Multiplicity of solutions for a class of elliptic problem in $\mathbb{R}^2$ with Neumann conditions

Claudianor Oliveira Alves

Departamento de Matemática e Estatística, Universidade Federal de Campina Grande, Aprigio Veloso, Bodocongo, Cep:58109-970, Campina Grande - PB, Brazil

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Abstract

In this work, we study the existence and multiplicity of solutions for a class of elliptic problems in exterior domains of $\mathbb{R}^2$ with Neumann boundary conditions and nonlinearity with critical growth.

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1. Introduction

In this paper, we are concerned with the existence and multiplicity of solutions for the following class of elliptic problem with Neumann conditions:

$$\begin{cases}
-\Delta u + u = Q(x)f(u) & \text{in } \mathbb{R}^2 \setminus \Omega, \\
\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial \Omega,
\end{cases} \tag{P}$$

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E-mail address: coalves@dme.ufcg.edu.br (C.O. Alves).

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where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $Q$ is a continuous function satisfying

$$Q(x) > 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Omega \quad \text{and} \quad \lim_{|x| \to \infty} Q(x) = \bar{Q} > 0,$$

and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function satisfying the following hypotheses:

$f$ has critical growth at both $+\infty$ and $-\infty$, that is, it behaves like $e^{\alpha_0 x^2}$ as $|s| \to \infty$ for some $\alpha_0 > 0$. More precisely,

$$\lim_{|s| \to \infty} \frac{|f(s)|}{e^{\alpha_0 s^2}} = 0 \quad \forall \alpha > \alpha_0, \quad \lim_{|s| \to \infty} \frac{|f(s)|}{e^{\alpha_0 s^2}} = +\infty \quad \forall \alpha < \alpha_0.$$

Moreover, we assume that

$$|f(s)| \leq Ce^{\alpha_0 s^2} \quad \text{for all} \quad s \in \mathbb{R}. \quad (f_1)$$

There is $\theta > 2$ verifying

$$0 < \theta F(s) \leq sf(s) \quad \text{for all} \quad s \in \mathbb{R}. \quad (f_2)$$

There exists $q > 1$ such that

$$\limsup_{|s| \to 0} \frac{|f(s)|}{|s|^q} < \infty. \quad (f_3)$$

The function $s \to \frac{f(s)}{s}$ is increasing in $(0, +\infty)$. \quad (f_4)

There are constants $p > 2$ and $C_p$ such that

$$f(s) \geq C_p s^{p-1} \quad \text{for all} \quad s \in [0, +\infty), \quad (f_5)$$

where

$$C_p > \left[ \frac{2\alpha_0^2 \theta (p-2)}{p(\theta-2)} \right]^{(p-2)/2} S_p^p,$$

$$S_p = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx}{(\bar{Q} \int_{\mathbb{R}^2} u^p \, dx)^{1/p}} \right)^{1/2}.$$
and \( \zeta > 0 \) is a positive constant such that the extension operator \( E : H^1(\mathbb{R}^2 \setminus \Omega) \to H^1(\mathbb{R}^2) \) satisfies

\[
\| Eu \|_{H^1(\mathbb{R}^2)} \leq \zeta \| u \|_{H^1(\mathbb{R}^2 \setminus \Omega)} \quad \forall u \in H^1(\mathbb{R}^2 \setminus \Omega).
\]

We recall that \( E \) exists because the set \( \Omega \) has smooth bounded boundary (see [1]).

In [6], Benci and Cerami studied problem (\( P \)) assuming \( N \geq 3 \), \( Q \equiv 1 \) and \( f(u) = |u|^\eta - 1 u \) with \( 1 < \eta < \frac{N+2}{N-2} \). They showed that (\( P \)), with Dirichlet condition, has not a ground state solution, that is, a solution of (\( P \)) with minima energy. However, Esteban in [10] proved that the same problem with Neumann condition has a ground state solution.

In [8], Cao also studied problem (\( P \)) for \( N \geq 3 \), \( f(u) = |u|^\eta - 1 u \) and \( Q \) satisfying condition (\( Q_1 \)). He showed that this problem has at least two solutions, a positive solution and a nodal solution, that is, a solution of (\( P \)) that changes of sign. In [3], Alves et al. showed that the results found in [8], also hold for the p-Laplacian operator and also for a larger class of nonlinearity.

Motivated by papers [3,8] and by some ideas developed in [4,7], we prove the existence of ground state and nodal solutions to (\( P \)). We used variational methods such as the Mountain Pass Theorem without Palais–Smale condition (see [5,14]) to obtain a positive ground state solution. In relation to nodal solution, we apply the implicit function Theorem. An important point in our work is that the nonlinearity has critical growth in \( \mathbb{R}^2 \), this fact implies that some estimates and arguments explored in [3,8] cannot be used. To overcome these difficulties, we used a version of a result due to Lions for the critical growth case in \( \mathbb{R}^2 \) proved by Alves et al. in [4].

