$ and $\mathscr{B}^*$. We will show that $\mathscr{B}^*$ is actually a proper fork algebra and so $\mathscr{A}$ is representable.

If we put (as we did in the proof of Corollary 4) p and q to be $(Id_U \triangledown E)\breve{}$ and $(E \triangledown Id_U)\breve{}$ in $\mathscr{B}^*$ respectively, then we have that $p\breve{}; p \subseteq Id_U$, $q\breve{}; q \subseteq Id_U$, $p\breve{}; q = E$.

Claim 6. *For any $\langle x, y\rangle \in E$ there is exactly one $u \in U$ with $\langle u,x\rangle \in p$, $\langle u, y\rangle \in q$.*

Proof. Since $\langle x, y\rangle \in E = p\breve{}; q$ there is a $u \in U$ with $\langle x,u\rangle \in p\breve{}$, $\langle u, y\rangle \in q$ and so $\langle u,x\rangle \in p$.

On the other hand, if $\langle u,x\rangle \in p$, $\langle u, y\rangle \in q$, $\langle u',x\rangle \in p$, $\langle u', y\rangle \in q$ for some $u, u' \in U$ then $\langle u, u'\rangle \in p; p\breve{} \cap q; q\breve{}$. But (1) and (3) imply that $Id_U \supseteq (Id_U \triangledown E)\breve{} \triangledown (E \triangledown Id_U)\breve{} = p \triangledown q = p; p\breve{} \cap q; q\breve{}$. So $\langle u, u'\rangle \in Id_U$, thus $u = u'$. $\square$

This claim ensures that we can define a function $*: E \to U$ as follows: $*(x, y) = u$ if $\langle u,x\rangle \in p$, $\langle u, y\rangle \in q$.

Claim 7. *$*$ is injective.*

Proof. Suppose that $*(x, y) = u = *(x', y')$. Then $\langle u,x\rangle \in p$, $\langle u,x'\rangle \in p$ so $\langle x,x'\rangle \in p\breve{}; p \subseteq Id_U$ (cf. the proof of Corollary 4), hence $x = x'$. Similarly, $y = y'$. $\square$

Finally, the following claim completes the proof of Theorem 5.

Claim 8. *For any $R, S \in \mathscr{B}^*$*

$$R \triangledown S = \{\langle x, *(y,z)\rangle \mid x, y, z \in U,\ xRy,\ xSz\}.$$

Proof. Fix $R, S \in B$. Then $R \triangledown S = (R; p\breve{}) \cap (S; q\breve{})$ by (1) of Definition 1.

First suppose that $\langle x, u\rangle \in R \triangledown S = (R; p\breve{}) \cap (S; q\breve{})$. Then there are elements y, z of U with xRy, xSz, $\langle y, u\rangle \in p\breve{}$, $\langle z, u\rangle \in q\breve{}$. Hence, $\langle u, y\rangle \in p$, $\langle u, z\rangle \in q$ and so $*(y,z) = u$.

On the other hand, let $x, y, z \in U$, xRy, xSz. Then $\langle y, z\rangle \in E$. Let $u = *(y, z)$. Therefore, $\langle y, u\rangle \in p\breve{}$, $\langle z, u\rangle \in q\breve{}$ so $\langle x, *(y,z)\rangle \in (R; p\breve{}) \cap (S; q\breve{}) = R \triangledown S$. $\square$

Corollary 9. *Let $\mathscr{A}$ be a fork algebra in which the formula $0 \neq x \Rightarrow 1; x; 1 = 1$ is valid. Then $\mathscr{A}$ is isomorphic to a proper fork algebra with square greatest element. (That is, its greatest element is $U \times U$, where U is the base set.)*

Proof. Assume that the formula is valid in $\mathscr{A}$. Then the formula forces the relation algebraic reduct of $\mathscr{A}$, say $\mathscr{C}$, to be simple (see e.g. [16]). Since any simple quasi-projective relation algebra is isomorphic to a proper relation algebra with square greatest element (see [32]), we conclude that there is a proper relation algebra $\mathscr{B}$ with square greatest element isomorphic to $\mathscr{C}$.

The same way we did in the proof of Theorem 5, $\mathscr{B}$ can be expanded to a proper fork algebra $\mathscr{B}^*$ that is isomorphic to $\mathscr{A}$, as desired. □

In papers discussing fork algebras, the notion of a proper fork algebra is defined in various ways. In the sense of Baum et al. [3] not all fork algebras are representable, i.e. isomorphic to a proper fork algebra. The same applies to e.g. Veloso et al. [33]. This was proved in [26] explicitly and in [18, 27] implicitly.

