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Geometric Singular Perturbation Theory for Ordinary Differential Equations

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I. INTRODUCTION

The aim of this paper is to present a geometric approach to singular perturbation theory for real ordinary differential equations. We divide singular perturbation theory into a local theory and a global theory. Local singular perturbation theory is concerned with the structure of the solutions of a singular perturbation problem near a point. The local theory is nontrivial precisely because of the presence of singularities. Global singular perturbation theory is concerned with the structure of the solutions of a singular perturbation problem in a large domain. In many applications the global theory gives information about the behavior of solutions during unbounded time intervals.

The singular perturbation problems we consider are characterized by two time scales, slow time t and fast time τ. These are related by $\tau = t/\epsilon$, where ϵ is a small parameter. Our problems are governed in slow time by systems of differential equations which are singular at $\epsilon = 0$. The singularities of the slow time systems appear as manifolds of equilibrium points of the fast time systems.

In the local theory we study a singular perturbation problem near a point in its manifold of equilibrium points. In the global theory we study a singular

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perturbation problem near a compact subset of its manifold of equilibrium points. The global theory is much richer than the local theory because the global theory includes limiting behavior as $t \to \pm\infty$ for fixed ϵ. The local theory includes such behavior only near an equilibrium point of the reduced system.

Sections II–IV of this paper form an extended introduction. After fixing notation, we study a singular perturbation problem whose reduced problem has a periodic solution. In Section III we sketch the construction of a center manifold for this problem. On the center manifold the singular perturbation problem restricts to a regular perturbation problem. By restricting attention to the center manifold we are able to reduce some hard problems in singular perturbation theory to easy problems in regular perturbation theory. In Section III we also sketch the construction of an invariant family of manifolds transversal to the center manifold. This family is defined in terms of asymptotic properties of the singular problem for large time. We use this invariant family to prove a new result about asymptotic phase. In Section IV we use the center manifold to reduce an asymptotic expansion computation for a singular perturbation problem to a Taylor series computation for a regular perturbation problem.

In order to study qualitative properties we require a coordinate-free notion of singular perturbation problem. In Sections V–VII we formulate this notion first in local form and then in global form. Then in Sections VIII–IX we state the main theorem. We interpret this theorem in terms of asymptotic expansions in Section X and in terms of normal forms in Section XI. A further interpretation in terms of Lyapunov functions follows directly from the existence of normal forms, but we do not pursue this question. Applications appear in Sections XII–XV and XVII. In Section XVI we recall the invariant manifold theory required for some applications and for the proof of the main theorem. Finally, in Section XVIII we prove the main theorem.

The main analytical tool used in this paper is the invariant manifold theory developed in Fenichel (1971, 1974, 1977). The hard analysis we require is developed in those papers. The application of invariant manifold theory is especially easy because of the presence of manifolds of equilibrium points for the fast time systems. The presence of these equilibrium points also facilitates the computation of asymptotic expansions.

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II. Notation

We use two time scales, slow time t and fast time τ. Differentiation with respect to t is denoted by $\cdot$ and differentiation with respect to τ is denoted by $'$. The solution operator of any system of differential equations is denoted by $\cdot t$ or $\cdot\tau$, depending on whether the independent variable is slow time or fast time. This means that $t \to p \cdot t$ (or $\tau \to p \cdot \tau$) is a solution of the system satisfying $p \cdot 0 = p$. All systems we consider have unique solutions, but the solutions need not exist for all time.

Differentiation with respect to any variable other than t or τ is denoted by D. For functions of several variables we use D_1, D_2, etc., to denote partial differentiation.

Single bars $|\ |$ denote the norm of a vector, double bars $\|\ \|$ denote the norm of a matrix or linear operator, and angle brackets $\langle\ ,\ \rangle$ denote any inner product. One single bar $|$ denotes restriction. This symbol is applied to mappings, vector fields, and tangent spaces.

The usual distance in Euclidean space is denoted by $d(\ ,\ )$. We also use this symbol for the distance associated with any Riemannian metric on a manifold.

III. The Geometry of Periodic Solutions

Our main results are expressed in terms of invariant manifolds and invariant families of manifolds. To motivate the introduction of these geometric objects we now give a heuristic outline of our results, as they apply to the special case of singular perturbation theory for periodic solutions. Invariant manifold theory was applied to this problem in Kopell (1977). We show in Section IV how our geometric constructions lead to the computation of asymptotic expansions. We return to the study of periodic solutions in Section XIII, where we generalize the results of this section and give precise smoothness conditions.

Let M be an open subset of $R^\mu \times R^\nu$, and let $\mathscr{E} = M \cap (R^\mu \times \{0\})$ be nonempty. In this section we consider a system of the form

$$\begin{aligned} \dot{x} &= f_0(x, y, \epsilon) \\ \epsilon\dot{y} &= g(x, y, \epsilon) \end{aligned} \tag{3.1$_\epsilon$}$$

defined for $(x, y) \in M$, for small, real ϵ. When $\epsilon = 0$ the system (3.1) degenerates to the reduced system

$$\begin{aligned} \dot{x} &= f_0(x, y, 0) \\ 0 &= g(x, y, 0). \end{aligned} \tag{3.2}$$

We assume that

$$g(x, 0, 0) = 0 \qquad \text{for all } (x, 0) \in \mathscr{E}, \tag{3.3}$$

so that (3.2) defines a flow in $\mathscr{E}$, and we assume that this flow has a periodic orbit γ_0: $x = p(t)$, $y = 0$. Our aim is to describe the orbit structure of $(3.1)_\epsilon$, for small nonzero ϵ.

The form of the singularity in (3.1) suggests the rescaling $\tau = t/\epsilon$. This transforms $(3.1)_\epsilon$ to

$$\begin{aligned} x' &= \epsilon f_0(x, y, \epsilon) \\ y' &= g(x, y, \epsilon). \end{aligned} \tag{$(3.4)_\epsilon$}$$

The set $\mathscr{E}$ consists entirely of equilibrium points of $(3.4)_0$.

Along with (3.1) and (3.4), it is convenient to consider the equivalent systems in which ϵ is introduced as a dummy variable in the phase space. These systems are

$$\begin{aligned} \dot{x} &= f_0(x, y, \epsilon) \\ \epsilon\dot{y} &= g(x, y, \epsilon) \\ \dot{\epsilon} &= 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} x' &= \epsilon f_0(x, y, \epsilon) \\ y' &= g(x, y, \epsilon) \\ \epsilon' &= 0. \end{aligned} \tag{3.6}$$

The set $\mathscr{E} \times \{0\}$ consists entirely of equilibrium points of (3.6).

Our plan is to relate the orbit structure of $(3.1)_\epsilon$ near γ_0, for small nonzero ϵ, to the orbit structure of the reduced system (3.2) near γ_0 and to the linearizations of $(3.4)_0$ and (3.6) at points of γ_0 and $\gamma_0 \times \{0\}$. The linearization of $(3.4)_0$ at $(x, 0) \in \mathscr{E}$ is

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & D_2 g(x, 0, 0) \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{3.7}$$

The second component satisfies

$$\delta y' = D_2 g(x, 0, 0)\, \delta y, \tag{3.8}$$

a linear equation parametrized by $(x, 0) \in \mathscr{E}$. We call (3.8) the initial layer equation. The linearization of (3.6) at $(x, 0, 0) \in \mathscr{E} \times \{0\}$ is

$$\begin{Bmatrix} \delta x \\ \delta y \\ \delta\epsilon \end{Bmatrix}' = \begin{Bmatrix} 0 & 0 & f_0(x, 0, 0) \\ 0 & D_2 g(x, 0, 0) & D_3 g(x, 0, 0) \\ 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} \delta x \\ \delta y \\ \delta\epsilon \end{Bmatrix}. \tag{3.9}$$

The first qualitative question to ask about (3.1) is whether $(3.1)_\epsilon$ has a periodic orbit γ_ϵ near γ_0 for ϵ near zero. This question has been studied by Friedrichs and Wasow (1955), Flatto and Levinson (1955), and Anosov (1963). Anosov's result is most general. He proved that γ_0 can be continued to a family γ_ϵ if: (i) γ_0, regarded as a periodic orbit of the reduced system (3.2), has 1 as a Floquet multiplier of multiplicity precisely one, and (ii) for each $(x, 0) \in \gamma_0$, the initial layer equation (3.8) has a hyperbolic equilibrium point at $\delta y = 0$.

Anosov's first condition is necessary for regular continuation of γ_0 as a periodic solution of the reduced equation. Anosov showed by example that it may be impossible to continue γ_0 if his second condition is violated.

We give a simple geometric proof of Anosov's theorem. It follows from our proof that γ_ϵ and the period of γ_ϵ depend smoothly on ϵ. To demonstrate that our geometric theory leads to nontrivial computations we find the first-order terms in the Taylor series for the location and period of γ_ϵ. In case γ_0 is a hyperbolic periodic orbit of the reduced system (3.2) we show that γ_ϵ is a hyperbolic periodic orbit of $(3.1)_\epsilon$, and we show that the stable manifold and the unstable manifold of γ_ϵ depend smoothly on ϵ. Any hyperbolic periodic orbit, considered as a subset of its stable manifold, is asymptotically stable with asymptotic phase (see Hale, 1969, p. 217). We show that the asymptotic phase depends smoothly on ϵ even at $\epsilon = 0$.

To simplify our exposition, we now assume that the eigenvalues of $D_2g(x, 0, 0)$ lie in the left half plane, for all $(x, 0) \in \gamma_0$, or equivalently that the initial layer equation (3.8) has an asymptotically stable equilibrium point at the origin, for all $(x, 0) \in \gamma_0$. The case in which $D_2g(x, 0, 0)$ has some eigenvalues in the right half plane is treated in Section XIII.

Let $(x, 0)$ be any point in $\mathscr{E}$ near γ_0. The eigenvalues of $D_2g(x, 0, 0)$ lie in the left half plane, so the coefficient matrix of (3.9) has ν eigenvalues in the left half plane and has zero as an eigenvalue of multiplicity $\mu + 1$. Let $E^c_{(x,0)}$ denote the invariant subspace associated with the eigenvalue zero. An easy computation shows that if $e_1, \ldots, e_\mu$ is any basis of R^μ, then

$$\begin{Bmatrix} e_1 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} e_2 \\ 0 \\ 0 \end{Bmatrix}, \ldots, \begin{Bmatrix} e_\mu \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ D_2g(x, 0, 0)^{-1} D_3g(x, 0, 0) \\ -1 \end{Bmatrix}$$

is a basis of $E^c_{(x,0)}$. The subspace $E^c_{(x,0)}$ is invariant and asymptotically stable under the flow of (3.9), depends continuously on $(x, 0)$, and is transversal to the plane through $(x, 0)$ parallel to the y-axis. We will show that there is a smooth $(\mu + 1)$-dimensional manifold $\mathscr{C}$ containing γ_0, invariant and asymptotically stable under the nonlinear system (3.6), and tangent to $E^c_{(x,0)}$ at $(x, 0)$, for each $(x, 0) \in \mathscr{E}$. The assertion that $\mathscr{C}$ exists is a global center manifold theorem. The center manifold theorem of Kelley (1967) shows that such a manifold exists near each $(x, 0) \in \mathscr{E}$; we show that there is one manifold which is a center manifold for all the points of $\mathscr{E}$.

Anosov's perturbation theorem for γ_ϵ follows immediately from the existence of $\mathscr{C}$. By examining the basis of $E^c_{(x,0)}$ we see that $g(x, y, \epsilon)$ is of order ϵ on $\mathscr{C}$. Hence $\epsilon^{-1}g(x, y, \epsilon)$ is smooth on $\mathscr{C}$, and so the restriction of (3.5) to $\mathscr{C}$ is smooth. Anosov's theorem then is reduced to Poincaré continuation on $\mathscr{C}$. This is typical of the applications of our geometric theory. The restriction of a singular perturbation problem to a suitable invariant submanifold is a regular perturbation problem whose solution is well known.

We postpone the discussion of the stable manifold and the unstable manifold of γ_ϵ to Section XIII, and turn now to the question of asymptotic phase. We make the additional assumption that γ_0, regarded as a periodic orbit of the reduced system (3.2), has all its nontrivial Floquet multipliers inside the unit circle. The case in which γ_0 has some Floquet multipliers outside the unit circle is treated in Section XIII.

Let $(x, 0)$ be any point in $\mathscr{E}$ near γ_0. Let $E^s_{(x,0)}$ be the invariant subspace associated with the eigenvalues of the coefficient matrix of (3.9) in the left half plane. If $\hat{e}_1, \ldots, \hat{e}_\nu$ is any basis of R^ν, then

$$\begin{Bmatrix} 0 \\ \hat{e}_1 \\ 0 \end{Bmatrix}, \ldots, \begin{Bmatrix} 0 \\ \hat{e}_\nu \\ 0 \end{Bmatrix}$$

is a basis of $E^s_{(x,0)}$. The subspace $E^s_{(x,0)}$ is invariant under the flow of (3.9), and all points in $E^s_{(x,0)}$ are asymptotic to the origin at a rate faster than $e^{K\tau}$, where $K < 0$ is any number greater than the real parts of all the eigenvalues of $D_2 g(x, 0, 0)$, for all $(x, 0) \in \gamma_0$. We will show, for the nonlinear system (3.6), that there is a neighborhood U of $\gamma_0 \times \{0\}$, and a smooth family of ν-dimensional manifolds $\mathscr{F}^s(x, y, \epsilon)$, $(x, y, \epsilon) \in U$, such that

(i) $\mathscr{F}^s(x, 0, 0)$ is tangent to $E^s_{(x,0)}$, for all $(x, 0, 0) \in U \cap (\mathscr{E} \times \{0\})$.

(ii) The family $\{\mathscr{F}^s(x, y, \epsilon) \colon (x, y, \epsilon) \in U\}$ is invariant in the sense that

$$(\mathscr{F}^s(x, y, \epsilon)) \cdot \tau \subset \mathscr{F}^s((x, y, \epsilon) \cdot \tau)$$

for all $(x, y, \epsilon) \in U$, for all $\epsilon \geqslant 0$, $\tau \geqslant 0$.

(iii) For $\epsilon \geqslant 0$, $\mathscr{F}^s(x, y, \epsilon)$ is uniquely characterized by

$$\mathscr{F}^s(x, y, \epsilon) = \{(\bar{x}, \bar{y}, \epsilon) \in U \colon |(\bar{x}, \bar{y}, \epsilon) \cdot \tau - (x, y, \epsilon) \cdot \tau\,|/e^{K\tau} \to 0 \text{ as } \tau \to \infty\}.$$

The meaning of the third condition is that the orbits through points in $\mathscr{F}^s(x, y, \epsilon)$ are asymptotically equivalent to the orbit through (x, y, ϵ) as $\tau \to \infty$, up to an error of order $e^{K\tau}$. For $\epsilon > 0$ this means that $(\bar{x}, \bar{y}, \epsilon) \in \mathscr{F}^s(x, y, \epsilon)$ if and only if

$$|(\bar{x}, \bar{y}) \cdot t - (x, y) \cdot t\,|/e^{Kt/\epsilon} \to 0 \qquad \text{as} \quad t \to \infty,$$

where now $\cdot t$ is the solution operator of $(3.1)_\epsilon$. Hence the manifolds $\mathscr{F}^s(x, y, \epsilon)$ are defined by asymptotic equivalence up to transcendentally small errors under the flow of $(3.1)_\epsilon$. We will use the manifolds $\mathscr{F}^s(x, y, \epsilon)$ to study asymptotic phase.

Let S be any small surface transversal to γ_0 in M. Let (p_ϵ, q_ϵ) be the intersection of γ_ϵ with S. Because γ_ϵ depends smoothly on ϵ, p_ϵ and q_ϵ also depend smoothly on ϵ. For $\epsilon > 0$ we say that (x, y, ϵ) has asymptotic phase $\theta = \theta(x, y, \epsilon)$ on γ_ϵ if

$$|(x, y, \epsilon) \cdot t - (p_\epsilon, q_\epsilon, \epsilon) \cdot (t + \theta)| \to 0 \qquad \text{as} \quad t \to \infty. \tag{3.10}$$

Here $\cdot t$ is the solution operator of (3.5). The asymptotic phase θ, if it exists, is unique modulo multiples of the period of γ_ϵ.

On $\mathscr{C}$, (3.5) is smooth even at $\epsilon = 0$. Hence we may use (3.10) to define asymptotic phase in $\mathscr{C}$ even for $\epsilon = 0$. It is an easy result that every point $(x, y, \epsilon) \in \mathscr{C}$, $\epsilon > 0$, has asymptotic phase on γ_ϵ (see Hale, 1969, p. 217). Furthermore, the asymptotic phase in $\mathscr{C}$ depends smoothly on (x, y, ϵ) even at $\epsilon = 0$. See Fenichel (1974, Theorem 5). We define, for $\epsilon > 0$,

$$\Theta(\theta, \epsilon) = \{(x, y, \epsilon) \in \mathscr{C}: \theta(x, y, \epsilon) = \theta\}.$$

Because $\theta(x, y, \epsilon)$ is smooth, each $\Theta(\theta, \epsilon)$ is a smooth $(\mu - 1)$-dimensional manifold. For any (θ, ϵ), all the points in $\Theta(\theta, \epsilon)$ have the same asymptotic behavior, up to an error of order $e^{K_1 t}$ as $t \to \infty$, where $K_1 < 0$ is larger than the real parts of all the nontrivial Floquet exponents of γ_0.

Define, for $\epsilon \geqslant 0$,

$$\Phi_\epsilon(\theta) = \{(\bar{x}, \bar{y}) \in M: (\bar{x}, \bar{y}, \epsilon) \in \mathscr{F}^s(x, y, \epsilon) \text{ for some } (x, y, \epsilon) \in \Theta(\theta, \epsilon)\}.$$

Then each $\Phi_\epsilon(\theta)$ is a smooth $(\mu + \nu - 1)$-dimensional submanifold of M, and all the points in $\Phi_\epsilon(\theta)$ have the same asymptotic phase. As θ increases through one period of γ_ϵ, the manifolds $\Phi_\epsilon(\theta)$ sweep out a neighborhood of γ_ϵ. This proves that γ_ϵ is asymptotically stable with asymptotic phase. The manifolds $\Phi_\epsilon(\theta)$ depend smoothly on θ and ϵ, and so the asymptotic phase is smooth.