Concerning the existence of ground state solution, we will prove the following result:

**Theorem 1.1.** Suppose that \( f \) satisfies (\( f_1 \))–(\( f_5 \)), \( Q \) satisfies (\( Q_1 \)) and

\[
Q(x) \geq \bar{Q} - Ce^{-m|x|} \quad |x| \geq R_0,
\]

where \( C, R_o \) are positive constants and \( m > 2 \). Then (\( P \)) has a positive ground state solution.

In order to get nodal solution, it is necessary the following additional conditions on \( f \): There exists \( \sigma \geq 2 \) such that

\[
f'(s)s^2 - f(s)s \geq C|s|^{\sigma} \quad \forall s \in \mathbb{R}
\]

and

\[
|f'(s)s| \leq Ce^{4\pi s^2} \quad \forall s \in \mathbb{R},
\]

for some positive constant \( C \).
Theorem 1.2. Suppose that $f$ satisfies $(f_1)$–$(f_7)$, $Q$ satisfies $(Q_1)$ and

$$Q(x) \geq \tilde{Q} + Ce^{-\mu|x|} \quad \forall x \in \mathbb{R}^N,$$

where $C$ is a positive constant and $\mu < \frac{1}{q + 1}$. Then $(P)$ has a nodal solution.

This paper is organized as follows: in Section 2, we recall some results involving the limit problem. In Section 3, we state some lemmas and propositions used in the proof of the main results. In Section 4, we prove the main results. In Section 5 we prove some technical lemmas and propositions stated in the Section 3.

To finish this section, we would like to cite also the papers of Adimurthi and Yadava [2] and de Figueiredo et al. [9] and the references therein, where elliptic problems in $\mathbb{R}^2$ have being considered.

2. The limit problem

In this work, we need to recall some results involving the limit problem

$$\begin{cases} -\Delta u + u = \tilde{Q} f(u) & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2). \end{cases} \quad (P_\infty)$$

Hereafter, if $h$ is a Lebesgue integrable function and $B$ is a measurable set, we write $\int_B h$ for $\int_B h \, dx$. Moreover, if $h \in H^1(\mathbb{R}^2 \setminus \Omega)$ we denote by $\|h\|$ its usual norm.

The energy functional $I_\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ associated to problem $(P_\infty)$ is given by

$$I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) - \int_{\mathbb{R}^2} \tilde{Q} F(u^+),$$

where $F(u) = \int_0^u f(t) \, dt$ and $u^+(x) = \max\{u(x), 0\}$. Using the hypotheses on function $f$, we have that $I_\infty \in C^1(H^1(\mathbb{R}^2), \mathbb{R})$ and the weak solutions of $(P_\infty)$ are nontrivial critical points of $I_\infty$.

Repeating the same arguments explored by Cao [7] and Alves et al. [4], it is possible to check that $I_\infty$ verifies the Mountain Pass Geometry and that there exists a positive function $\tilde{u} \in B_1(0) \setminus \{0\} \subset H^1(\mathbb{R}^2)$ verifying

$$I_\infty(\tilde{u}) = c_\infty \quad \text{and} \quad I_\infty'(\tilde{u}) = 0,$$

where $c_\infty$ is the minimax level of the Mountain Pass Theorem applied to $I_\infty$. In this case, the function $\tilde{u}$ is a ground state solution to $(P_\infty)$. Moreover, we have the following result.
Theorem 2.1. Assume that \((f_1)\) and \((f_3)\) hold. Then, any positive solution \(\bar{u}\) of problem \((P_\infty)\) with \(\|\bar{u}\|_{H^1(R^2)} < 1\) satisfies:

\[
\text{(I) } \lim_{|x| \to \infty} \bar{u}(x) = 0
\]

and

\[
\text{(II) } C_1 e^{-a|x|} \leq \bar{u}(x) \leq C_2 e^{-b|x|} \text{ in } R^2,
\]

where \(C_1, C_2 > 0\) are positive constants and \(0 < b < 1 < a\). Moreover, we can be chosen \(a = 1 + \delta, b = 1 - \delta\) for \(\delta > 0\).