On the other hand, in the sense of [10] or [8] all fork algebras are (relatively) representable, since the notion of proper fork algebra in [10] or [8] coincides with our notion.

Corollary 10. *Any subclass of the class of fork algebras is relatively representable. Hence, all fork algebras in the sense of Frias et al.* [10] (*or any of the quoted papers*) *are relatively representable.*

Finally, we would like to recall from the mathematical literature a description of the different types of representation theorems.

Remark 11 (*On relative and absolute representability*). Since we are dealing with representability of algebras, Tarski's views on what a satisfactory representation theorem is seem to be relevant to cite here. In this connection Henkin–Monk–Tarski [15, 2.7.46, pp. 459–461] distinguishes absolute representability and relative representability in the following way. [6]

A class K_1 of algebras is called *abstract* if it is closed under taking isomorphic copies. In contrast, a class K_2 is *concrete* if for any $\mathscr{A}, \mathscr{B} \in K_2$ whenever $\mathscr{A}$ and $\mathscr{B}$ have the same universe, then $\mathscr{A} = \mathscr{B}$. In mathematics, absolute representation theorems are of the following form. An abstract class K_1 of algebras is considered together with a concrete class $K_2 \subseteq K_1$. Now, an absolute representation theorem says that every algebra $\mathscr{A}$ in K_1 is isomorphic to a member $\mathscr{B}$ of K_2. Technically, we say that $\mathscr{A}$ is *represented* by $\mathscr{B}$.

Examples for pairs $\langle K_1, K_2 \rangle$ that support absolute representation theorems are semigroups (as K_1) and transformation semigroups (as K_2), groups and permutation groups, distributive lattices and set lattices, Boolean algebras and Boolean set algebras, locally finite cylindric algebras and locally finite cylindric set algebras.

In some cases, when illuminating absolute representation theorems are not available in the literature, relative representation theorems are used as a (useful) substitute. In *relative representation theorems*, instead of requiring K_2 to be concrete, we require K_2 to be only relatively concrete with respect to a third class K_3 of still *abstract* structures. That is, we require the existence of a map f from K_2 to K_3 such that

[6] For more on this issue, cf. [22] (the subsections devoted to the finitization problems) and [26].

for any $\mathscr{A},\mathscr{B}\in K_2$, $\mathscr{A}=\mathscr{B}$ holds whenever $\mathscr{A}$ and $\mathscr{B}$ have the same universe and $f(\mathscr{A})=f(\mathscr{B})$.[7]

In the discussion in [15] cited above, the Jónsson–Tarski representation of Boolean algebras with operators is mentioned as an example for a relative representation theorem [15, p. 460].[8]

In the relative representation theorem of fork algebras (which was proved in the present paper) the underlying abstract class K_3 is

$$K_3=\{\langle U,*\rangle \mid U \text{ is a set, } *:U\times U\rightarrow U \text{ is an injective function}\}.$$

In both examples the corresponding K_3 can be viewed as the class of underlying *Kripke structures* or *Kripke frames*. The same approach was used in [15] to give relative representation for cylindric algebras.

As a theoretical observation we show that for every class K_1 we can find K_2 that gives an absolute representation. Given an algebra $\mathscr{A}=\langle A,f_i\rangle_{i\in I}\in K_1$, define $\mathscr{A}'=\langle A\times\{\mathscr{A}\},f_i'\rangle_{i\in I}$ where $f_i'(\langle x_1,\mathscr{A}\rangle,\ldots,\langle x_k,\mathscr{A}\rangle)=\langle f_i(x_1,\ldots,x_k),\mathscr{A}\rangle$. Now $K_2=\{\mathscr{A}':\mathscr{A}\in K_1\}$ suffices.

This illustrates that K_2 being a concrete class is only a necessary (but not sufficient) condition for $\langle K_1,K_2\rangle$ supporting a satisfactory representation theorem. Further, the value of an absolute representation theorem lies in the simplicity and illuminating power (of the definition) of K_2. It is preferred that all operations of algebras of K_2 should be set theoretically defined and motivated by intuition.

The class **RA** is not absolute representable over **RRA** as it was demonstrated by Lyndon and Monk. Later research revealed that every relation algebra is isomorphic to a relativized relation algebra. Since relativized **RA**s form a concrete class this gives a satisfactory absolute representation for **RA**. Similar positive representation results hold for cylindric algebras; cf. [22, 1, 16].

In view of the above discussion, absolute representation theorems bridge the gap between the abstract and concrete, while relative representation theorems do not bridge such a conceptually important gap. Their purpose is to interpret an abstract class K_1 in terms of *simpler* or *more basic* but still abstract class K_3.