IV. Asymptotic Expansions for Periodic Solutions

We now show that the geometry outlined in the previous section leads to nontrivial computations. Under the hypotheses of Anosov's perturbation theorem we compute the first order terms in the Taylor series for the location and period of γ_ϵ. These are asymptotic computations because the center manifold is not unique, but only is unique to infinite order. The nonuniqueness of the center manifold is discussed in detail in later sections.

Before studying the singular perturbation problem we recall the regular perturbation theory for a closed orbit in R^μ. Smooth means C^r for some large finite r. Let

$$\dot{x} = \phi(x, \epsilon) \tag{4.1}$$

be a smooth differential equation defined in an open subset of R^μ, and suppose

$$\dot{x} = \phi(x, 0) \tag{4.2}$$

has a periodic orbit γ_0. We denote the solution operator of (4.1) by $F^t(x, \epsilon)$.

To simplify the construction of the Poincaré map we assume that γ_0 passes through the origin, and that $\phi(0, 0)$ is the unit vector along the μth coordinate axis. Let T be the period of γ_0, and let $\bar{x}(t) = F^t(0, 0)$ be the solution of (4.2) through the origin. The variational equation of (4.2) along $\bar{x}(t)$ is

$$\dot{\delta x} = D_1\phi(\bar{x}(t), 0)\, \delta x. \tag{4.3}$$

Let $U(t)$ be the $\mu \times \mu$ matrix-valued solution of

$$\dot{U}(t) = D_1\phi(\bar{x}(t), 0)\, U(t) \tag{4.4}$$

satisfying $U(0) = I$. This is just the derivative $D_1F^t(0, 0)$. We assume $U(t)$ is known for $0 \leqslant t \leqslant T$. Because $F^T(0, 0) = 0$ and $\phi(0, 0)$ is the unit vector along the μth coordinate axis, $U(T) = D_1F^T(0, 0)$ has the form

$$\begin{vmatrix} A & 0 \\ r & 1 \end{vmatrix}, \tag{4.5}$$

where A is a $(\mu - 1) \times (\mu - 1)$ invertible matrix, 0 is the zero column vector, and r is a row vector. The Floquet multipliers of γ_0 are the eigenvalues of $U(T)$.

Define j: $R^{\mu-1} \to R^\mu$, k: $R^\mu \to R^{\mu-1}$, and π_μ: $R^\mu \to R$ by

$$j(x_1, ..., x_{\mu-1}) = (x_1, ..., x_{\mu-1}, 0)$$

$$k(x_1, ..., x_\mu) = (x_1, ..., x_{\mu-1})$$

and

$$\pi_\mu(x_1, ..., x_\mu) = x_\mu .$$

Define $\tau(x, \epsilon)$ by requiring that τ be continuous and satisfy $\tau(0, 0) = T$ and

$$\pi_\mu F^{\tau(x,\epsilon)}(x, \epsilon) = 0. \tag{4.6}$$

By the implicit function theorem, τ is well defined and smooth for (x, ϵ) near $(0, 0) \in R^\mu \times R$. Define the Poincaré map $\Pi(\xi, \epsilon)$ by

$$\Pi(\xi, \epsilon) = kF^{\tau(j(\xi),\epsilon)}(j(\xi), \epsilon). \tag{4.7}$$

By the implicit function theorem, Π is well defined and smooth for (ξ, ϵ) near $(0, 0) \in R^{\mu-1} \times R$.

Now assume that 1 has multiplicity one as a Floquet multiplier of γ_0, or equivalently that 1 is not an eigenvalue of A. Then by the implicit function theorem we can solve

$$\Pi(\xi, \epsilon) = \xi \tag{4.8}$$

for ξ as a smooth function of ϵ, for ϵ near zero. This shows that γ_0 can be continued to a smooth family of closed orbits γ_ϵ. The period of γ_ϵ is

$$T(\epsilon) = \tau(j(\xi(\epsilon)), \epsilon). \tag{4.9}$$

Differentiating (4.7) and (4.8) at $\epsilon = 0$ gives

$$k \circ \phi(0, 0)[D_1\tau(0, 0) \circ j(D\xi(0)) + D_2\tau(0, 0)] + k \circ D_1F^t(0, 0) \circ j(D\xi(0)) + k \circ D_2F^T(0, 0) = D\xi(0).$$

Noting that $k \circ \phi(0, 0) = 0$ and $k \circ D_1F^T(0, 0) \circ j = A$, we have

$$D\xi(0) = (I - A)^{-1}k \circ D_2F^T(0, 0). \tag{4.10}$$

Differentiating (4.9) gives

$$DT(0) = D_1\tau(0, 0) \circ j \circ D\xi(0) + D_2\tau(0, 0). \tag{4.11}$$

Differentiating (4.6) with respect to x gives

$$\pi_u[\phi(0, 0)\, D_1\tau(0, 0) + D_1F^T(0, 0)] = 0,$$

so

$$D_1\tau(0, 0) = -[r, 1]$$

and

$$D_1\tau(0, 0) \circ j = -r. \tag{4.12}$$

Differentiating (4.6) with respect to ϵ gives

$$\pi_u[\phi(0, 0)\, D_2\tau(0, 0) + D_2F^T(0, 0)] = 0$$

so

$$D_2\tau(0, 0) = -\pi_u D_2F^T(0, 0). \tag{4.13}$$

Combining (4.10), (4.11), (4.12), and (4.13) gives

$$DT(0) = -r(I - A)^{-1}k \circ D_2F^T(0, 0) - \pi_u D_2F^T(0, 0). \tag{4.14}$$

With these computations we know the location and period of γ_ϵ to first order in ϵ, except for the factor $D_2F^T(0, 0)$ which enters in (4.10) and (4.14). Let $v(t) = D_2F^t(0, 0)$. From (4.1) we see that $v(t)$ satisfies the inhomogeneous variational equation

$$\dot{v}(t) = D_1\phi(\bar{x}(t), 0)\, v(t) + D_2\phi(\bar{x}(t), 0), \tag{4.15}$$

with the initial value $v(0) = 0$. Equation (4.15) can be integrated by using the matrix-valued solution $U(t)$ of the homogeneous variational equation, yielding

$$v(t) = \int_0^t U(t)\, U(s)^{-1}\, D_2\phi(\bar{x}(s), 0)\, ds,$$

and so

$$D_2F^T(0, 0) = \int_0^T U(T)\, U(s)^{-1}\, D_2\phi(\bar{x}(s), 0)\, ds. \tag{4.16}$$

Now we return to the singular perturbation problem (3.1). The center manifold $\mathscr{C}$ is the graph of a smooth function $y = u(x, \epsilon)$ satisfying

$$u(x, 0) = 0. \tag{4.17}$$

On $\mathscr{C}$, (3.1) reduces to

$$\begin{aligned} \dot{x} &= f_0(x, y, \epsilon) \\ \dot{y} &= \epsilon^{-1} g(x, y, \epsilon) \end{aligned} \tag{4.18}$$

with

$$y = u(x, \epsilon). \tag{4.19}$$

Combining (4.18) and (4.19) gives a regular system in R^μ,

$$\dot{x} = f_0(x, u(x, \epsilon), \epsilon). \tag{4.20$_\epsilon$}$$

We will apply regular perturbation theory to (4.20).

The reduced system (3.2) is equivalent to (4.20) with $\epsilon = 0$. We have assumed that (3.2) has a periodic solution γ_0. Without loss of generality we may assume that γ_0 passes through the origin and that $f_0(0, 0, 0)$ is the unit vector along the μth coordinate axis. Let $\bar{x}(t)$ be the solution of $(4.20)_0$ through the origin, let T be the period of $\bar{x}(t)$, and let $U(t)$ be the solution of the variational equation of the reduced equation $(4.20)_0$ along $\bar{x}(t)$,

$$\begin{aligned} \dot{U}(t) &= D_1 f_0(\bar{x}(t), 0, 0)\, U(t) \\ U(0) &= I. \end{aligned}$$

We can apply our computations from regular perturbation theory and simply write down the first-order terms in the series for the location and period, if we know

$$\partial/\partial\epsilon\, f_0(x, u(x, \epsilon), \epsilon)\,|_{\epsilon=0} = D_2 f_0(x, 0, 0)\, D_2 u(x, 0) + D_3 f_0(x, 0, 0) \tag{4.21}$$

for $x \in \gamma_0$. This requires the computation of $D_2u(x, 0)$ for $x \in \gamma_0$. Differentiate (4.19) with respect to t to get

$$\dot{y} = D_1 u(x, \epsilon)\dot{x}. \tag{4.22}$$

Substitute (4.18) and (4.19) in (4.22)

$$g(x, u(x, \epsilon), \epsilon) = \epsilon D_1 u(x, \epsilon) f_0(x, u(x, \epsilon), \epsilon). \tag{4.23}$$

Differentiate (4.23) with respect to ϵ

$$\begin{aligned} D_2g(x, u(x, \epsilon), \epsilon)\, D_2u(x, \epsilon) &+ D_3g(x, u(x, \epsilon), \epsilon) \\ &= D_1u(x, \epsilon)\, f_0(x, u(x, \epsilon), \epsilon) \\ &\quad + \epsilon D_{12}u(x, \epsilon)\, f_0(x, u(x, \epsilon), \epsilon) \\ &\quad + \epsilon D_1u(x, \epsilon)\{D_2f_0(x, u(x, \epsilon), \epsilon)\, D_2u(x, \epsilon) \\ &\qquad + D_3f_0(x, u(x, \epsilon), \epsilon)\}. \end{aligned} \tag{4.24}$$

Set $\epsilon = 0$ in (4.24), and note that by (4.17), $D_1u(x, 0) \equiv 0$. Then

$$D_2g(x, 0, 0)\, D_2u(x, 0) + D_3g(x, 0, 0) = 0$$

so

$$D_2u(x, 0) = -D_2g(x, 0, 0)^{-1}\, D_3g(x, 0, 0) \tag{4.25}$$

We now sum up our computations. Let $\xi(\epsilon)$ be the projection on $R^{\mu-1}$ of the intersection of γ_ϵ with the plane $x_\mu = 0$, and let $T(\epsilon)$ be the period of γ_ϵ. Let j, k and π_μ be the linear operators defined above. Then

$$D\xi(0) = (I - A)^{-1}\, kv$$

and

$$DT(0) = -r(I - A)^{-1}\, kv - \pi_\mu v,$$

where

$$A = k \circ U(T) \circ j$$

$$r = \pi_\mu \circ U(T) \circ j$$

and

$$v = \int_0^T U(T)\, U(s)^{-1}\{-D_2f_0(\bar{x}(s), 0, 0)\, D_2g(\bar{x}(s), 0, 0)^{-1}\, D_3g(\bar{x}(s), 0, 0)$$
$$+ D_3f_0(\bar{x}(s), 0, 0)\}\, ds.$$

The computation of higher order terms proceeds in the same manner. We find the higher order terms in the asymptotic expansion for u by differentiating (4.24) repeatedly at $\epsilon = 0$, and then substitute these into the Taylor series computed using regular perturbation theory.

V. The Equations of Local Singular Perturbation Theory

Many local singular perturbation problems take the form (3.1), with condition (3.3) satisfied. This form is not natural, however, because it depends upon the choice of special coordinates. In place of (3.1) and (3.3) we study more general systems in a form which is independent of the choice of coordinates.

To motivate our definition of local singular perturbation problem we consider two more examples. The Van der Pol equation with a large parameter takes the form

$$\begin{aligned} \dot{x} &= y \\ \epsilon\dot{y} &= y - \tfrac{1}{3}y^3 - x, \end{aligned} \tag{5.1}$$

where ϵ is a small parameter. (See Stoker, 1950; we have changed Stoker's notation.) After the rescaling $\tau = t/\epsilon$, (5.1) is transformed to

$$\begin{aligned} x' &= \epsilon y \\ y' &= y - \tfrac{1}{3}y^3 - x. \end{aligned} \tag{5.2}$$

The equations for traveling wave solutions of Nagumo's equation are

$$\begin{aligned} x' &= \epsilon y_1 \\ y_1' &= y_2 \\ y_2' &= \sigma y_2 - f(y_1) + x, \end{aligned} \tag{5.3}$$

where σ is a nonzero parameter, ϵ is a small parameter, and $f(y_1)$ typically has the qualitative properties of $-y_1(y_1 - \frac{1}{2})(y_1 - 1)$. (See Conley, 1977a, 1977b; we have changed Conley's notation.) After the rescaling $t = \epsilon\tau$, (5.3) is transformed to

$$\begin{aligned} \dot{x} &= y_1 \\ \epsilon\dot{y}_1 &= y_2 \\ \epsilon\dot{y}_2 &= \sigma y_2 - f(y_1) + x. \end{aligned} \tag{5.4}$$

The singularities in (3.1), (5.1), and (5.4) are reflected in the presence of manifolds of equilibrium points for (3.4), (5.2), and (5.3). Generically equilibrium points are isolated; for the fast time systems of singular perturbation problems they are not. We take this as our definition of singular perturbation problem.

Let M be an open subset of $R^\mu \times R^\nu$, let $\mathscr{E}$ be a μ-dimensional submanifold of M, let r be a positive integer or infinity, and let ϵ_0 be a small positive number. In the local theory we study C^r systems of differential equations of the form

$$z' = h(z, \epsilon) \tag*{(5.5)$_\epsilon$}$$

defined for $z \in M$, $\epsilon \in (-\epsilon_0, \epsilon_0)$, subject to the condition

$$h(z, 0) = 0 \qquad \text{for all } z \in \mathscr{E}. \tag{5.6}$$

System (3.4) is a special case of (5.5), with $z = (x, y)$ and $h = (\epsilon f_0, g)$. Along with (5.5) we study the slow time system

$$\epsilon \dot{z} = h(z, \epsilon) \tag{5.7$_\epsilon$}$$

and the extended systems

$$\begin{aligned} z' &= h(z, \epsilon) \\ \epsilon' &= 0 \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} \epsilon \dot{z} &= h(z, \epsilon) \\ \dot{\epsilon} &= 0. \end{aligned} \tag{5.9}$$

Note that $\mathscr{E}$ consists entirely of equilibrium points of $(5.5)_0$, although we do not require that $\mathscr{E}$ contain all the equilibrium points of $(5.5)_0$ in M. For applications to exchange of stability problems, as in Section XV, the equilibrium points of $(5.5)_0$ do not form a manifold.

For some applications we are interested in (5.5)–(5.9) only for small positive ϵ. Then it is natural to assume that h is defined only for small positive ϵ, and that h and its derivatives up to order r have limits as $\epsilon \to 0+$. Under these assumptions, however, we can extend h to a C^r function defined for ϵ in a neighborhood of zero. Then the assumption that (5.5)–(5.9) are defined for ϵ in a neighborhood of zero may be regarded as a convenient way to keep track of one-sided limits as $\epsilon \to 0+$. The reader is cautioned, however, that in such applications our results for $\epsilon < 0$ have no natural meaning.

The linearization of $(5.5)_0$ at $z \in \mathscr{E}$ is

$$\delta z' = D_1 h(z, 0)\, \delta z. \tag{5.10}$$

It follows from (5.6) that zero is an eigenvalue of $D_1 h(z, 0)$ of multiplicity at least μ. We call the μ zeros corresponding to the tangent space of $\mathscr{E}$ the trivial eigenvalues, and we call the remaining eigenvalues the nontrivial eigenvalues.

Let $\mathscr{E}_R \in \mathscr{E}$ be the open set where all the nontrivial eigenvalues are nonzero. For each $z \in \mathscr{E}_R$ the kernel of $D_1 h(z, 0)$ has a unique invariant complement, so there is a well-defined projection on the kernel. We denote this projection by $\pi^{\mathscr{E}}$, because it is associated with the tangent space of $\mathscr{E}$. The kernel and its invariant complement are C^{r-1}, so $\pi^{\mathscr{E}}$ also is C^{r-1}. Let $\mathscr{E}_H \subset \mathscr{E}_R$ be the open set where all the nontrivial eigenvalues have nonzero real parts. For $z \in \mathscr{E}_H$, the linearization of $(5.5)_0$ normal to $\mathscr{E}$ has a hyperbolic fixed point.

For the local theory near a point $z_0 \in \mathscr{E}$ we can choose an (x, y)-coordinate system with origin at z_0 and with $\mathscr{E}$ tangent to the x-axis at the origin. In such coordinates $(5.5)_\epsilon$ takes the form

$$\begin{aligned} x' &= f(x, y, \epsilon) \\ y' &= g(x, y, \epsilon) \end{aligned} \tag{5.11$_\epsilon$}$$

with

$$\begin{aligned} f(0, 0, 0) &= 0 \\ g(0, 0, 0) &= 0 \\ D_1 f(0, 0, 0) &= 0 \\ D_1 g(0, 0, 0) &= 0. \end{aligned} \tag{5.12}$$

Suppose there are k_1 nontrivial eigenvalues in the left half plane, k_2 nontrivial eigenvalues on the imaginary axis, and k_3 nontrivial eigenvalues in the right half plane. Then we can decompose y as $(y_1, y_2, y_3) \in R^{k_1} \times R^{k_2} \times R^{k_3}$, and decompose $g(x, y, \epsilon)$ as $(g_1(x, y_1, y_2, y_3, \epsilon), g_2(x, y_1, y_2, y_3, \epsilon), g_3(x, y_1, y_2, y_3, \epsilon)) \in R^{k_1} \times R^{k_2} \times R^{k_3}$, with the requirement that

$$\left.\frac{\partial(g_1, g_2, g_3)}{\partial(y_1, y_2, y_3)}\right|_{\substack{x=0 \\ y=0 \\ \epsilon=0}} = \begin{vmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{vmatrix}, \tag{5.13}$$

where the eigenvalues of A_1 are in the left half plane, the eigenvalues of A_2 are on the imaginary axis, and the eigenvalues of A_3 are in the right half plane.