Proof. Using conditions \((f_1)\) and \((f_3)\), for each \(\tau > 1\) and \(\varepsilon > 0\), there exists \(C_\varepsilon > 0\) such that

\[
|f(s)s|, |F(s)| \leq \varepsilon |s|^2 + C_\varepsilon(e^{4\pi \tau s^2} - 1)|s| \quad \forall s \in R.
\]

Using the fact that \(\|\bar{u}\|_{H^1(R^2)} < 1\) and arguments found in [4,7], there exists \(q\) near 1, \(q > 1\) such that

\[
h(x) = f(\bar{u}(x)) \in L^q(R^2).
\]

By bootstrap arguments, for \(x \in R^2\) and \(R > 0\), it follows that \(\bar{u} \in W^{2,q}(B_R(x))\) with

\[
\|\bar{u}\|_{W^{2,q}(B_R(x))} \leq C \{ |h|_{L^q(B_{2R}(x))} + |\bar{u}|_{L^q(B_{2R}(x))}\}
\]

which implies,

\[
\|\bar{u}\|_{W^{2,q}(B_R(x))} \leq C \{ |h|_{L^q(B_{2R}(x))} + |\bar{u}|_{L^2(B_{2R}(x))}\}.
\]

Since the imbedding \(W^{2,q}(B_R(x)) \hookrightarrow C(\bar{B}_R(x))\) is continuous,

\[
\|\bar{u}\|_{L^\infty(B_R(x))} \leq C \{ |h|_{L^q(B_{2R}(x))} + |\bar{u}|_{L^2(B_{2R}(x))}\}.
\]

The last inequality implies that \(\bar{u} \in L^\infty(R^2)\) and \(\lim_{|x| \to \infty} \bar{u}(x) = 0\).

The inequalities in (II) involving the exponential functions follow with the same arguments found in Li and Yan [11]. \(\square\)

Remark 2.1. (i) Theorem 2.1 completes the result proved in [11], because our nonlinearity has a different behavior at infinity.

(ii) With the same arguments used in the proof of Theorem 2.1, we can show that all positive weak solutions \(u_1\) of \((P)\), with \(\|u_1\| < \frac{1}{\varepsilon}\), has exponential decaying.
3. Statement of lemmas and propositions

In this section we state some necessary results to prove Theorems 1.1 and 1.2. The proofs of some of them are in Section 5.

3.1. Technical results to get ground state solution

The first lemma can be found in Alves et al. [3].

Lemma 3.1. Let $F \in C^2(\mathbb{R}, \mathbb{R}^+)$ be a convex and even function such that $F(0) = 0$ and $f(s) = F'(s) \geq 0 \forall s \in [0, \infty)$. Then, for all $u, v \geq 0$

$$|F(u - v) - F(u) - F(v)| \leq 2(f(u)v + f(v)u).$$

The next lemma is related to the Mountain Pass Geometry and we do not make its proof because it is well known. See for example Alves et al. [4]. Hereafter, let us denote by $I : H^1(\mathbb{R}^2 \setminus \Omega) \to \mathbb{R}$ the energy functional related to $(P)$, that is,

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} (|\nabla u|^2 + u^2) - \int_{\mathbb{R}^2 \setminus \Omega} Q(x)F(u).$$

Lemma 3.2. The functional $I$ verifies the Mountain Pass Geometry, that is,

(i) There exist $r, \rho > 0$ such that $I(u) \geq r, \|u\| = \rho$.

(ii) There exists $e \in B^c_\rho(0)$ such that $I(e) < 0$.

Using a version of Mountain Pass Theorem without Palais–Smale condition (see [14, Theorem 1.15]) and $(f_4)$, there exists $u_n \in H^1(\mathbb{R}^2 \setminus \Omega)$ satisfying

$$I(u_n) \to c_1 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$c_1 = \inf \left\{ \sup_{t \geq 0} I(tu); u \in H^1(\mathbb{R}^2 \setminus \Omega) \setminus \{0\} \right\}. \quad (3.1)$$

The next result establishes a relation between the levels $c_1$ and $c_\infty$.

Proposition 3.1. Assume that $Q$ satisfies $(Q_1)$–$(Q_2)$. Then

$$0 < c_1 < c_\infty.$$
Proof. See Section 5.

The following result may be proved in much the same way as in Lions [12].

Lemma 3.3. Let \( \{u_n\} \subset H^1(\mathbb{R}^2 \setminus \Omega) \) be a bounded sequence such that

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{U_{R,y}} |u_n|^2 = 0,
\]

for some \( R > 0 \) and \( U_{R,y} = B_R(y) \cap (\mathbb{R}^2 \setminus \Omega) \) with \( U_{R,y} \neq \emptyset \). Then,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2 \setminus \Omega} |u_n|^{q+1} = 0 \quad \text{for all } q > 1.
\]

Proposition 3.2. Let \( \{u_n\} \subset H^1(\mathbb{R}^2 \setminus \Omega) \) be a sequence with \( u_n \rightharpoonup 0 \) and

\[
\limsup_{n \to \infty} \|u_n\|^{2} \leq m < \frac{1}{2^{\frac{1}{2}}}.
\]

If there exists \( R > 0 \) such that

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{U_{R,y}} |u_n|^2 = 0,
\]

and \((f_1)\)–\((f_5)\) hold, we have

\[
\int_{\mathbb{R}^2 \setminus \Omega} F(u_n), \quad \int_{\mathbb{R}^2 \setminus \Omega} f(u_n)u_n \to 0 \quad \text{as } n \to \infty.
\]

Proof. See Section 5.