As a final illustration we mention the following.

Theorem 3 herein says that for any $\mathscr{A}\in\mathbf{RA}$ if

$$\mathscr{A}\models(\exists p,q)\ p \text{ and } q \text{ are quasi-projections (in the sense of Definition 1)}$$

then $\mathscr{A}$ is representable. This is an absolute representation theorem since all the operations of the algebra in question are represented.

[7] We note that this will not force K_2 to be concrete.

[8] One of the main problems discussed there was that this does not provide an absolute representation (but only a relative one). The desirability of having an absolute representation theorem was explained in detail there.

Consider the following reformulation of Theorem 3:

Let $\mathscr{A} = \langle A, +, \cdot, -, 0, 1, ;, \breve{}, Id, \triangledown, p, q\rangle$ where the p, q free reduct of $\mathscr{A}$ is in **RA** and p, q are 0-ary operations. If

$$\mathscr{A} \models p \text{ and } q \text{ are quasi-projections}$$

then the p, q free reduct of $\mathscr{A}$ is an **RRA**. This is not a representation theorem for the expanded algebras since p, q are operations of $\mathscr{A}$ that are not represented. [9]

In case of absolute representation theorems the members of K_2 are called proper members of K_1. In case of relative representation theorems they are called weakly proper.

Related works and some conclusions

The class of proper fork algebras, as defined in this paper, is a weakly proper class. If we adjust the definition such that we require that the hidden function $*$ should coincide with the set theoretic pairing function (that sends the points u and v to the ordered pair $\langle u, v\rangle$), then the new version PFA of the class of proper fork algebras becomes a concrete class. Now, PFA (as K_2) could support an absolute representation theorem, but it turns out that the variety generated by PFA is not axiomatizable anymore (cf. e.g. [26, 18]). This yields that neither our class of fork algebras nor any axiomatizable subclass of it can be represented. However, Németi showed in [19–21] that in a non-well-founded set theory one can obtain an *absolute* representation theorem for fork algebras. (The methods in [19–21] are strongly related to the ones in [25, 24], where an absolute representation theorem is proved for a natural kind of algebras of relations.)

It is demonstrated in [26] that the only possible method (known so far) of making fork algebras absolutely representable in a natural way goes via the non-well-founded set theory. In particular, [26] investigates all the possible absolute representations (set theoretic pairing, free groupoids for $*$, finite trees for $*$, etc.) suggested in the literature and proves that none of them works in our usual set theory ZFC.

So, absolute representability is available for fork algebras if we are willing to switch to the special set theory in [19]. But what happens if we stick with relative representability?

What inconvenience can be caused by relative representability? In [26] it is proved that we cannot avoid loops in relatively represented fork algebras (FAs). This means that the non-standard pairing function $* : (U \times U) \to U$ built into our relative representation might contain loops, i.e. $*(a,b) = a$ or in general $*(\ldots * (a,b), \ldots) = a$ may happen. But, in our opinion, this contradicts the programmer's intuition concerning ordered pairs or forking.

[9] If we would call the above reformulation a representation theorem for $\mathscr{A}$ then, by the same token, Stone's representation theorem for Boolean algebras could be called a representation theorem for **RA**s.

Consider the following infinite set of equations:

$$Id \bigtriangledown Id \leqslant -Id,$$
$$(Id \bigtriangledown Id) \bigtriangledown Id \leqslant -Id,$$
$$((Id \bigtriangledown Id) \bigtriangledown Id) \bigtriangledown Id \leqslant -Id,$$
$$\vdots$$

All these equations are intuitively true properties of fork. But none of these equations is true in FAs or in proper FAs. Moreover, if we add any finite number of these to the FA-axioms, the rest will remain unprovable from the enriched theory. (There are many other infinite sets of equations behaving the above way, e.g. we could have started with $(Id \bigtriangledown -Id) \leqslant -Id$.) It was proved in [26] that in the language of FAs there is no finite set of axioms which would avoid the above outlined "looping". Is there an alternative to FAs which can avoid looping? The weakly higher-order cylindric algebras of ternary relations in Németi, Simon [23] provide a strictly stronger formalism than that of FAs where looping can be excluded by a single axiom.

Summary. We saw that our main theorem, i.e. relative representability of FAs, is an immediate corollary of an old theorem of Tarskian algebraic logic concerning quasi-projective relation algebras (QRAs). This together with other observations in the paper seem to point to the direction that developing the theory of fork algebras as integral part of Tarskian algebraic logic would be desirable. We note that in the textbook on Tarskian algebraic logic [16] the theory of relation algebras (and their alternatives like polyadic algebras, etc.) is discussed in a separate chapter.

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