If $z_0 \in \mathscr{E}_R$, the kernel of $D_1 h(z_0, 0)$ has a unique invariant complement, and we may take the y-axis tangent to be this complement. Then we have

$$D_2 f(0, 0, 0) = 0. \tag{5.14}$$

The reduction of (5.5) to (5.11) involves only linear algebra. The next three lemmas show that we can simplify (5.5) still further using elementary analysis. In Section XI we will use invariant manifold theory to reduce (5.5) to a local normal form.

LEMMA 5.1. *There is a C^r local coordinate system $z = \phi(x, y)$ such that $z_0 = \phi(0, 0)$, transforming* (5.5) *into*

$$\begin{aligned} x' &= \epsilon f_0(x, y, \epsilon) + f_1(x, y, \epsilon) \cdot y \\ y' &= g(x, y, \epsilon) \end{aligned} \tag{5.15}$$

where $f_0(0, 0, 0) = 0$, $f_1(0, 0, 0) = 0$, and $g(x, 0, 0) \equiv 0$. The functions f_0 and f_1 are C^{r-1}.

Proof. In the construction of the coordinate system for (5.11), (5.12) we may take for x a local coordinate in $\mathscr{E}$ and for y a local coordinate normal to $\mathscr{E}$. Then, because h vanishes identically on $\mathscr{E}$, we have

$$\begin{aligned} f(x, 0, 0) &\equiv 0 \\ g(x, 0, 0) &\equiv 0 \end{aligned} \tag{5.16}$$

for all x near the origin in R^{μ}. Following Milnor (1963, p. 5), we write

$$\begin{aligned} f(x, y, \epsilon) &= \int_0^1 \frac{\partial}{\partial\alpha} f(x, \alpha y, \alpha\epsilon)\, d\alpha \\ &= \epsilon \int_0^1 D_3 f(x, \alpha y, \alpha\epsilon)\, d\alpha \\ &\quad + \int_0^1 D_2 f(x, \alpha y, \alpha\epsilon)\, d\alpha \cdot y \\ &= \epsilon f_0(x, y, \epsilon) + f_1(x, y, \epsilon) \cdot y \end{aligned}$$

LEMMA 5.2. *Suppose* $z_0 \in \mathscr{E}_R$. *Then there is a* C^{r-1} *coordinate system as in Lemma* 5.1, *satisfying the additional conditions*

$$D_2 f(x, 0, 0) \equiv 0. \tag{5.17}$$

Condition (5.17) *is equivalent to* $f_1(x, 0, 0) \equiv 0$.

Proof. In the construction of the (x, y)-coordinates we require, for each $(x, 0)$ near $(0, 0)$, that the image of the plane through $(x, 0)$ parallel to the y-axis be tangent to the invariant complement of $T\mathscr{E}$ at $(x, 0)$.

LEMMA 5.3. *If all the nontrivial eigenvalues of* $D_1 h(z_0, 0)$ *lie in the left half plane or in the right half plane, there is a* C^{r-1} *local coordinate system in which* (5.5) *takes the form* (3.4).

Proof. We do not require this lemma, so we omit the proof. The idea is that for $\epsilon = 0$, for each z near z_0, there is a unique manifold consisting of points near z whose forward (backward) orbit is asymptotic to z. See Hadamard (1901) and Fenichel (1974, Theorem 1; 1977, Theorem 1). Using these special manifolds one can eliminate the term $f(x, y, \epsilon) \cdot y$ in Lemma 5.1.

Remark. Lemma 5.3 follows directly from Theorem 9.1.

In $\mathscr{E}_R$ we define the reduced system

$$\dot{z} = \pi^{\mathscr{E}} D_2 h(z, 0). \tag{5.18}$$

We will show in Section VII that (5.18) defines a vector field on $\mathscr{E}_R$. For now we note that in the special coordinates of Lemma 5.1, (5.18) reduces to

$$\dot{x} = f(x, 0, 0) \tag{5.19}$$

with $y = 0$. Hence also (5.18) agrees with (3.2) for systems of the form (3.1).

In general we do not define the reduced system in $\mathscr{E} - \mathscr{E}_R$. An exception appears in Section XV.

The following lemma gives the reduced system in local coordinates in which $\mathscr{E}$ appears as the graph of a function $y = u(x)$.

LEMMA 5.4. *Consider a system*

$$\begin{aligned} x' &= f(x, y, \epsilon) \\ y' &= g(x, y, \epsilon), \end{aligned}$$

defined for (x, y) in an open set M in $R^\mu \times R^\nu$, for ϵ near zero. Let $y = u(x)$ be a function defined for x near x_0, such that

$$\begin{aligned} f(x, u(x), 0) &\equiv 0 \\ g(x, u(x), 0) &\equiv 0. \end{aligned}$$

Suppose $(x_0, u(x_0)) \in \mathscr{E}_R$, so that the matrix

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} D_1 f(x_0, u(x_0), 0) & D_2 f(x_0, u(x_0), 0) \\ D_1 g(x_0, u(x_0), 0) & D_2 g(x_0, u(x_0), 0) \end{vmatrix}$$

has rank ν. Let $\kappa = Du(x_0)$. Then the projection $\pi^{\mathscr{E}} = \pi^{\mathscr{E}}(x_0, u(x_0))$ is multiplication by the matrix

$$\begin{vmatrix} I + \beta(\delta - \kappa\beta)^{-1}\kappa & -\beta(\delta - \kappa\beta)^{-1} \\ \kappa + \kappa\beta(\delta - \kappa\beta)^{-1}\kappa & -\kappa\beta(\delta - \kappa\beta)^{-1} \end{vmatrix}$$

and the reduced system is

$$\begin{aligned} \dot{x} &= (I + \beta(\delta - \kappa\beta)^{-1}\kappa)\, D_3 f(x, u(x), 0) - \beta(\delta - \kappa\beta)^{-1}\, D_3 g(x, u(x), 0) \\ \dot{y} &= (\kappa + \kappa\beta(\delta - \kappa\beta)^{-1}\kappa)\, D_3 f(x, u(x), 0) - \kappa\beta(\delta - \kappa\beta)^{-1}\, D_3 g(x, u(x), 0). \end{aligned}$$

Proof. This is an elementary computation.

To exhibit the formulas of Lemma 5.4 in an explicit example we now compute the projection and the reduced system for Van der Pol's equation. We reverse the rôles of x and y, so that on $\mathscr{E}$ we can solve for y globally as a function of x. Then we have

$$\begin{aligned} x' &= x - \tfrac{1}{3}x^3 - y \\ y' &= \epsilon x \end{aligned}$$

and

$$u(x) = x - \tfrac{1}{3}x^3.$$

We find

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} 1 - x^2 & -1 \\ 0 & 0 \end{vmatrix}$$

and

$$k = 1 - x^2$$

so $\pi^{\mathscr{E}}$ is multiplication by

$$\begin{vmatrix} 0 & 1/(1 - x^2) \\ 0 & 1 \end{vmatrix}$$

and the reduced system is

$$\begin{aligned} x' &= x/(1 - x^2) \\ y' &= x. \end{aligned}$$

$\mathscr{E} - \mathscr{E}_R$ consists of the points $(1, 2/3)$ and $(-1, -2/3)$. The reduced system is singular at these points.

VI. Global Flows

In the following section we introduce a coordinate-free notion of singular perturbation problem. First, however, we recall the coordinate-free notion of ordinary differential equation, the flow of a vector field on a manifold.

Let M be a C^{r+1} n-dimensional manifold, $1 \leqslant r \leqslant \infty$. Let X be a C^r vector field on M, a function $X: M \to TM$ assigning to each $m \in M$ a tangent vector $X(m) \in T_mM$. Tangent vectors act on functions by directional differentiation. For any differentiable function ϕ defined near m, $X\phi(m)$ is defined as $D\phi(m)\, X(m)$. We define a flow $(m, t) \to m \cdot t$ by requiring that

$$d/dt\, \phi(m \cdot t) = X\phi(m \cdot t) \tag{6.1}$$

for all differentiable functions ϕ. See Lang (1962) for a proof that (6.1) defines a flow.

Let U be an open set in M, and let ϕ: $U \to V \subset R^n$ be a C^{r+1} diffeomorphism. Then $z = \phi(m)$ defines a local coordinate system. By (6.1), the flow of X satisfies

$$\dot{z} = X\phi(\phi^{-1}(z)) \tag{6.2}$$

in z-coordinates.

The variational equation of (6.2) along an orbit $z \cdot t$ is

$$\mathring{\delta z} = D^2\phi(\phi^{-1}(z \cdot t))(D\phi^{-1}(z \cdot t)\, \delta z, X(\phi^{-1}(z \cdot t)) \\ + D\phi(\phi^{-1}(z \cdot t))\, DX(\phi^{-1}(z \cdot t))\, D\phi^{-1}(z \cdot t)\, \delta z. \tag{6.3}$$

Here we follow Dieudonné (1960) in writing $D^2\phi$ as a bilinear map. Suppose now that $m = \phi^{-1}(z)$ is a zero of $X(m)$, an equilibrium point of the flow of X. Then (6.3) simplifies to

$$\mathring{\delta z} = D\phi(m)\, DX(m)\, D\phi(m)^{-1}\, \delta z. \tag{6.4}$$

The coefficient matrix of (6.4) transforms by similarity, and hence defines a linear map $TX(m)\colon T_mM \to T_mM$.

VII. The Equations of Global Singular Perturbation Theory

Let M be a C^{r+1} manifold, $1 \leqslant r \leqslant \infty$. Let X^ϵ be a family of vector fields on M, parametrized by $\epsilon \in (-\epsilon_0, \epsilon_0)$, such that $X^\epsilon(m)$ is a C^r function of (m, ϵ). Let $\mathscr{E}$ be a C^r submanifold of M consisting entirely of equilibrium points of X^0. In the global theory we study the flow of X^ϵ near $\mathscr{E}$, for small nonzero ϵ.

Let $z = \phi(m)$ be a C^{r+1} local coordinate in M. In z-coordinates the flow of X^ϵ satisfies

$$z' = X^\epsilon\phi(\phi^{-1}(z)) \tag*{$(7.1)_\epsilon$}$$

subject to the condition

$$X^\epsilon\phi(\phi^{-1}(z)) = 0 \qquad \text{for} \quad z \in \phi(\mathscr{E}). \tag{7.2}$$

This equation has the form of $(5.5)_\epsilon$. After the rescaling $t = \epsilon\tau$, $(7.1)_\epsilon$ takes the form

$$\epsilon\dot{z} = X^\epsilon\phi(\phi^{-1}(z)). \tag*{$(7.3)_\epsilon$}$$

This is the local form of $\epsilon^{-1}X^\epsilon$. Let $\mathbf{0}$ denote the zero vector field on $(-\epsilon_0, \epsilon_0)$. The vector fields $X^\epsilon \times \mathbf{0}$ and $\epsilon^{-1}X^\epsilon \times \mathbf{0}$ are defined on $M \times (-\epsilon_0, \epsilon_0)$. In local coordinates the flows of $X^\epsilon \times \mathbf{0}$ and $\epsilon^{-1}X^\epsilon \times \mathbf{0}$ satisfy

$$\begin{aligned} z' &= X^\epsilon\phi(\phi^{-1}(z)) \\ \epsilon' &= 0 \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} \epsilon\dot{z} &= X^\epsilon\phi(\phi^{-1}(z)) \\ \dot{\epsilon} &= 0. \end{aligned} \tag{7.5}$$

The flows of X^ϵ and $X^\epsilon \times \mathbf{0}$ are denoted by $\cdot t$, and the flows of $\epsilon^{-1}X^\epsilon$ and $\epsilon^{-1}X^\epsilon \times \mathbf{0}$ are denoted by $\cdot \tau$.

Let μ be the dimension of $\mathscr{E}$, and let ν be the codimension of $\mathscr{E}$ in M. Let m be any point in $\mathscr{E}$. We have linear maps

$$TX^0(m)\colon T_mM \to T_mM \tag{7.6}$$

and

$$T(X^0 \times \mathbf{0})(m, 0)\colon T_{(m,0)}\{M \times (-\epsilon_0, \epsilon_0)\} \to T_{(m,0)}\{M \times (-\epsilon_0, \epsilon_0)\} \tag{7.7}$$

on linear spaces of dimension $\mu + \nu$ and $\mu + \nu + 1$, respectively. Because X^0 vanishes identically on $\mathscr{E}$, $T_m\mathscr{E}$ is in the kernel of $TX^0(m)$. The subspace $T_m\mathscr{E}$ is invariant under $TX^0(m)$, and so $TX^0(m)$ induces a linear map

$$QX^0(m)\colon T_mM/T_m\mathscr{E} \to T_mM/T_m\mathscr{E}$$

on the quotient space. The eigenvalues of $QX^0(m)$ are the nontrivial eigenvalues of the linearization of $(7.1)_0$ at $z = \phi(m)$.

Let $\mathscr{E}_R \subset \mathscr{E}$ be the open set where QX^0 is invertible. For each $m \in \mathscr{E}_R$, $T_m\mathscr{E}$ has a unique complement N_m which is invariant under $TX^0(m)$. The complement is a concrete realization of the quotient space $T_mM/T_m\mathscr{E}$. Let $\pi^{\mathscr{E}}$ denote the projection on $T\mathscr{E}$ defined by the splitting $TM \mid \mathscr{E}_R = T\mathscr{E} \mid \mathscr{E}_R \oplus N$. The splitting is C^{r-1}, and so $\pi^{\mathscr{E}}$ is C^{r-1}. Let $\mathscr{E}_H \subset \mathscr{E}_R$ be the open subset where QX^0 has no pure imaginary eigenvalues.

In $\mathscr{E}_R$ the reduced vector field X_R is defined by

$$X_R(m) = \pi^{\mathscr{E}}\, \partial/\partial\epsilon\, X^\epsilon(m) \,|_{\epsilon=0}\,. \tag{7.8}$$

In the form (7.8) it is clear that X_R is a C^{r-1} vector field on $\mathscr{E}$. This justifies the local definition (5.12).

VIII. More Notation

Throughout this section there are parallel definitions for slow time and for fast time. We state explicitly only the slow time definitions.

Let M be a manifold, and let $\cdot t$ be a flow on M. For any subset $V \subset M$ and any subset $J \subset R$, let $V \cdot J = \{p \cdot t\colon p \in V,\, t \in J\}$. We say that V is positively invariant if $V \cdot [0, \infty) \subset V$, negatively invariant if $V \cdot (-\infty, 0] \subset V$, and invariant if $V \cdot (-\infty, \infty) \subset V$. Define

$$A^+(V) = \{p \in V\colon \overline{p \cdot [0, \infty)} \subset V\}$$

$$A^-(V) = \{p \in V\colon \overline{p \cdot (-\infty, 0]} \subset V\}$$

$$I(V) = \{p \in V\colon \overline{p \cdot (-\infty, \infty)} \subset V\},$$

where the bar denotes closure. We call $A^+(V)$, $A^-(V)$, and $I(V)$ the maximal positively invariant set in V, the maximal negatively invariant set in V, and the maximal invariant set in V, respectively. In case V is closed our definitions follow Conley and Easton (1971).

A manifold with boundary $\bar{V} = V \cup \partial V$ is called overflowing invariant if $\bar{V}$ is negatively invariant and orbits which intersect ∂V cross ∂V transversally. Similarly, $\bar{V}$ is inflowing invariant if $\bar{V}$ is positively invariant and orbits which meet ∂V cross ∂V transversally.

Suppose $V \subset U \subset M$. We say that V is invariant relative to U if orbit segments which leave V also leave U. More precisely, this means that for all $p \in V$, if $t \geqslant 0$ and $p \cdot [0, t] \subset U$, then $p \cdot [0, t] \subset V$, and if $t \leqslant 0$ and $p \cdot [t, 0] \subset U$, then $p \cdot [t, 0] \subset V$. We say that V is locally invariant if V is invariant relative to some neighborhood of V.

Suppose V is locally invariant, and let $\{S(p): p \in V\}$ be a family of subsets of M parametrized by $p \in V$. We say that $\{S(p): p \in V\}$ is locally positively invariant if $S(p) \cdot t \subset S(p \cdot t)$ for all $p \in V$ and all $t \geqslant 0$ such that $p \cdot [0, t] \subset V$. We say that $\{S(p): p \in V\}$ is locally negatively invariant if $S(p) \cdot t \subset S(p \cdot t)$ for all $p \in V$ and all $t \leqslant 0$ such that $p \cdot [t, 0] \subset V$.

Suppose V is a locally invariant submanifold of M, and $\{S(p): p \in V\}$ is a family of submanifolds of M parametrized by $p \in V$. Let

$$S^* = \{(p, p'): p \in V, p' \in S(p)\}.$$

We say that $\{S(p): p \in V\}$ is a C^r family of manifolds if S^* is a C^r submanifold of $M \times M$.

Suppose $\{V_\epsilon: \epsilon \in (-\epsilon_1, \epsilon_1)\}$ is a family of submanifolds of M parametrized by $\epsilon \in (-\epsilon_1, \epsilon_1)$. Let

$$V^* = \{(p, \epsilon): \epsilon \in (-\epsilon_1, \epsilon_1), p \in V_\epsilon\}.$$

We say that $\{V_\epsilon: \epsilon \in (-\epsilon_1, \epsilon_1)\}$ is a C^r family if V^* is a C^r submanifold of $M \times (-\epsilon_1, \epsilon_1)$. Let $\{S(p, \epsilon): \epsilon \in (-\epsilon_1, \epsilon_1), p \in V_\epsilon\}$ be a family of manifolds parametrized by $(p, \epsilon) \in V^*$. Let

$$S^* = \{(p, p', \epsilon): \epsilon \in (-\epsilon_1, \epsilon_1), p \in V_\epsilon, p' \in S(p, \epsilon)\}.$$

We say that $\{S(p, \epsilon): \epsilon \in (-\epsilon_1, \epsilon_1), p \in V_\epsilon\}$ is a C^r family if S^* is a C^r submanifold of $M \times M \times (-\epsilon_1, \epsilon_1)$.

A family of vector fields on a manifold M is called a C^r family if $X^\epsilon(m)$ is a C^r function of (m, ϵ).

IX. Global Singular Perturbation Theory

In this section we state the global version of our main result. Let M be a C^{r+1} manifold, $1 \leqslant r < \infty$, let X^ϵ, $\epsilon \in (-\epsilon_0, \epsilon_0)$ be a C^r family of vector fields on M, and let $\mathscr{E}$ be a C^r manifold consisting entirely of equilibrium points of X^0. Let k_1, k_2 and k_3 be fixed integers, and let $K \subset \mathscr{E}$ be a compact subset such that $QX^0(m)$ has k_1 eigenvalues in the left half plane, k_2 eigenvalues on the imaginary axis, and k_3 eigenvalues in the right half plane, for all $m \in K$. Then $T(X^\epsilon \times \mathbf{0})(m, 0)$ has k_1 eigenvalues in the left half plane, $k_2 + \mu + 1$ eigenvalues on the imaginary axis, and k_3 eigenvalues in the right half plane, for all $m \in K$.

For each $m \in K$, let $E_m{}^s$, $E_m{}^c$ and $E_m{}^u$ denote the invariant subspaces of $T_{(m,0)}M \times (-\epsilon_0, \epsilon)$ associated with the eigenvalues of $T(X^\epsilon \times \mathbf{0})(m, 0)$ in the left half plane, on the imaginary axis, and in the right half plane, respectively. We call a manifold $\mathscr{C}^s$ a center-stable manifold for $X^\epsilon \times \mathbf{0}$ near K if $K \times \{0\} \subset \mathscr{C}^s$, $\mathscr{C}^s$ is locally invariant under the flow of $X^\epsilon \times \mathbf{0}$, and for all $(m, 0) \in K \times \{0\}$, $\mathscr{C}^s$ is tangent to $E_m{}^s \oplus E_m{}^c$ at $(m, 0)$. We define center-unstable manifold and center manifold the same way, with $E_m{}^s \oplus E_m{}^c$ replaced by $E_m{}^c \oplus E_m{}^u$ and $E_m{}^c$, respectively.

Let $\mathscr{C}^s$ be a center-stable manifold for $X^\epsilon \times \mathbf{0}$ near K. We say that a family $\{\mathscr{F}^s(p): p \in \mathscr{C}^s\}$ is a C^{r_2} family of C^{r_1} stable manifolds for $\mathscr{C}^s$ near K if

(i) $\mathscr{F}^s(p)$ is a C^{r_1} manifold for each $p \in \mathscr{C}^s$.

(ii) $p \in \mathscr{F}^s(p)$, for each $p \in \mathscr{C}^s$.

(iii) $\mathscr{F}^s(p)$ and $\mathscr{F}^s(q)$ are disjoint or identical, for each p and q in $\mathscr{C}^s$.

(iv) $\mathscr{F}^s(m, 0)$ is tangent to $E_m{}^s$ at $(m, 0)$, for each $m \in K$.

(v) $\{\mathscr{F}^s(p): p \in \mathscr{C}^s\}$ is a positively invariant C^{r_2} family of manifolds.

Let $\mathscr{C}^u$ be a center-unstable manifold for $X^\epsilon \times \mathbf{0}$ near K. We say that a family $\{\mathscr{F}^u(p): p \in \mathscr{C}^u\}$ is a C^{r_2} family of C^{r_1} unstable manifolds for $\mathscr{C}^u$ near K if

(i) $\mathscr{F}^u(p)$ is a C^{r_1} manifold for each $p \in \mathscr{C}^u$.

(ii) $p \in \mathscr{F}^u(p)$, for each $p \in \mathscr{C}^u$.

(iii) $\mathscr{F}^u(p)$ and $\mathscr{F}^u(q)$ are disjoint or identical, for each p and q in $\mathscr{C}^u$.

(iv) $\mathscr{F}^u(m, 0)$ is tangent to $E_m{}^u$ at $(m, 0)$, for each $m \in K$.

(v) $\{\mathscr{F}^u(p): p \in \mathscr{C}^u\}$ is a negatively invariant C^{r_2} family of manifolds.

Figure 1 may help the reader keep track of these definitions. Near a point $m \in \mathscr{E}$ we choose (x, y) coordinates so that X^ϵ takes the form (5.11) with conditions (5.12) satisfied. We break up the vector y into components y_1, y_2, y_3 in (5.13), so that the matrix $D_2 g(0, 0, 0)$ takes the form

$$\begin{vmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{vmatrix}$$

with the eigenvalues of A_1 in the left half plane, the eigenvalues of A_2 on the imaginary axis, and the eigenvalues of A_3 in the right half plane. The center manifold $\mathscr{C}$ is tangent to the (x, y_2, ϵ)-plane, the center-stable manifold $\mathscr{C}^s$

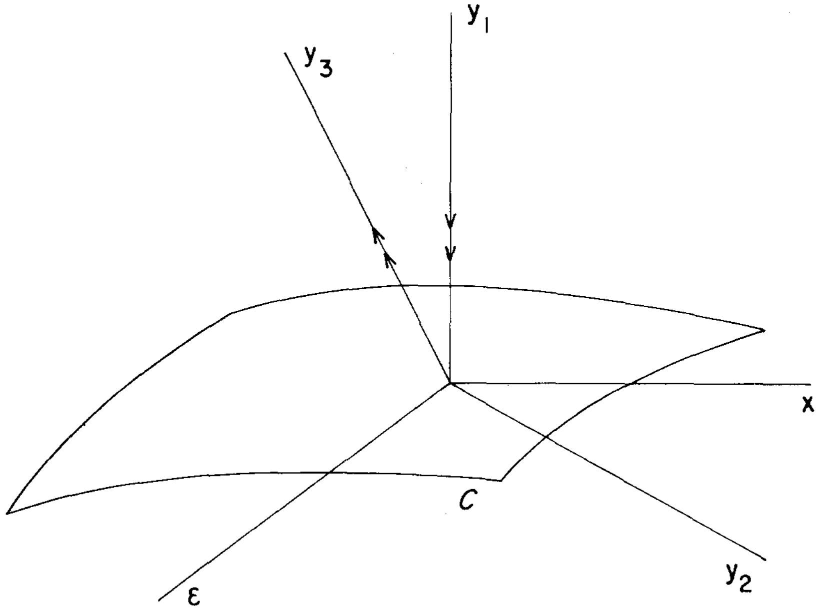

FIGURE 1

is tangent to the (x, y_1, y_2, ϵ)-plane, and the center-unstable manifold $\mathscr{C}^u$ is tangent to the (x, y_2, y_3, ϵ)-plane. The family of stable manifolds lies in $\mathscr{C}^s$; fibers are roughly parallel to the y_1-axis. The family of unstable manifolds lies in $\mathscr{C}^u$; fibers are roughly parallel to the y_3-axis.

In the following theorem A^+, A^-, and I are defined using the flow of $X^\epsilon \times \mathbf{0}$.

THEOREM 9.1. *Let M be a C^{r+1} manifold, $1 \leqslant r < \infty$. Let X^ϵ, $\epsilon \in (-\epsilon_0, \epsilon_0)$, be a C^r family of vector fields on M, and let $\mathscr{E}$ be a C^r submanifold of M consisting entirely of equilibrium points of X^0. Let k_1, k_2, and k_3 be fixed integers, and let $K \subset \mathscr{E}$ be a compact subset such that $QX^0(m)$ has k_1 eigenvalues in the left half plane, k_2 eigenvalues on the imaginary axis, and k_3 eigenvalues in the right half plane, for all $m \in K$. Then*

(i) *There is a C^r center-stable manifold $\mathscr{C}^s$ for $X^\epsilon \times \mathbf{0}$ near K. There is a C^r center-unstable manifold $\mathscr{C}^u$ for $X^\epsilon \times \mathbf{0}$ near K. There is a C^r center manifold $\mathscr{C}$ for $X^\epsilon \times \mathbf{0}$ near K. There is a neighborhood U of K such that $A^+(U) \subset \mathscr{C}^s$, $A^-(U) \subset \mathscr{C}^u$, and $I(U) \subset \mathscr{C}$.*

(ii) *There is a C^{r-1} family $\{\mathscr{F}^s(p): p \in \mathscr{C}^s\}$ of C^r stable manifolds for $\mathscr{C}^s$ near K. If $p \in M \times \{\epsilon\}$, then $\mathscr{F}^s(p) \subset M \times \{\epsilon\}$. Each manifold $\mathscr{F}^s(p)$ intersects $\mathscr{C}$ transversally, in exactly one point. There is a C^{r-1} family $\{\mathscr{F}^u(p): p \in \mathscr{C}^u\}$ of C^r unstable manifolds for $\mathscr{C}^u$ near K. If $p \in M \times \{\epsilon\}$, then $\mathscr{F}^u(p) \subset M \times \{\epsilon\}$. Each manifold $\mathscr{F}^u(p)$ intersects $\mathscr{C}$ transversally, in exactly one point.*

(iii) *Let $K_s < 0$ be larger than the real parts of the eigenvalues of $QX^0(m)$ in the left half plane, for all $m \in K$. Then there is a constant C_s such that if $p \in \mathscr{C}^s$ and $q \in \mathscr{F}^s(p)$, then*

$$d(p \cdot \tau, q \cdot \tau) \leqslant C_s e^{K_s \tau} d(p, q)$$

for all $\tau \geqslant 0$ such that $p \cdot [0, \tau] \subset \mathscr{C}^s$. If $p \in A^+(\mathscr{C}^s)$, then

$$\mathscr{F}^s(p) = \{q \in U: d(p \cdot \tau, q \cdot \tau) \leqslant C_s e^{K_s \tau} d(p, q) \text{ for all } \tau \geqslant 0\}.$$

Let $K_u > 0$ be smaller than the real parts of the eigenvalues of $QX^0(m)$ in the right half plane, for all $m \in K$. Then there is a constant C_u such that if $p \in \mathscr{C}^u$ and $q \in \mathscr{F}^u(p)$, then

$$d(p \cdot \tau, q \cdot \tau) \leqslant C_u e^{K_u \tau} d(p, q)$$

for all $\tau \leqslant 0$ such that $p \cdot [\tau, 0] \subset \mathscr{C}^u$. If $p \in A^-(U)$, then

$$\mathscr{F}^u(p) = \{q \in U: d(p \cdot \tau, q \cdot \tau) \leqslant C_u e^{K_u \tau} d(p, q) \text{ for all } \tau \leqslant 0\}.$$

(iv) *Let $\mathscr{D}^s$ be any C^r manifold which is invariant relative to U and tangent to $\mathscr{C}^s$ at a point $p_0 \in A^+(U)$. Then $\mathscr{C}^s$ and $\mathscr{D}^s$ have contact of order r at p_0. If $\{\mathscr{G}^s(p): p \in \mathscr{D}^s\}$ is any C^{r-1} locally positively invariant family of manifolds such that $\mathscr{F}^s(p_0)$ is tangent to $\mathscr{G}^s(p_0)$ at p_0, then $\mathscr{F}^{s*}$ and $\mathscr{G}^{s*}$ have contact of order $r-1$ at (p_0, p_0). Let $\mathscr{D}^u$ be any C^r manifold which is invariant relative to U and tangent to $\mathscr{C}^u$ at a point $p_0 \in A^-(U)$. Then $\mathscr{C}^u$ and $\mathscr{D}^u$ have contact of order r at p_0. If $\{\mathscr{G}^u(p): p \in \mathscr{D}^u\}$ is any C^r locally negatively invariant family of manifolds such that $\mathscr{F}^u(p_0)$ is tangent to $\mathscr{G}^u(p_0)$ at p_0, then $\mathscr{F}^{u*}$ and $\mathscr{G}^{u*}$ have contact of order $r-1$ at (p_0, p_0). Let $\mathscr{D}$ be any C^r manifold which is invariant relative to U and tangent to $\mathscr{C}$ at a point $p_0 \in I(U)$. Then $\mathscr{C}$ and $\mathscr{D}$ have contact of order r at p_0.*

(v) *If $K \subset \mathscr{E}_H$, define for $(m, \epsilon) \in \mathscr{C}$,*

$$\begin{aligned} X_{\mathscr{C}}(m, \epsilon) &= \epsilon^{-1} X^\epsilon(m) \times \{0\} \qquad &\text{if } \epsilon \neq 0 \\ &= X_R(m) \times \{0\} \qquad &\text{if } \epsilon = 0 \end{aligned}$$

Then $X_{\mathscr{C}}$ is a C^{r-1} vector field on $\mathscr{C}$ near $K \times \{0\}$.

Proof. The proof of Theorem 9.1 appears in Section XVIII.

Remarks. (i) In the simplest case, K contains just one point $m \in \mathscr{E}$. Then Theorem 9.1 describes the local behavior of the initial layer problem near m. In Section X we interpret this description in terms of local normal forms.

(ii) In the second simplest case K is a finite orbit segment of the reduced equation. In this case Theorem 9.1 includes and extends the results of Levin and Levinson (1954) and Levin (1956).

(iii) In the first really interesting case, $K \subset \mathscr{E}_H$ and K is a periodic orbit of the reduced vector field. We have discussed this case in Sections III and IV, and will return to it in Section XIII.

(iv) In case $K \subset \mathscr{E}_H$, Theorem 9.1.v reduces many singular perturbation problems to regular perturbation problems.

(v) In case M and $\mathscr{E}$ are C^∞, and X^ϵ is C^∞, the results of Theorem 9.1 hold for all r. Furthermore, the manifolds $\mathscr{F}^s(p)$ and $\mathscr{F}^u(p)$ are C^∞ manifolds, for each fixed p (see Fenichel, 1974, Theorem 1).

(vi) For C^r diffeomorphisms, for arbitrary r, Wan (1977) proves uniqueness of $\mathscr{C}$, $\mathscr{C}^s$, and $\mathscr{C}^u$ to order r at points of $K \times \{0\}$. The same idea works for flows.

If $r = \infty$, the manifolds $\mathscr{C}$, $\mathscr{C}^s$, and $\mathscr{C}^u$, and the families $\mathscr{F}^s$ and $\mathscr{F}^u$ may not be C^∞, even though Theorem 9.1 holds for all r, because one may construct different manifolds and families for different values of r. At points of $K \times \{0\}$, however, corresponding manifolds or families which are C^r have contact of order r, for any fixed r. This means that $\mathscr{C}$, $\mathscr{C}^s$, $\mathscr{C}^u$, $\mathscr{F}^s$, and $\mathscr{F}^u$ have unique asymptotic expansions at points of $K \times \{0\}$.

The functional equations used in the construction of $\mathscr{C}$, $\mathscr{C}^s$, $\mathscr{C}^u$, $\mathscr{F}^s$, and $\mathscr{F}^u$ have particularly simple forms at points of $K \times \{0\}$, because such points are equilibrium points if time is scaled appropriately. The functional equations lead directly to recursive computation schemes for the asymptotic expansions of $\mathscr{C}$, $\mathscr{C}^s$, $\mathscr{C}^u$, $\mathscr{F}^s$, and $\mathscr{F}^u$ in terms of the Taylor series of X^ϵ at points of $K \times \{0\}$. It is possible to use the functional equations to estimate the errors caused by truncation of the asymptotic expansions.

X. Local Singular Perturbation Theory: Partial Differential Equations and Asymptotic Expansions

In local coordinates the manifolds $\mathscr{C}$, $\mathscr{C}^s$, and $\mathscr{C}^u$ and the families $\mathscr{F}^s$ and $\mathscr{F}^u$ are given as graphs of functions. These functions satisfy partial differential equations which may be used to compute asymptotic expansions, as in Section IV. We remark that the asymptotic expansions at a point $m \in \mathscr{E}$ are independent of K, by Theorem 9.1.iv. Hence the computation of asymptotic expansions for $\mathscr{C}$, $\mathscr{C}^s$, $\mathscr{C}^u$, $\mathscr{F}^s$, and $\mathscr{F}^u$ belongs entirely to the local theory.

Assume the hypotheses of Theorem 9.1, and let $m \in K$ be given. We can choose local coordinates near m in which the flow of $X^\epsilon \times \mathbf{0}$ satisfies

$$\begin{aligned}
x' &= f(x, y_1, y_2, y_3, \epsilon) \\
y_1' &= g_1(x, y_1, y_2, y_3, \epsilon) \\
y_2' &= g_2(x, y\ \ , y_2, y_3, \epsilon) \\
y_3' &= g_3(x, y_1, y_2, y_3, \epsilon) \\
\epsilon' &= 0
\end{aligned} \tag{10.1}$$

with

$$\begin{aligned} f(0,0,0,0,0) &= 0 \\ g_i(0,0,0,0,0) &= 0, \qquad i = 1,2,3 \\ D_1 f(0,0,0,0,0) &= 0 \\ D_1 g_i(0,0,0,0,0) &= 0, \qquad i = 1,2,3 \end{aligned} \tag{10.2}$$

and

$$\left.\frac{\partial(g_1, g_2, g_3)}{\partial(y_1, y_2, y_3)}\right|_{\substack{x=0\\ y=0\\ \epsilon=0}} = \begin{vmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{vmatrix}, \tag{10.3}$$

as in (5.11), (5.12), and (5.13). The eigenvalues of A_1 lie in the left half plane, the eigenvalues of A_2 lie on the imaginary axis, and the eigenvalues of A_3 lie in the right half plane. The choice of such local coordinates depends only on linear algebra, and hence is computable.

In local coordinates the center manifold $\mathscr{C}$ is the graph of a pair of functions

$$\begin{aligned} y_1 &= u_1(x, y_2, \epsilon) \\ y_3 &= u_3(x, y_2, \epsilon). \end{aligned} \tag{10.4}$$

Because $\mathscr{C}$ is invariant, (10.4) is preserved under the flow of (10.1). Hence

$$\begin{aligned} y_1' &= D_1 u_1(x, y_2, \epsilon)x' + D_2 u_1(x, y_2, \epsilon)y_2' \\ y_3' &= D_1 u_3(x, y_2, \epsilon)x' + D_2 u_3(x, y_2, \epsilon)y_2'. \end{aligned} \tag{10.5}$$

We substitute (10.1) into (10.5), to get

$$\begin{aligned} g_1 &= D_1 u_1 f + D_2 u_1 g_2 \\ g_3 &= D_1 u_3 f + D_2 u_3 g_2, \end{aligned} \tag{10.6}$$

where the arguments of u_1 and u_3 are (x, y_2, ϵ) and the arguments of f, g_1, g_2, g_3 are $(x, u_1(x, y, \epsilon), y_2, u_3(x, y, \epsilon), \epsilon)$. Because $m \in \mathscr{E}$, $(m, 0) \in I(U) \subset \mathscr{C}$, and so

$$\begin{aligned} u_1(0,0,0) &= 0 \\ u_3(0,0,0) &= 0. \end{aligned} \tag{10.7}$$

To compute series expansions for $\mathscr{C}$ we differentiate (10.6) repeatedly with respect to all its arguments and set $(x, y_2, \epsilon) = (0, 0, 0)$. Then also $(y_1, y_3) = 0$ because of (10.7), so all the equations we derive depend only on the Taylor series of X^ϵ at m for $\epsilon = 0$. Unique solvability up to some order is guaranteed by Theorem 9.1.iv.

In local coordinates the center-stable manifold $\mathscr{C}^s$ is the graph of a function $y_3 = y_3(x, y_1, y_2, \epsilon)$ and the center-unstable manifold is the graph of a function $y_1 = y_1(x, y_2, y_3, \epsilon)$. These functions satisfy partial differential equations which may be used for series computations.

The equations for the invariant families $\mathscr{F}^s$ and $\mathscr{F}^u$ are more complicated than the equations for the invariant manifolds $\mathscr{C}$, $\mathscr{C}^s$ and $\mathscr{C}^u$. We consider $\mathscr{F}^s$ and omit the parallel discussion for $\mathscr{F}^u$. Each manifold $\mathscr{F}^s(p)$ intersects $\mathscr{C}$ in exactly one point. Hence we may parametrize the family in terms of local coordinates (ξ, η_2, ϵ) in $\mathscr{C}$, by means of (10.4). Each manifold $\mathscr{F}^s(p)$ is the graph of four functions

$$\begin{aligned} x &= v(\eta_1\,;\ \xi, \eta_2, \epsilon) \\ y_1 &= \eta_1 + u_1(\epsilon, \eta_2, \epsilon) \\ y_2 &= v_2(\eta_1\,;\ \xi, \eta_2, \epsilon) \\ y_3 &= v_3(\eta_1\,;\ \xi, \eta_2, \epsilon), \end{aligned} \tag{10.8}$$

satisfying

$$\begin{aligned} v(0;\ \xi, \eta_2, \epsilon) &= \xi \\ v_2(0;\ \xi, \eta_2, \epsilon) &= \eta_2 \\ v_3(0;\ \xi, \eta_2, \epsilon) &= u_3(\xi, \eta_2, \epsilon). \end{aligned} \tag{10.9}$$

The second equation of (10.8) simply means that η_1 measures the deviation of the y_1-component from its value on $\mathscr{C}$. Repeated differentiation of (10.8) and (10.9) at (0, 0, 0, 0) leads to equations relating the series expansions of v, v_2, v_3 and the Taylor series of X^ϵ at m for $\epsilon = 0$. Unique solvability up to some order is guaranteed by Theorem 9.1.iv.

Note that ξ and η_2 evolve according to (10.1), the flow of X^ϵ on $\mathscr{C}$.

We remark that the partial differential equations derived in this section can, in principle, be used to estimate the errors caused by truncation of the series expansions.

XI. Local Singular Perturbation Theory: Normal Forms

In this section we derive a local normal form for X^ϵ near a point $m_0 \in K$. We assume for simplicity that M and X^ϵ are C^∞. Smooth means C^r, where $r < \infty$ is arbitrary. The reader can derive precise smoothness conditions from Theorem 9.1.

THEOREM 11.1. *Assume the hypotheses of Theorem* 9.1, *and let* $m_0 \in K$. *Then there is a smooth* ϵ*-dependent local coordinate system* $(x, y_1, y_2, y_3) = \phi(m, \epsilon)$

near m_0, *with* $(0, 0, 0, 0) = \phi(m_0, \epsilon)$ *such that in* (x, y_1, y_2, y_3)*-coordinates the flow of* X^ϵ *satisfies*

$$\begin{aligned} x' &= f(x, y_1, y_2, y_3, \epsilon) \\ y_1' &= g_1(x, y_1, y_2, y_3, \epsilon) \\ y_2' &= g_2(x, y_1, y_2, y_3, \epsilon) \\ y_3' &= g_3(x, y_1, y_2, y_3, \epsilon) \end{aligned} \tag{11.1$_\epsilon$}$$

subject to the conditions

$$\begin{aligned} f(x, 0, 0, 0, 0) &= 0 \\ g_1(x, 0, 0, 0, 0) &= 0 \\ g_2(x, 0, 0, 0, 0) &= 0 \\ g_3(x, 0, 0, 0, 0) &= 0; \end{aligned} \tag{11.2}$$

$$\frac{\partial(g_1, g_2, g_3)}{\partial(y_1, y_2, y_3)}(x, 0, 0, 0, 0) = \begin{vmatrix} A_1(x) & 0 & 0 \\ 0 & A_2(x) & 0 \\ 0 & 0 & A_3(x) \end{vmatrix} \tag{11.3}$$

with the eigenvalues of $A_1(0)$ *in the right half plane, the eigenvalues of* $A_2(0)$ *on the imaginary axis, and the eigenvalues of* $A_3(0)$ *in the left half plane*;

$$g_1(x, 0, y_2, y_3, \epsilon) = 0; \tag{11.4}$$

$$g_3(x, y_1, y_2, 0, \epsilon) = 0; \tag{11.5}$$

$$\begin{aligned} D_2 f(x, y_1, y_2, 0, \epsilon) &= 0 \\ D_2 g(x, y_1, y_2, 0, \epsilon) &= 0; \end{aligned} \tag{11.6}$$

$$\begin{aligned} D_4 f(x, 0, y_2, y_3, \epsilon) &= 0 \\ D_4 g(x, 0, y_2, y_3, \epsilon) &= 0. \end{aligned} \tag{11.7}$$

If $m_0 \in \mathscr{E}_R$, *the normal form* (11.1) *also satisfies*

$$D_3 f(x, 0, 0, 0, 0) = 0. \tag{11.8}$$

Proof. We construct the normal form by means of a sequence of coordinate transformations. First, let x be a local coordinate on $\mathscr{E}$, with $x = 0$ at m_0, and let y be a coordinate normal to $\mathscr{E}$. Then (11.2) expresses the assumption that $\mathscr{E}$ consists entirely of equilibrium points of X^0. Next, decompose y using the invariant subspaces of the matrix

$$\frac{\partial(g_1, g_2, g_3)}{\partial(y_1, y_2, y_3)}(x, 0, 0, 0, 0),$$

to achieve (11.3). If $m_0 \in \mathscr{E}_R$, we also require

$$D_2 f(x, 0, 0, 0, 0) = 0$$
$$D_3 f(x, 0, 0, 0, 0) = 0$$

and

$$D_4 f(x, 0, 0, 0, 0) = 0.$$

To achieve (11.4) and (11.5) we transform the coordinates so that $\mathscr{C}^u$ is the (x, y_2, y_3, ϵ)-plane and $\mathscr{C}^s$ is the (x, y_1, y_2, ϵ)-plane. To achieve (11.6) and (11.7) we transform coordinates so that the manifolds $\mathscr{F}^s$ are planes parallel to the y_1-axis and the manifolds $\mathscr{F}^u$ are planes parallel to the y_3-axis. The condition (11.6) means that in $\mathscr{C}^s$ the evolutions of x and y are independent of y_1. This is the invariance condition for $\mathscr{F}^s$. Similarly, (11.7) means that in $\mathscr{C}^u$ the evolutions of x and y_2 are independent of y_3. This is the invariance condition for $\mathscr{F}^u$.

Remark. The transformations required for the proof of Theorem 11.1 may be expressed explicitly in terms of the functions whose graphs specify $\mathscr{C}$, $\mathscr{C}^s$, $\mathscr{C}^u$, $\mathscr{F}^s$, and $\mathscr{F}^u$. Hence we can compute these transformations using the techniques of Section X.

XII. Local Theory Near an Equilibrium Point of the Reduced Vector Field

As an application of the results of Sections IX–X we now prove two easy theorems about singular perturbations of systems with equilibrium points.

THEOREM 12.1. *Let M be a C^{r+1} manifold, $2 \leqslant r < \infty$. Let X^ϵ, $\epsilon \in (-\epsilon_0, \epsilon_0)$, be a C^r family of vector fields on M, and let $\mathscr{E}$ be a C^r submanifold of M consisting entirely of equilibrium points of X^0. Let $m \in \mathscr{E}_H$ be an equilibrium point of the reduced vector field X_R. Suppose that 1 is not an eigenvalue of $TX_R(m)$. Then there exists $\epsilon_1 > 0$ and there exists a C^{r-1} family of points m_ϵ, $\epsilon \in (-\epsilon_1, \epsilon_1)$, such that $m_0 = m$ and m_ϵ is an equilibrium point of X^ϵ.*

Proof. Choose an (x, y) coordinate system centered at m, as in Theorem 10.1. Then y_2 is absent, so $y = (y_1, y_3)$. The flow of X^ϵ satisfies

$$\begin{aligned} x' &= f(x, y_1, y_3, \epsilon) \\ y_1' &= g_1(x, y_1, y_3, \epsilon) \\ y_3' &= g_3(x, y_1, y_3, \epsilon) \end{aligned}$$

with

$$\begin{aligned} g_1(x, 0, \epsilon) &= 0 \\ g_3(x, 0, \epsilon) &= 0. \end{aligned}$$

The manifold $y = 0$ is invariant, and on $y = 0$ we have

$$f(x, 0, 0, \epsilon) = \epsilon f_0(x, 0, 0, \epsilon),$$

where $f_0(x, 0, 0, \epsilon)$ is C^{r-1}. The theorem follows from the regular perturbation theorem for an equilibrium point of

$$\dot{x} = f_0(x, 0, 0, \epsilon).$$

In the following theorem the invariant sets A^-, A^+, and I are defined using the flow of $\epsilon^{-1}X^\epsilon$, and $\cdot t$ is the solution operator of $\epsilon^{-1}X^\epsilon$.

THEOREM 12.2. *Under the hypotheses of Theorem* 12.1, *suppose* $TX_R(m)$ *has* j_1 *eigenvalues in the right half plane, no eigenvalues on the imaginary axis, and* j_3 *eigenvalues in the left half plane. Suppose* $QX^0(m)$ *has* k_1 *eigenvalues in the left half plane and* k_3 *eigenvalues in the right half plane. Then there exists* $\epsilon_1 > 0$ *such that*:

(i) *There is a* C^{r-1} *family of points* $\{m_\epsilon : \epsilon \in (-\epsilon_1, \epsilon_1)\}$ *such that* $m_0 = m$ *and for* $\epsilon \neq 0$, m_ϵ *is a hyperbolic equilibrium point of* X^ϵ. *There is a family of neighborhoods of* m, $\{U_\epsilon, \epsilon \in (-\epsilon_1, \epsilon_1)\}$, *such that for* $\epsilon \neq 0$, $I(U_\epsilon) = \{m_\epsilon\}$.

(ii) *There are* C^{r-1} *families of* $(j_1 + k_1)$*-dimensional and* $(j_3 + k_3)$*-dimensional manifolds* $\{W_\epsilon^s : \epsilon \in (-\epsilon_1, \epsilon_1)\}$ *and* $\{W_\epsilon^u : \epsilon \in (-\epsilon_1, \epsilon_1)\}$ *such that for* $\epsilon > 0$, $A^+(U_\epsilon) = W_\epsilon^s$ *and* $A^-(U_\epsilon) = W_\epsilon^u$.

(iii) *There is a* C^{r-1} *family of* k_1*-dimensional manifolds* $\{\mathscr{F}_\epsilon^s(p) : \epsilon \in (-\epsilon_1, \epsilon_1)$, $p \in W_\epsilon^s\}$ *such that for each* $\epsilon > 0$, $\{\mathscr{F}_\epsilon^s(p) : p \in W_\epsilon^s\}$ *is a positively invariant family of manifolds. There are constants* $C_s > 0$ *and* $K_s < 0$ *such that for each* $\epsilon \in (0, \epsilon_1)$ *and* $p \in W_\epsilon^s$, $\mathscr{F}_\epsilon^s(p)$ *is uniquely characterized by*

$$\mathscr{F}_\epsilon^s(p) = \{q \in U_\epsilon : d(q \cdot t, p \cdot t) \leqslant C_s e^{K_s t/\epsilon} d(q, p) \text{ for all } t \geqslant 0\}.$$

There is a C^{r-1} *family of* k_3*-dimensional manifolds* $\{\mathscr{F}_\epsilon^u(p) : \epsilon \in (-\epsilon_1, \epsilon_1)$, $p \in W_\epsilon^u\}$ *such that for each* $\epsilon > 0$, $\{\mathscr{F}_\epsilon^u(p) : p \in W_\epsilon^u\}$ *is a negatively invariant family of manifolds. There are constants* $C_u > 0$ *and* $K_u > 0$ *such that for each* $\epsilon \in (0, \epsilon_1)$ *and* $p \in W_\epsilon^u$, $\mathscr{F}_\epsilon^u(p)$ *is uniquely characterized by*

$$\mathscr{F}_\epsilon^u(p) = \{q \in U_\epsilon : d(q \cdot t, p \cdot t) \leqslant C_u e^{K_u t/\epsilon} d(q, p) \text{ for all } t \leqslant 0\}.$$

Remarks. (i) For $\epsilon > 0$ the manifolds W_ϵ^s and W_ϵ^u are local stable and unstable manifolds of m_ϵ.

(ii) For $\epsilon > 0$ the equilibrium point, the invariant manifolds, and the invariant families of manifolds all are uniquely determined by the asymptotic properties of the flow of $\epsilon^{-1}X^\epsilon$ as $t \to \pm\infty$ for fixed ϵ.

(iii) Because the equilibrium point, the invariant manifolds, and the invariant families of manifolds are embedded in families which are smooth at $\epsilon = 0$, we can compute their asymptotic expansions in powers of ϵ.

Proof of Theorem 12.2. Choose special coordinates as in the proof of Theorem 12.1. The flow of $\dot{x} = f(x_0, 0, 0, \epsilon)$ has a hyperbolic equilibrium point with a j_1-dimensional stable manifold Γ_ϵ^s and a j_3-dimensional unstable manifold Γ_ϵ^u. Then W_ϵ^s is represented in (x, y)-coordinates as $\{(x, y_1, y_3): x \in \Gamma_\epsilon^s, y_3 = 0\}$, and W_ϵ^u is represented in (x, y)-coordinates as $\{(x, y_1, y_3): x \in \Gamma_\epsilon^u, y_1 = 0\}$. The families $\mathscr{F}_\epsilon^s$ and $\mathscr{F}_\epsilon^u$ are represented in (x, y)-coordinates by holding $x \in \Gamma_\epsilon^s$ constant for $y_3 = 0$ and by holding $x \in \Gamma_\epsilon^u$ constant for $y_1 = 0$. The estimates in Theorem 10.2.iii follow from Theorem 9.1, or from an easy Lyapunov function argument using the normal form.

XIII. Global Theory Near a Periodic Orbit of the Reduced Vector Field

We have discussed periodic orbits in Sections III–IV. We now use Theorem 9.1 to make these results precise.

Theorem 13.1. *Let M be a C^{r+1} manifold, $2 \leqslant r < \infty$. Let X^ϵ, $\epsilon \in (-\epsilon_0, \epsilon_0)$, be a C^r family of vector fields, and let $\mathscr{E}$ be a C^r submanifold of M consisting entirely of equilibrium points of X^0. Let $\gamma \in \mathscr{E}_H$ be a periodic orbit of the reduced vector field X_R, and suppose that γ_0, as a periodic orbit of X_R, has 1 as a Floquet multiplier of multiplicity precisely one. Then there exists $\epsilon_1 > 0$ and there exists a C^{r-1} family of closed curves γ_ϵ, $\epsilon \in (-\epsilon_1, \epsilon_1)$, such that $\gamma_0 = \gamma$ and ϵ is a periodic orbit of $\epsilon^{-1}X^\epsilon$. The period of γ_ϵ is a C^{r-1} function of ϵ.*

Proof. Construct $\mathscr{C}$ as in Theorem 9.1. The vector field $X_{\mathscr{C}}$ is C^{r-1}, and Theorem 12.1 follows from Poincaré continuation on $\mathscr{C}$.

In the following theorem the invariant sets A^-, A^+, and I are defined using the flow of $\epsilon^{-1}X^\epsilon$, and $\cdot t$ denotes the solution operator of $\epsilon^{-1}X^\epsilon$.

Theorem 13.2. *Under the hypotheses of Theorem* 13.1, *suppose γ, as a periodic orbit of X_R, has j_1 Floquet multipliers inside the unit circle, no nontrivial Floquet multipliers on the unit circle, and j_3 Floquet multipliers outside the unit circle. Suppose, for each $m \in \gamma$, that $QX^0(m)$ has k_1 eigenvalues in the left half plane, no eigenvalues on the imaginary axis, and k_3 eigenvalues in the right half plane. Then there exists $\epsilon_1 > 0$ such that*:

(i) *There is a C^{r-1} family of closed curves $\{\gamma_\epsilon, \epsilon \in (-\epsilon_1, \epsilon_1)\}$ in M such that $\gamma_0 = \gamma$ and for $\epsilon \neq 0$, γ_ϵ is a periodic orbit of $\epsilon^{-1}X^\epsilon$. The period of γ_ϵ is a C^{r-1} function of ϵ. There is a family of neighborhoods $\{U_\epsilon: \epsilon \in (-\epsilon_1, \epsilon_1)\}$ such that for $\epsilon \neq 0$, $I(U_\epsilon) = \gamma_\epsilon$.*

(ii) *For $\epsilon > 0$, γ_ϵ has $j_1 + k_1$ Floquet multipliers inside the unit circle and $j_3 + k_3$ Floquet multipliers outside the unit circle.*

(iii) *There are C^{r-1} families of manifolds in M, $\{W_\epsilon^s\colon \epsilon \in (-\epsilon_1, \epsilon_1)\}$ and $\{W_\epsilon^u\colon \epsilon \in (-\epsilon_1, \epsilon_1)\}$, such that $W_\epsilon^s \cap W_\epsilon^u = \gamma_\epsilon$. For each ϵ, W_ϵ^s is $(j_1 + k_1 + 1)$-dimensional and W_ϵ^u is $(j_3 + k_3 + 1)$-dimensional. For $\epsilon > 0$, $A^+(U_\epsilon) = W_\epsilon^s$ and $A^-(U_\epsilon) = W_\epsilon^u$; W_ϵ^s is a local stable manifold of γ_ϵ and W_ϵ^u is a local unstable manifold of γ_ϵ.*

(iv) *There are constants $K_s < 0$ and $K_u > 0$, and there are C^{r-1} families of manifolds $\{\mathscr{F}_\epsilon^s(m)\colon \epsilon \in (-\epsilon_1, \epsilon_1),\ m \in W_\epsilon^s\}$ and $\{\mathscr{F}_\epsilon^u(m)\colon \epsilon \in (-\epsilon_1, \epsilon_1),\ m \in W_\epsilon^u\}$, characterized for $\epsilon > 0$ by*

$$\mathscr{F}_\epsilon^s(m) = \{\hat{m} \in A^+(U_\epsilon)\colon e^{-K_s t/\epsilon} d(m \cdot t, \hat{m} \cdot t) \to 0 \text{ as } t \to \infty\}$$

and

$$\mathscr{F}_\epsilon^u(m) = \{\hat{m} \in A^-(U_\epsilon)\colon e^{-K_u t/\epsilon} d(m \cdot t, \hat{m} \cdot t) \to 0 \text{ as } t \to -\infty\}.$$

Each manifold $\mathscr{F}_\epsilon^s$ has dimension k_1, and each manifold $\mathscr{F}_\epsilon^u$ has dimension k_3.

(v) *There are C^{r-2} families of manifolds in M, $\{\Phi_\epsilon^s(m)\colon \epsilon \in (-\epsilon_1, \epsilon_1),\ m \in W_\epsilon^s\}$ and $\{\Phi_\epsilon^u(m)\colon \epsilon \in (-\epsilon_1, \epsilon_1),\ m \in W_\epsilon^u\}$, characterized for $\epsilon > 0$ by*

$$\Phi_\epsilon^s(m) = \{\hat{m} \in A^+(U_\epsilon)\colon d(m \cdot t, \hat{m} \cdot t) \to 0 \text{ as } t \to \infty\}$$

and

$$\Phi_\epsilon^u(m) = \{\hat{m} \in A^-(U_\epsilon)\colon d(m \cdot t, \hat{m} \cdot t) \to 0 \text{ as } t \to -\infty\}.$$

Each manifold $\mathscr{F}_\epsilon^s(m)$ has dimension $j_1 + k_1$, and each manifold $\Phi_\epsilon^u(m)$ has dimension $j_3 + k_3$. Each manifold $\Phi_\epsilon^s(m)$ or $\Phi_\epsilon^u(m)$ intersects γ_ϵ transversally in exactly one point.

Proof. Construct $\mathscr{C}$, $\mathscr{C}^s$, $\mathscr{C}^u$, $\{\mathscr{F}^s(p)\colon p \in \mathscr{C}^s\}$, and $\{\mathscr{F}^u(p)\colon p \in \mathscr{C}^u\}$ as in Theorem 9.1. Let $\gamma_\epsilon \times \{\epsilon\}$ be a closed orbit of $\epsilon^{-1}X^\epsilon \times \mathbf{0}$ in $\mathscr{C} \cap M \times \{\epsilon\}$, constructed by Poincaré continuation as in Theorem 12.1. Let $\Gamma^s(\epsilon)$ and $\Gamma^u(\epsilon)$ be the local stable manifold and the local unstable manifold of $\gamma_\epsilon \times \{\epsilon\}$ in $\mathscr{C}$. These are characterized by

$$\Gamma^s(\epsilon) = \{p \in \mathscr{C}\colon d(p \cdot t, \gamma_\epsilon \times \{\epsilon\}) \to 0 \text{ as } t \to \infty\}$$

$$\Gamma^u(\epsilon) = \{p \in \mathscr{C}\colon d(p \cdot t, \gamma_\epsilon \times \{\epsilon\}) \to 0 \text{ as } t \to -\infty\}.$$

By regular perturbation theory it is known that $\Gamma^s(\epsilon)$ and $\Gamma^u(\epsilon)$ are C^{r-1} functions of ϵ, that $\Gamma^s(\epsilon)$ is a $(j_1 + 1)$-dimensional manifold and that $\Gamma^u(\epsilon)$ is a $(j_3 + 1)$-dimensional manifold. Because ϵ is constant on orbits, $\Gamma^s(\epsilon) \subset M \times \{\epsilon\}$, and $\Gamma^u(\epsilon) \subset M \times \{\epsilon\}$. Define

$$W_\epsilon^s = \left\{\hat{m} \in M\colon (\hat{m}, \epsilon) \in \bigcup \{\mathscr{F}^s(p)\colon p \in \Gamma^s(\epsilon)\}\right\}$$

$$W_\epsilon^u = \left\{\hat{m} \in M\colon (\hat{m}, \epsilon) \in \bigcup \{\mathscr{F}^u(p)\colon p \in \Gamma^u(\epsilon)\}\right\}.$$

It is easy to verify from Theorem 9.1 that $W_\epsilon{}^s$ and $W_\epsilon{}^u$ satisfy the properties listed in (iii), and then (ii) follows immediately. Define, for $m \in W_\epsilon{}^s$,

$$\mathscr{F}_\epsilon{}^s(m) = \{\hat{m} \in M: (\hat{m}, \epsilon) \in \mathscr{F}^s(m, \epsilon)\}.$$

Define, for $m \in W_\epsilon{}^u$,

$$\mathscr{F}_\epsilon{}^u(m) = \{\hat{m} \in M: (\hat{m}, \epsilon) \in \mathscr{F}^u(m, \epsilon)\}.$$

The manifolds $\mathscr{F}_\epsilon{}^s(m)$ and $\mathscr{F}_\epsilon{}^u(m)$ are just projections onto M of the manifolds $\mathscr{F}^s(m, \epsilon)$ and $\mathscr{F}^u(m, \epsilon)$, so the properties listed in (iv) follow immediately from Theorem 9.1.

Define, for $m \in \gamma_\epsilon$,

$$\Delta^s(m, \epsilon) = \{p \in \Gamma^s(\epsilon): d(p \cdot t, (m, \epsilon) \cdot t) \to 0 \text{ as } t \to \infty\}$$
$$\Delta^u(m, \epsilon) = \{p \in \Gamma^s(\epsilon): d(p \cdot t, (m, \epsilon) \cdot t) \to 0 \text{ as } t \to -\infty\}.$$

These sets are defined by asymptotic phase conditions for the regular vector field $X_{\mathscr{C}}$. Existence and smoothness follow from the regular theory (see Hale, 1969, p. 217; and Fenichel, 1977, Theorem 5). Define, for $m \in \gamma_\epsilon$,

$$\Phi_\epsilon{}^s(m) = \left\{\hat{m} \in M: (\hat{m}, \epsilon) \in \bigcup \{\mathscr{F}^s(p): p \in \Delta^s(m, \epsilon)\}\right\}$$
$$\Phi_\epsilon{}^u(m) = \left\{\hat{m} \in M: (\hat{m}, \epsilon) \in \bigcup \{\mathscr{F}^u(p): p \in \Delta^u(m, \epsilon)\}\right\}.$$

The properties listed in (v) follow immediately from this definition.

XIV. Outer Solutions and Inner Corrections

An important topic in singular perturbation theory is the computation of outer solutions and inner corrections. The outer solutions are families of solutions which are smooth to some order at $\epsilon = 0$, but do not have enough free parameters to satisfy arbitrary initial conditions. The inner corrections are error terms which decay transcendentally as $\epsilon \to 0+$ for fixed t. Addition of the inner corrections takes account of the initial layers, and provides enough free parameters to allow open sets of initial data. Outer solutions and inner corrections usually are constructed formally using asymptotic expansions, as in Hoppensteadt (1971).

Theorem 9.1 gives a geometric construction of outer solutions and inner corrections, in case $k_2 = 0$. We say that $p \cdot t$ is an outer solution for $\epsilon^{-1}X^\epsilon \times 0$ if $p \in \mathscr{C}$. We say that two solutions $p \cdot t$ and $q \cdot t$ differ by an inner correction as $t \to \infty$ (as $t \to -\infty$) if $q \in \mathscr{F}^s(p)$ $(q \in \mathscr{F}^u(p))$. Theorem 9.1.v shows that the outer solutions depend smoothly on ϵ, and Theorem 9.1.iii shows that the inner corrections are transcendentally small as $\epsilon \to 0+$.

Theorem 9.1.iv is a uniqueness result for outer solutions and inner corrections, at points of $A^+(U)$, $A^-(U)$, and $I(U)$. Uniqueness, rather than uniqueness to finite or infinite order, requires control of the flow of X^ϵ as $t \to \pm\infty$ for fixed ϵ. Computation of asymptotic expansions which are uniformly valid over infinite time intervals is extremely difficult, so our uniqueness results generally lie outside the scope of asymptotic series methods.

Theorem 9.1 leads to computations of asymptotic expansions for outer solutions and inner corrections (see Sections IV and X).

XV. Exchange of Stability

The purpose of this section is to fit the exchange of stability problem treated by Lebovitz and Schaar (1975) into the framework we have developed. In this problem $\mathscr{E} - \mathscr{E}_R$ is a codimension one submanifold of $\mathscr{E}$. We show that it is possible to define the reduced field even in $\mathscr{E} - \mathscr{E}_R$, with the loss of one derivative.

Let X^ϵ, $\epsilon \in (-\epsilon_0, \epsilon_0)$, be a C^r family of vector fields on a C^{r+1} manifold M. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be μ-dimensional submanifolds of M consisting entirely of equilibrium points of X^0. Suppose that $\mathscr{E}_1 \cap \mathscr{E}_2$ is a $(\mu - 1)$-dimensional manifold, and that $\mathscr{E}_1$ and $\mathscr{E}_2$ intersect nontangentially. Suppose also that as a point moves across $\mathscr{E}_1 \cap \mathscr{E}_2$ in $\mathscr{E}_1$ or in $\mathscr{E}_2$, one of the nontrivial eigenvalues of TX^0 moves across zero while the other nontrivial eigenvalues of TX^0 remain in the left half plane. Under these assumptions and under certain additional assumptions, Lebovitz and Schaar showed that there are trajectories of X^ϵ, for small ϵ, which move toward $\mathscr{E}_1 \cap \mathscr{E}_2$ near the stable part of $\mathscr{E}_1$ and then move away from $\mathscr{E}_1 \cap \mathscr{E}_2$ near the stable part of $\mathscr{E}_2$. In this case we say that an exchange of stability has occurred (see Fig. 2).

We consider the exchange of stability problem near a point $m_0 \in \mathscr{E}_1 \cap \mathscr{E}_2$. Theorem 9.1 is applicable, with $\mathscr{E} = \mathscr{E}_1$ and $k_1 = \dim M - \mu - 1$, $k_2 = 1$, $k_3 = 0$. There is a center manifold $\mathscr{C}$, and all orbits near $m_0 \times \{0\}$ behave like orbits in $\mathscr{C}$, up to transcendentally small errors. Hence we may restrict attention to the flow in $\mathscr{C}$. By passing to the normal form of Theorem 11.1, we see that to study the local exchange of stability problem in a manifold of arbitrary dimension it is sufficient to study this problem near the origin in $R^{\mu+1}$.

We choose special coordinates x_1, $\xi = (x_2, \ldots, x_\mu)$, and y near the origin in $R^{\mu+1}$, such that $\mathscr{E}_1$ is the (x_1, ξ)-plane and $\mathscr{E}_2$ is the (ξ, y) plane. In these coordinates the flow of X^ϵ satisfies a system of the form

$$\begin{aligned} \dot{x} &= f(x, \xi, y, \epsilon) \\ \dot{\xi} &= F(x, \xi, y, \epsilon) \\ \dot{y} &= g(x, \xi, y, \epsilon). \end{aligned} \tag{15.1}$$

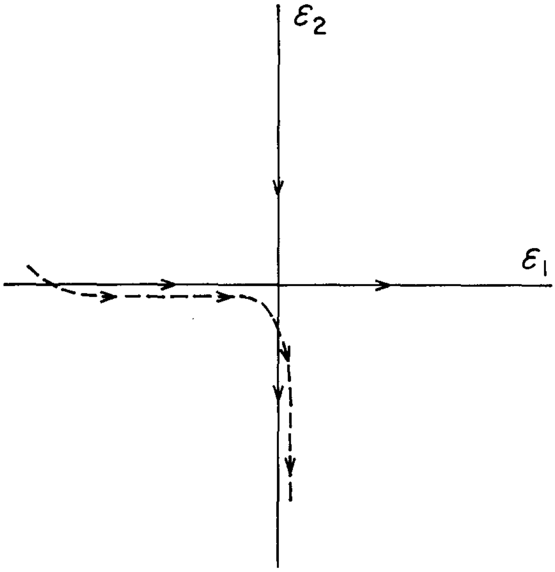

FIGURE 2

Because X^0 vanishes on $x_1 = 0$ and on $y = 0$, all terms in the Taylor series of f, F, and g for $\epsilon = 0$ are divisible by $x_1 y$. It follows that (15.1) takes the form

$$\begin{aligned} \dot{x}_1 &= ax_1 y + c\epsilon + \cdots \\ \dot{\xi} &= Ax_1 y + C\epsilon + \cdots \\ \dot{y} &= bx_1 y + d\epsilon + \cdots. \end{aligned} \tag{15.2}$$

We assume that a and b are nonzero. This is a nondegeneracy condition on the eigenvalues of TX^0 near $\mathscr{E}_1 \cap \mathscr{E}_2$.

We now apply Lemma 5.4 to compute the projection $\pi^{\mathscr{E}_1}$ and the reduced flow in $\mathscr{E}_1$. We note that $u \equiv 0$, so $\kappa = 0$, and $\pi^{\mathscr{E}_1}$ is multiplication by the matrix

$$\begin{vmatrix} I & D_3\begin{pmatrix} f \\ F \end{pmatrix}(x_1, \xi, 0, 0)\, D_3 g(x_1, \xi, 0, 0)^{-1} \\ 0 & 0 \end{vmatrix}.$$

As $x_1 \to 0$, both factors in the upper right hand corner of this matrix vanish to first order in x_1, so the projection is C^{r-2} at $x_1 = 0$. At the origin $\pi^{\mathscr{E}_1}$ is multiplication by the matrix

$$\begin{pmatrix} I & \begin{pmatrix} a/b \\ A/b \end{pmatrix} \\ 0 & 0 \end{pmatrix},$$

and the reduced system is

$$\begin{aligned} \dot{x}_1 &= c - (a/b)d + \cdots \\ \dot{\xi} &= C - (A/b)d + \cdots \\ \dot{y} &= 0. \end{aligned} \tag{15.3}$$

The natural nondegeneracy condition $\dot{x}_1 \neq 0$ reduces to

$$ad - bc \neq 0. \tag{15.4}$$

This condition is symmetric in x_1 and y, so it also is the natural nondegeneracy condition for the reduced flow in $\mathscr{E}_2$. An easy computation shows that (15.4) is just the first condition in hypothesis $H1$ of Lebovitz and Schaar (1975).

XVI. Invariant Manifold Theory

In this section we recall some definitions and results of invariant manifold theory from Fenichel (1971, 1974, 1977). We also indicate some straightforward extensions to parameter-dependent families of manifolds. Throughout this section M is a C^{r+1} manifold, where r is a positive integer, X is a C^r vector field on M with flow F^τ, and $\bar{\Lambda} = \Lambda \cup \partial\Lambda \subset M$ is a C^r compact submanifold with boundary, locally invariant under X. We choose a Riemannian metric for TM, and let $|\ |$ denote the norm associated with the metric.

The results we recall may be classed as perturbation theorems and invariant family theorems. The perturbation theorems are based on a construction which requires that $\bar{\Lambda}$ be overflowing invariant and asymptotically stable in a sense we make precise below. The invariant family theorems are based on a construction which requires that $\bar{\Lambda}$ be overflowing invariant under the flow of X and asymptotically stable under the flow of $-X$. We are able to use these constructions together precisely because center manifolds are defined by neutral growth conditions.

We begin by recalling the perturbation theorem for overflowing invariant manifolds. Assume $\bar{\Lambda}$ is overflowing invariant. Choose any complement N of $T\Lambda$ in $TM \mid \bar{\Lambda}$, and let π^N denote the projection on N corresponding to the splitting $TM \mid \bar{\Lambda} = T\Lambda \oplus N$. For any $m \in \bar{\Lambda}$, $v^0 \in T_m\Lambda$, $w^0 \in N_m$, and $\tau \leqslant 0$, let

$$v^\tau = DF^\tau(m)\, v^0$$

and

$$w^\tau = \pi^N DF^\tau(m)\, w^0.$$

Define

$$\nu^*(m) = \inf\{\nu > 0 \colon \nu^\tau / |\, w^\tau\,| \to 0 \text{ as } \tau \to -\infty \text{ for all } w^0 \in N_m\}.$$

If $\nu^*(m) < 1$ define

$$\sigma^*(m) = \inf\{\sigma\colon |v^\tau|/|w^\tau|^\sigma \to 0 \text{ as } \tau \to -\infty \text{ for all } v^0 \in T_m\Lambda,\ w^0 \in N_m\}.$$

The functions ν^* and σ^*, as well as the functions λ^*, α^*, ρ_1^*, ρ_2^*, τ_1^*, and τ_2^* defined below, are called generalized Lyapunov type numbers. The functions ν^* and σ^* are independent of the choice of the metric and the choice of N. The condition $\nu^* < 1$ is an asymptotic stability condition for $\bar{\Lambda}$ under the linearized flow of X.

The perturbation theorem for overflowing invariant manifolds, Fenichel (1971, Theorem 1) asserts that if $\nu^*(m) < 1$ and $\sigma^*(m) < 1/r$ for all $m \in \bar{\Lambda}$, then for any C^r vector field Y, C^1-close to X, there is a C^r manifold with boundary $\bar{\Lambda}_Y$, C^1-close to $\bar{\Lambda}$, such that $\bar{\Lambda}_Y$ is overflowing invariant under Y. The manifold $\bar{\Lambda}_Y$ is constructed by means of a contraction mapping. It follows from the uniqueness of the fixed point of the contraction mapping that there is a neighborhood U of Λ_Y such that $\Lambda_Y = A^-(U)$, the maximal subset of U which is negatively invariant under Y. We remark that the perturbation theorem is valid also for manifolds with corners provided X points strictly outward on all the smooth surfaces of $\partial\Lambda$.

The unstable manifold theorem for overflowing invariant manifolds is a variant of the perturbation theorem. Instead of starting with an overflowing invariant manifold, we start with its tangent space and then prove that the manifold exists. The following is an improved statement of Fenichel (1971, Theorem 4). Let $\bar{\Lambda}$ be overflowing invariant under X. Let $N^u \subset TM \mid \bar{\Lambda}$ be a negatively invariant subbundle containing $T\Lambda$. Let $I \subset N^u$ be any complement of $T\Lambda$, and let $J \subset TM \mid \bar{\Lambda}$ be any complement of N^u. Then $TM \mid \bar{\Lambda}$ splits as $T\Lambda \oplus I \oplus J$. Let π^I and π^J be the projections on I and J corresponding to this splitting. For any $m \in \bar{\Lambda}$, $v^0 \in T_m\Lambda$, $w^0 \in I_m$, $x^0 \in J_m$, and $\tau \leqslant 0$, let

$$\begin{aligned} v^\tau &= DF^\tau(m)\, v^0 \\ w^\tau &= \pi^I DF^\tau(m)\, w^0 \\ x^\tau &= \pi^J DF^\tau(m)\, x^0. \end{aligned}$$

Define

$$\lambda^*(m) = \inf\{\lambda > 0\colon |x^\tau|\,\lambda^\tau \to 0 \text{ as } \tau \to -\infty \text{ for all } x^0 \in I_m\}$$

and

$$\nu^*(m) = \inf\{\nu > 0\colon \nu^\tau/|w^\tau| \to 0 \text{ as } \tau \to -\infty \text{ for all } w^0 \in J_m\}.$$

If $\nu^*(m) < 1$ define

$$\sigma^*(m) = \inf\{\sigma\colon |v^\tau|/|w^\tau|^\sigma \to 0 \text{ as } \tau \to -\infty \text{ for all } v^0 \in T_m\Lambda,\ w^0 \in J_m\}.$$

The unstable manifold theorem for overflowing invariant manifolds asserts that if $\lambda^*(m) < 1$, $\nu^*(m) < 1$, and $\sigma^*(m) < 1/r$, then there is a C^r overflowing invariant manifold (with corners) $W_X{}^u$ containing $\bar{\Lambda}$ and tangent to N along $\bar{\Lambda}$. This manifold is called the unstable manifold (or more precisely a local unstable manifold) of $\bar{\Lambda}$.

The unstable manifold $W_X{}^u$ satisfies the hypotheses of the perturbation theorem, so for any Y near X there exists $W_Y{}^u$ near $W_X{}^u$, overflowing invariant under Y. Furthermore, there is a neighborhood U of the interior of $W_Y{}^u$ such that $W_Y{}^u = A^+(U)$, the maximal subset of U which is overflowing invariant under Y.

Suppose now that Λ is a compact manifold without boundary, invariant under X. Let N^s and N^u be subbundles of $TM \mid \Lambda$ such that $N^s + N^u = TM \mid \Lambda$, and $N^s \cap N^u = T\Lambda$. We say that Λ is r-normally hyperbolic if Λ and N^u satisfy the hypotheses of the unstable manifold theorem, and Λ and N^s satisfy the hypotheses of the unstable manifold theorem for the flow $F^{-\tau}$ of $-X$. If Λ is r-normally hyperbolic, and Y is a C^r vector field, C^1 close to X, then there are C^r manifolds $W_Y{}^u$ and $W_Y{}^s$, overflowing invariant under the flows of Y and $-Y$, respectively. The manifold $\Lambda_Y = W_Y{}^u \cap W_Y{}^s$ is invariant under Y, and is C^r diffeomorphic to Λ. We call $W_Y{}^u$ and $W_Y{}^s$ the unstable manifold and the stable manifold of Λ_Y respectively. There is a neighborhood U of Λ such that $W_Y{}^u = A^-(U)$, $W_Y{}^s = A^+(U)$, and $\Lambda_Y = I(U)$.

We now recall the expanding family theorems. Let $\bar{\Lambda} = \Lambda \cup \partial\Lambda$ be overflowing invariant. Let $I \subset TM \mid \bar{\Lambda}$ be a negatively invariant subbundle containing $T\Lambda$. Choose any complement J of I in $TM \mid \bar{\Lambda}$ and any complement N of $T\Lambda$ in $TM \mid \bar{\Lambda}$. Let π^J and π^N be the projections on N and J corresponding to the splitting $TM \mid \bar{\Lambda} = T\Lambda \oplus J \oplus N$. For any $m \in \bar{\Lambda}$, $v^0 \in T_m\Lambda$, $\hat{v}^0 \in T_m\Lambda$, $w^0 \in J_m$, $x^0 \in N_m$, and $\tau \leqslant 0$, let

$$\begin{aligned} v^\tau &= DF^\tau(m)\, v^0 \\ \hat{v}^\tau &= DF^\tau(m)\, \hat{v}^0 \\ w^\tau &= \pi^J DF^\tau(m)\, w^0 \\ x^\tau &= \pi^N DF^\tau(m)\, x^0. \end{aligned}$$

Define

$$\alpha^*(m) = \inf\{\alpha > 0: \mid w^\tau \mid \alpha^\tau \to 0 \text{ as } \tau \to -\infty \text{ for all } w^0 \in J_m\}.$$

If $\alpha^*(m) < 1$ define

$$\rho_1^*(m) = \inf\{\rho > 0: [\mid w^\tau \mid / \mid v^\tau \mid] \rho^\tau \to 0 \text{ as } \tau \to -\infty \text{ for all } v^0 \in T_m\Lambda,\ w^0 \in J_m\}$$

$$\rho_2^*(m) = \inf\{\rho > 0: [\mid w^\tau \mid / \mid x^\tau \mid] \rho^\tau \to 0 \text{ as } \tau \to -\infty \text{ for all } w^0 \in J_m,\ x^0 \in N_m\}.$$

If $\rho_1^*(m) < 1$ for all $m \in \bar{\Lambda}$, then by Fenichel (1974, Lemma 3) there is a unique negatively invariant complement E of $T\Lambda$ in I. We call $(\bar{\Lambda}, I)$ an invariant manifold with expanding structure if $\alpha_1^*(m) < 1$, $\rho_1^*(m) < 1$, and $\rho_2^*(m) < 1$ for all $m \in \bar{\Lambda}$. It is known that if $(\bar{\Lambda}, I)$ is an invariant manifold with expanding structure, then there is a family of manifolds $\{\overline{W}^E_{loc}(m) = W^E_{loc}(m) \cup \partial W^E_{loc}(m): m \in \bar{\Lambda}\}$, invariant in the sense that $F^\tau(\overline{W}^E_{loc}(m)) \subset W^E_{loc}(F^\tau(m))$ for all $\tau < 0$ and $m \in \bar{\Lambda}$, such that $W^E_{loc}(m)$ is tangent to E_m at m. See Fenichel (1974, Theorem 1). We call the family $\{\overline{W}^E_{loc}(m): m \in \bar{\Lambda}\}$ an expanding family. This family satisfies some estimates Fenichel (1974, Theorem 3) which we do not repeat here. These estimates give Theorem 9.1.iii.

Let $(\bar{\Lambda}, I)$ be an invariant manifold with expanding structure, as above. Define

$$\tau_1^*(m) = \inf\{e: [| w^\tau |/| v^\tau |]^e \, | \hat{v}^\tau | \to 0 \text{ as } \tau \to -\infty$$
$$\text{for all } v^0 \in T_m\Lambda,\ \hat{v}^0 \in T_m\Lambda,\ w^0 \in J_m\}$$

and

$$\tau_2^*(m) = \inf\{e: [| w^\tau |/| x^\tau |]^e \, | \hat{v}^\tau | \to 0 \text{ as } \tau \to -\infty$$
$$\text{for all } \hat{v}^0 \in T_m\Lambda,\ w^0 \in J_m,\ x^0 \in N_m\}.$$

By Fenichel (1977, Theorem 3), if $r' \leqslant r - 1$ and $\tau_1^*(m) < 1/r'$, $\tau_2^*(m) < 1/r'$ for all $m \in \bar{\Lambda}$, then the family $\{\overline{W}^E_{loc}(m): m \in \bar{\Lambda}\}$ is a $C^{r'}$ family of manifolds.

We require the following extension of the perturbation theorem for overflowing invariant manifolds. Suppose $\{X^\epsilon: \epsilon \in (-\epsilon_0, \epsilon_0)\}$ is a C^r family of vector fields on M. Suppose $\bar{\Lambda}^0$ is overflowing invariant under X^0, and $\nu^*(m) < 1$ and $\sigma^*(m) < 1/r$ for all $m \in \bar{\Lambda}^0$. Then we assert that there is a C^r family of manifolds with boundary $\{\bar{\Lambda}^\epsilon: \epsilon \in (-\epsilon_1, \epsilon_1)\}$, for some $\epsilon_1 > 0$ such that $\bar{\Lambda}^\epsilon$ is overflowing invariant under X^ϵ. To prove this result we add ϵ as a dummy space variable satisfying $\dot{\epsilon} = 0$. For any $\epsilon_1 > 0$, $\bar{\Lambda}^0 \times [-\epsilon_1, \epsilon_1]$ is negatively invariant under $X^0 \times \mathbf{0}$, and $\nu^*(m, \epsilon) < 1$, $\sigma^*(m, \epsilon) < 1/r$ for all $(m, \epsilon) \in \bar{\Lambda}^0 \times [-\epsilon_1, \epsilon_1]$. If ϵ_1 is small, $X^\epsilon \times \mathbf{0}$ is close to $X^0 \times \mathbf{0}$ in a neighborhood of $\bar{\Lambda}^0 \times [-\epsilon_1, \epsilon_1]$. Because of the special role of ϵ as a parameter, the construction of the perturbation theorem for overflowing invariant manifolds can be applied in this case to show that $X^\epsilon \times \mathbf{0}$ has a C^r negatively invariant manifold with corners $\tilde{\Lambda}$ near $\bar{\Lambda}^0 \times [-\epsilon_1, \epsilon_1]$. The manifolds defined by $\bar{\Lambda}^\epsilon = \{m \in M: (m, \epsilon) \in \tilde{\Lambda}\}$ form a C^r family, and $\bar{\Lambda}^\epsilon$ is overflowing invariant under X^ϵ.

Suppose $\{X^\epsilon: \epsilon \in (-\epsilon_0, \epsilon_0)\}$ is a C^r family of vector fields on M. Suppose $\{\bar{\Lambda}^\epsilon: \epsilon \in (-\epsilon_0, \epsilon_0)\}$ as a C^r family of manifolds with boundary, such that $\bar{\Lambda}^\epsilon$ is overflowing invariant under X^ϵ. Suppose $(\bar{\Lambda}^0, I^0)$ is an invariant manifold with expanding structure. Then the argument which shows that $T\Lambda^0$ has a unique negatively invariant complement E^0 also shows that there are unique negatively invariant bundles E^ϵ near E^0, for all small $| \epsilon |$. A continuity argument then shows that $(\bar{\Lambda}^\epsilon, T\Lambda^\epsilon \oplus E^\epsilon)$ is an invariant manifold with expanding structure. If, for some $r' \leqslant r - 1$, $\tau_1^*(m) < 1/r'$ and $\tau_2^*(m) < 1/r'$ for the flow of X^0, for all $m \in \bar{\Lambda}^0$, then also $\tau_1^*(m) < 1/r'$ and $\tau_2^*(m) < 1/r'$ for the flow of X^ϵ, for all $m \in \bar{\Lambda}^\epsilon$ for all small ϵ. We find, then, that $\{\overline{W}^E_{loc}(m): m \in \bar{\Lambda}^\epsilon, \epsilon \in (-\epsilon_1, \epsilon_1)\}$ is a $C^{r'}$ family of manifolds, if ϵ_1 is sufficiently small.

XVII. Extensions

The theory developed so far admits several straightforward extensions. We mention two of these.

Our first extension is a global theory near a normally hyperbolic invariant manifold for the reduced vector field. Let M be a C^{r+1} manifold, $2 \leqslant r \leqslant \infty$. Let X^ϵ, $\epsilon \in (-\epsilon_0, \epsilon_0)$, be a C^r family of vector fields, and let $\mathscr{E}$ be a C^r submanifold of M consisting entirely of equilibrium points of X^0. Then any structure in $\mathscr{E}_H$ which persists under regular perturbations also persists under singular perturbations. In particular, suppose $\Lambda \subset \mathscr{E}_H$ is a normally hyperbolic invariant manifold. Then we can imitate the theory developed in Section XIII for periodic orbits, with the periodic orbit γ replaced by Λ. Asymptotic stability with asymptotic phase must be defined as in Fenichel (1974, 1977).

Our second extension is a refinement of the invariant family construction. Assume the hypotheses of Theorem 9.1. Then $QX^0(m)$ has k_1 eigenvalues in the left half plane, for all $m \in K$. Suppose ℓ_1 of these eigenvalues have real parts smaller than the real parts of the remaining $k_1 - \ell_1$ eigenvalues, for all $m \in K$. Then the invariant subspaces corresponding to the ℓ_1 eigenvalues with smallest real parts determine an expanding structure over K, and so determine an invariant family, a subfamily of $\mathscr{F}^s$. See Fenichel (1974, Theorem 3) for rate estimates for this invariant family.

XVIII. Proof of Theorem 9.1

We prove Theorem 9.1 by modifying the given vector field X^ϵ so that the invariant manifold theory outlined in Section XVI is applicable. Verification of the hypotheses of the invariant manifold theorems is especially simple because we compute the Lyapunov type numbers only at equilibrium points.

In order to display the geometric ideas underlying the proof of Theorem 9.1, we present the detailed proof first in case K consists of just one point. Then we sketch the modifications necessary for the proof of the general result.

Assume now that K consists of just one point $m_0 \in \mathscr{E}$. Choose local coordinates as in (11.1)–(11.3), so that the flow of $X^\epsilon \times \mathbf{0}$ satisfies

$$\begin{aligned} x' &= f(x, y_1, y_2, y_3, \epsilon) \\ y_1' &= g_1(x, y_1, y_2, y_3, \epsilon) \\ y_2' &= g_2(x, y_1, y_2, y_3, \epsilon) \\ y_3' &= g_3(x\ y_1\ y_2, y_3, \epsilon) \\ \epsilon' &= 0 \end{aligned} \tag{18.1}$$

with the conditions

$$\begin{aligned} f(0,0,0,0,0) &= 0 & & \\ g_i(0,0,0,0,0) &= 0 & i &= 1,2,3 \\ D_1 f(0,0,0,0,0) &= 0 & & \\ D_1 g_i(0,0,0,0,0) &= 0, & i &= 1,2,3 \end{aligned} \tag{18.2}$$

and

$$\left.\frac{\partial(g_1, g_2, g_3)}{\partial(y_1, y_2, y_3)}\right|_{\substack{x=0\\y=0\\\epsilon=0}} = \begin{vmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{vmatrix}, \tag{18.3}$$

where the eigenvalues of A_1 are in the left half plane, the eigenvalues of A_2 are on the imaginary axis, and the eigenvalues of A_3 are in the right half plane. Now replace the variables $(x, y_1, y_2, y_3, \epsilon)$ by $(x/\delta_1, y_1/\delta_1^2, y_2/\delta_1^2, y_3/\delta_1^2, \epsilon/\delta_1^3)$, where δ_1 is a small positive number. This makes (18.1) arbitrarily C^1-close to

$$\begin{aligned} x' &= 0 \\ y_1' &= A_1 y_1 \\ y_2' &= A_2 y_2 \\ y_3' &= A_3 y_3 \\ \epsilon' &= 0, \end{aligned} \tag{18.4}$$

uniformly in $U_1 = \{(x, y_1, y_2, y_3, \epsilon): |x| \leqslant 1, |y_1| \leqslant 1, |y_2| \leqslant 1, |y_3| \leqslant 1, |\epsilon| \leqslant 1\}$.

Let $\Lambda^s = \{(x, y_1, y_2, y_3, \epsilon): y_3 = 0\}$, $\Lambda^u = \{(x, y_1, y_2, y_3, \epsilon): y_1 = 0\}$, and $\Lambda = \Lambda^s \cap \Lambda^u$. These manifolds are locally invariant under (18.4), but they are not compact. The manifolds with corners $\Lambda^s \cap U_1$, $\Lambda^u \cap U_1$, and $\Lambda \cap U_1$ are compact, but do not have the overflowing invariance properties required for the invariant manifold theorems summarized in Section XVI. We will modify (18.1) and (18.4) in order to arrange the required overflowing invariance properties. First, however, it is instructive to compute the generalized Lyapunov type numbers for Λ^s, Λ^u, and Λ, using the usual metric on Euclidean space and ignoring the noncompactness. The variational system of (18.4) along any orbit is

$$\begin{aligned} \delta x' &= 0 \\ \delta y_1' &= A_1\, \delta y_1 \\ \delta y_2' &= A_2\, \delta y_2 \\ \delta y_3' &= A_3\, \delta y_3 \\ \delta\epsilon' &= 0. \end{aligned} \tag{18.5}$$

The flow of (18.5) is given explicitly by

$$\begin{pmatrix} \delta x \\ \delta y_1 \\ \delta y_2 \\ \delta y_3 \\ \delta\epsilon \end{pmatrix} \cdot \tau = \begin{pmatrix} \delta x \\ e^{A_1\tau}\,\delta y_1 \\ e^{A_2\tau}\,\delta y_2 \\ e^{A_3\tau}\,\delta y_3 \\ \delta\epsilon \end{pmatrix}. \tag{18.6}$$

Let ℓ_1 be the largest of the real parts of the eigenvalues of A_1, and let ℓ_3 be the smallest of real parts of the eigenvalues of A_3. Then $\ell_1 < 0 < \ell_3$. We can choose bases in R^{k_1}, R^{k_2}, and R^{k_3}, such that

$$\begin{aligned} \langle y_1, A_1 y_1\rangle &\leqslant (\ell_1 + \delta_2)\,|\,y_1\,|^2 && \text{for all } y_1 \in R^{k_1} \\ \langle y_2, A_2 y_2\rangle &\leqslant \delta_2\,|\,y_2\,|^2 && \text{for all } y_2 \in R^{k_2} \\ \langle y_2, -A_2 y_2\rangle &\leqslant \delta_2\,|\,y_2\,|^2 && \text{for all } y_2 \in R^{k_2} \\ \langle y_3, -A_3 y_3\rangle &\leqslant (-\ell_3 + \delta_2)\,|\,y_3\,|^2 && \text{for all } y_3 \in R^{k_3}, \end{aligned}$$

where δ_2 is an arbitrary positive number. Hence

$$\begin{aligned} \|\, e^{A_1\tau} \,\| &\leqslant e^{(\ell_1+\delta_2)\tau} && \text{for all } \tau \geqslant 0 \\ \|\, e^{A_2\tau} \,\| &\leqslant e^{\delta_2\tau} && \text{for all } \tau \geqslant 0 \\ \|\, e^{A_2\tau} \,\| &\leqslant e^{-\delta_2\tau} && \text{for all } \tau \leqslant 0 \\ \|\, e^{A_3\tau} \,\| &\leqslant e^{(\ell_3-\delta_2)\tau} && \text{for all } \tau \leqslant 0. \end{aligned}$$

These observations enable us to compute the generalized Lyapunov type numbers for Λ^s, Λ^u, and Λ.

Consider Λ^u and the flow (18.6). The function ν^* measures the growth of δy_1, and the function σ^* measures the comparison between the growth of δy_1 and the growth of the remaining components. We find

$$\begin{aligned} \nu^*(m) &= e^{\ell_1} \\ \sigma^*(m) &= 0 \end{aligned}$$

for all $m \in \Lambda^u$. Similarly, for the flow (18.6) with time reversed we find

$$\begin{aligned} \nu^*(m) &= e^{-\ell_3} \\ \sigma^*(m) &= 0 \end{aligned}$$

for all $m \in \Lambda^s$. The family of planes parallel to the y_3-axis forms an expanding structure for Λ in Λ^u, for the flow (18.6). We find

$$\begin{aligned}\alpha^*(m) &= e^{-\ell_3}\\ \rho_1^*(m) &= e^{-\ell_3}\\ \tau_1^*(m) &= 0\end{aligned}$$

for all $m \in \Lambda$. The type numbers ρ_2^* and τ_2^* are undefined. The family of planes parallel to the y_1-axis forms an expanding structure for Λ in Λ^s, for the flow (18.6) with time reversed. We find

$$\begin{aligned}\alpha^*(m) &= e^{\ell_1}\\ \rho_1^*(m) &= e^{\ell_1}\\ \tau_1^*(m) &= 0\end{aligned}$$

for all $m \in \Lambda$. The type numbers ρ_2^* and τ_2^* are undefined.

Now we modify (18.1) and (18.5) in order to satisfy the overflowing invariance conditions required for application of the invariant manifold theorems. Choose real numbers a_1, a_2, a_3, a_4, and a_5, satisfying $0 < a_5 < a_4 < a_3 < a_2 < a_1 = 1$, and choose a C^∞ "bump" function ω: $[0, 1] \to R$ such that $\omega(a) = 0$ for $a \in [0, a_4]$, $\omega(a_3) > 0$, $\omega(a_2) < 0$, and $\omega(a_1) > 0$ (see Fig. 3). Let $U_i = \{(x, y_1, y_2, y_3, \epsilon): |x| \leqslant a_i, |y_1| \leqslant a_i, |y_2| \leqslant a_i, |y_3| \leqslant a_i, |\epsilon| \leqslant 1\}$. Consider the systems

$$\begin{aligned}x' &= f(x, y_1, y_2, y_3, \epsilon) + \delta_3\omega(|x|)x\\ y_1' &= g_1(x, y_1, y_2, y_3, \epsilon)\\ y_2' &= g_2(x, y_1, y_2, y_3, \epsilon) + \delta_3\omega(|y_2|)y_2\\ y_3' &= g_3(x, y_1, y_2, y_3, \epsilon)\\ \epsilon' &= 0\end{aligned} \tag{18.7}$$

and

$$\begin{aligned}x' &= \delta_3\omega(|x|)x\\ y_1' &= A_1y_1\\ y_2' &= A_2y_2 + \delta_3\omega(|y_2|)y_2\\ y_3' &= A_3y_3\\ \epsilon' &= 0\end{aligned} \tag{18.8}$$

where δ_3 is a small positive number. Choose δ_3 small, and δ_2 small relative to δ_3. Then $\Lambda^u \cap U_1$ is overflowing invariant under (18.8) and satisfies

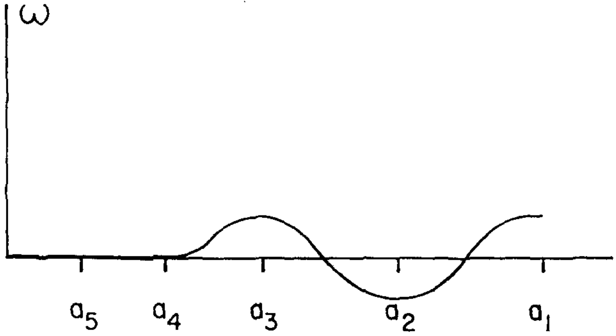

FIGURE 3

$\nu^*(m) < 1$, $\sigma^*(m) < 1/r$ for all $m \in \Lambda^u \cap U_1$. Similarly $\Lambda^s \cap U_2$ is overflowing invariant under (18.8) with time reversed, and satisfies $\nu^*(m) < 1$, $\sigma^*(m) < 1/r$ for all $m \in \Lambda^s \cap U_2$. If δ_1 is sufficiently small, (18.8) is C^1-close to (18.8). It follows from the perturbation theorem for overflowing invariant manifolds that there is a C^r manifold $\tilde{\Lambda}^u$ near $\Lambda^u \cap U_1$ which is overflowing invariant under (18.7), and there is a C^r manitold $\tilde{\Lambda}^s$ near $\Lambda^s \cap U_2$ which is overflowing invariant under (18.7) with time reversed. It follows from the proof of the perturbation theorem that $\tilde{\Lambda}^u$ is the maximal negatively invariant set in U_1, and $\tilde{\Lambda}^s$ is the maximal positively invariant set in U_2, for the flow of (18.7). In particular, since the origin is an equilibrium point of (18.7), $\tilde{\Lambda}^u$ and $\tilde{\Lambda}^s$ both contain the origin. Because the tangent spaces of $\tilde{\Lambda}^u$ and $\tilde{\Lambda}^s$ at the origin are invariant and are close to the tangent spaces of Λ^u and Λ^s at the origin, it follows that the corresponding tangent spaces coincide. Hence $\tilde{\Lambda}^u$ is a center-unstable manifold for (18.7) at the origin and $\tilde{\Lambda}^s$ is a center-stable manifold for (18.7) at the origin. In U_4, (18.1) and (18.7) coincide, so $\tilde{\Lambda}^u \cap U_4$ is a center-unstable manifold for (18.1) at the origin, and $\tilde{\Lambda}^s \cap U_4$ is a center-stable manifold for (18.1) at the origin. Hence also $\tilde{\Lambda}^s \cap \tilde{\Lambda}^u \cap U_4$ is a center manifold for (18.1) at the origin. (These are manifolds with corners, but of course we are free to round the corners.)

Now consider the flow of (18.8) with time reversed, restricted to $\Lambda^u \cap U_2$. If δ_3 is sufficiently small, and δ_2 is sufficiently small relative to δ_3, then $\Lambda \cap U_2$ is an invariant manifold with expanding structure for this flow, and $\alpha^*(m) < 1$, $\rho_1^*(m) < 1$, $\tau_1^*(m) < 1/(r-1)$ for all $m \in \Lambda \cap U_2$. If δ_1 is sufficiently small relative to δ_2 and δ_3, then (18.1) is C^1-close to (18.7), $\tilde{\Lambda}^u \cap U_2$ is C^1-close to $\Lambda^u \cap U_2$, and $\tilde{\Lambda}^u \cap \tilde{\Lambda}^s \cap U_2$ is C^1-close to $\Lambda \cap U_2$. The inequalities for the type numbers α^*, ρ_1^*, and τ_1^* persist under C^1-small perturbations, so $\tilde{\Lambda}^u \cap \tilde{\Lambda}^s \cap U_2$ is an invariant manifold with expanding structure for the flow of (18.1) with time reversed, restricted to $\tilde{\Lambda}^u \cap U_2$. This expanding family determines a C^{r-1} family of manifolds $\{\mathscr{F}^u(p): p \in \tilde{\Lambda}^u \cap \tilde{\Lambda}^s \cap U_2\}$. The manifolds $\mathscr{F}^u(p)$ are disjoint and fill a neighborhood of $\tilde{\Lambda}^u \cap \tilde{\Lambda}^s \cap U_2$ in

$\tilde{\Lambda}^u \cap U_2$. If δ_1 is sufficiently small, this neighborhood is all of $\tilde{\Lambda}^u \cap U_2$. Then we define, for arbitrary $q \in \tilde{\Lambda}^u \cap U_2$, $\tilde{\mathscr{F}}^u(q) = \tilde{\mathscr{F}}^u(p(q))$, where $p(q)$ is the unique point $p \in \tilde{\Lambda}^u \cap \tilde{\Lambda}^s \cap U_2$ such that $q \in \tilde{\mathscr{F}}^u(p)$.

The family $\tilde{\mathscr{F}}^s$ is constructed in the same manner, except that $\Lambda^u \cap U_2$ is replaced by $\Lambda^s \cap U_3$, and we consider the flows of (18.1) and (18.7) without reversing time.

Define

$$\mathscr{C} = \tilde{\Lambda}^u \cap \tilde{\Lambda}^s \cap U_5$$
$$\mathscr{C}^u = \bigcup \{\tilde{\mathscr{F}}^u(p) \cap U_5 : p \in \mathscr{C}\}$$
$$\mathscr{C}^s = \bigcup \{\tilde{\mathscr{F}}^s(p) \cap U_5 : p \in \mathscr{C}\}.$$

Then $\mathscr{C}^u$ is an open set in $\tilde{\Lambda}^u \cap U_2$, and $\mathscr{C}^u$ contains the origin, so $\mathscr{C}^u$ is a center-unstable manifold for (18.1) at the origin. Similarly, $\mathscr{C}^s$ is a center-stable manifold for (18.1) at the origin. By a transversality argument, $\mathscr{C} = \mathscr{C}^u \cap \mathscr{C}^s$. Hence $\mathscr{C}$ is a center manifold for (18.1) at the origin. For $p \in \mathscr{C}^u$, define

$$\mathscr{F}^u(p) = \tilde{\mathscr{F}}^u(p) \cap U_5 .$$

If δ_1 is sufficiently small, $\mathscr{F}^u(p) \subset U_2$ for all $p \in \mathscr{C}^u$. Hence $\mathscr{F}^u$ defines an unstable manifold family for (18.1) at the origin. For $p \in \mathscr{C}^s$, define

$$\mathscr{F}^s(p) = \tilde{\mathscr{F}}^s(p) \cap U_5 .$$

By a similar argument $\mathscr{F}^s$ defines a stable manifold family for (18.1) at the origin. We leave the definition of U to the reader. The easiest way to define U is to note that the proof of Theorem 11.1 requires only the existence of the invariant manifolds and the invariant families. In the coordinates in which $X^\epsilon \times \mathbf{0}$ is in normal form, U is just a product of open balls.

With these constructions we have proved Theorem 9.1.i and most of Theorem 9.1.ii. It remains to show that if $p \in \mathscr{C}^s \cap (M \times \{\epsilon\})$, then $\mathscr{F}^s(p) \subset M \times \{\epsilon\}$, and that if $p \in \mathscr{C}^u \cap (M \times \{\epsilon\})$, then $\mathscr{F}^u(p) \subset M \times \{\epsilon\}$. But these assertions follow from Fenichel (1974, Theorem 3), because $\mathscr{F}^s$ and $\mathscr{F}^u$ are determined by asymptotic equivalence under the flow of (18.7) as $t \to \pm\infty$, and ϵ is constant on orbits of (18.7). Theorem 9.1.iii also follows from Fenichel (1974, Theorem 3), applied to orbit segments which lie entirely in the region where (18.1) coincides with (18.7).

The uniqueness results, Theorem 9.1.iv, follow from the uniqueness of the invariant manifolds and invariant families constructed in Fenichel (1971, 1974, 1977). These uniqueness results follow from the use of contraction mappings. The restrictions to $A^+(U)$, $A^-(U)$, and $I(U)$ guarantee that (18.1) and (18.7) coincide at all points involved in the constructions of the invariant manifolds and invariant families.

To prove Theorem 9.1.v it is sufficient to prove that

$$X_R(m) = \lim_{\substack{(\tilde{m},\epsilon)\to(m,0)\\(m,\epsilon)\mathscr{C}}} \epsilon^{-1}X^\epsilon(\tilde{m}),$$

for all m in $\mathscr{E}$ near K. Fix a point m in $\mathscr{E}$ near K, and choose coordinates near m such that X^ϵ takes the form (18.1) with y_2 missing, subject to (18.2) and (18.3). Because m is near $\mathscr{E}_H$, we may assume that m is in $\mathscr{E}_H$, and hence that $\partial f/\partial y_1$ and $\partial f/\partial y_3$ vanish at the origin. In these coordinates $\mathscr{C}$ is represented as the graph of a pair of functions $y_1 = u_1(x, \epsilon)$, $y_3 = u_3(x, \epsilon)$ which vanish to first order in x at $\epsilon = 0$. The flow of (18.1) on $\mathscr{C}$ satisfies

$$\begin{aligned}\dot{x} &= \epsilon^{-1}f(x, u_1(x, \epsilon), u_3(x\ \ \epsilon), \epsilon)\\ \dot{y}_1 &= \epsilon^{-1}g_1(x, u_1(x, \epsilon), u_3(x, \epsilon), \epsilon)\\ \dot{y}_3 &= \epsilon^{-1}g_3(x, u_1(x, \epsilon), u_3(x, \epsilon), \epsilon)\\ \dot{\epsilon} &= 0\end{aligned} \tag{18.9}$$

for $\epsilon \neq 0$, subject to

$$\begin{aligned}y_1 &= u_1(x, \epsilon)\\ y_3 &= u_3(x, \epsilon).\end{aligned} \tag{18.10}$$

From (18.9) and (18.10) we have

$$\begin{aligned}\epsilon u_1(x, \epsilon) &= g_1(x, u_1(x, \epsilon), u_3(x, \epsilon), \epsilon)\\ \epsilon u_3(x, \epsilon) &= g_3(x, u_1(x, \epsilon), u_3(x, \epsilon), \epsilon).\end{aligned} \tag{18.11}$$

It follows from (18.11) and our assumptions about $\partial f/\partial y_1$ and $\partial f/\partial y_3$ that as $(x, \epsilon) \to (0, 0)$, the system (18.9) tends to

$$\begin{aligned}\dot{x} &= \partial f/\partial\epsilon\,(0, 0, 0, 0)\\ \dot{y}_1 &= 0\\ \dot{y}_3 &= 0\\ \dot{\epsilon} &= 0.\end{aligned} \tag{18.12}$$

By Lemma 5.4, (18.12) is just the local form of $X_R(m)$. This proves Theorem 9.1.v, and completes the proof of Theorem 9 in case K is a point.

In case K is not just a point we construct small neighborhoods U_i of K, $i = 1,\ldots, 5$, satisfying $\overline{U}_{i+1} \subset U_i$, $i = 1,\ldots, 4$. As in the previous case, we construct each neighborhood as a smooth manifold with corners. Then we construct a "bump" vector field Ω which vanishes on U_4, points outward in directions near $T\mathscr{E} \oplus E^c$ on the boundary of U_3, points inward in directions near $T\mathscr{E} \oplus E^c$ on the boundary of U_2, and points outward in directions near $T\mathscr{E} \oplus E^c$ on the boundary of U_1. We replace X^ϵ by $X^\epsilon + \delta\Omega$, with δ and $|\epsilon|$ small, and proceed as in the case in which K is just a point.

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