Proposition 3.3. If \( \{u_n\} \subset H^1(\mathbb{R}^2 \setminus \Omega) \) satisfies

\[
I(u_n) \to c_1 \quad \text{and} \quad I'(u_n) \to 0,
\]

we have that \( \limsup_{n \to \infty} \|u_n\|_{\mathbb{R}^2 \setminus \Omega} < \frac{1}{\sqrt{2\varepsilon}} \). Moreover, the weak limit \( u_1 \) of \( \{u_n\} \) in \( H^1(\mathbb{R}^2 \setminus \Omega) \) is a nontrivial critical point of \( I \) with \( I(u_1) = c_1 \).

Proof. See Section 5.
3.2. Technical results to get nodal solutions

Consider the closed set
\[ \mathcal{M} := \{ u \in H^1(\mathbb{R}^2 \setminus \Omega) \mid u^+ \neq 0, \ I'(u^+)u^+ = 0 \} \]
and \( \widehat{c} \) the following real number
\[ \widehat{c} = \inf_{u \in \mathcal{M}} I(u). \]

The proof of the next lemma follows by similar arguments explored in [3] and we omit it.

**Lemma 3.4.** Assume that \((f_1), (f_3), (f_6)\) and \((f_7)\) hold. Then, there exists a sequence \((u_n) \subset \mathcal{M}\) satisfying
\[ I(u_n) \to \widehat{c} \quad \text{and} \quad I'(u_n) \to 0. \]

The next proposition is a key point in our arguments to find nodal solution, because it gives a good estimate to \( \widehat{c} \).

**Proposition 3.4.** Suppose that \(Q\) satisfies \((Q_1)-(Q_3)\). Then
\[ 0 < \widehat{c} < c_1 + c_\infty. \] (3.2)

**Proof.** See Section 5.

4. Proof of the main theorems

**Proof of Theorem 1.1.** First of all, to find a positive ground state solution we will assume that
\[ f(t) = 0 \quad \forall t \leq 0. \]

By Proposition 3.3 and the Mountain Pass Theorem (see [5, 14]), \(I\) has a critical point \(u_1\) at the level \(c_1\). We claim that \(u_1\) is nonnegative. Indeed, we know that \(I'(u_1)u_1^- = 0\), thus \(\|u^-\| = 0\) and \(u_1^- = 0\). Using the maximum principle, we have \(u_1 > 0\) in \(\mathbb{R}^2 \setminus \Omega\). □
Proof of Theorem 1.2. Let \((u_n) \subset \mathcal{M}\) be the sequence obtained in Lemma 3.4. Then, the weak limit \(u\) of \(\{u_n\}\) in \(H^1(\mathbb{R}^2 \setminus \Omega)\) is a nontrivial critical point of \(I\) and \(u^\pm \neq 0\).

To check the above claim, remember that

\[
I(u_n) \to \hat{c} \quad \text{and} \quad I'(u_n) \to 0.
\]

Then

\[
\frac{(\theta - 2)}{2\theta} \limsup_{n \to \infty} \|u_n\|^2 \leq c_1 + c_\infty \leq 2c_\infty
\]

which gives

\[
\limsup_{n \to \infty} \|u_n\|^2 \leq \sqrt{\gamma_*} = \frac{40c_\infty}{\theta - 2}.
\]

From \((f_1)-(f_5)\) (see [4]), it follows that

\[
c_\infty < \frac{(\theta - 2)}{4 \xi^2 \theta},
\]

then

\[
\limsup_{n \to \infty} \|v_n\|_{H^1(\mathbb{R}^2)} \leq \sqrt{\gamma_*} \xi < 1, \quad \text{for } v_n = E_{u_n}.
\]

Using an inequality of Trudinger–Moser type showed by Cao in [7] and repeating the same arguments used in the proof of Proposition 3.3, we can conclude that \(u\) is a critical point of \(I\). Now, we will prove that \(u^\pm \neq 0\).

We have three cases to consider:

(I) \(u^+ = u^- = 0\).
(II) \(u^+ \neq 0\) and \(u^- = 0\).
(III) \(u^+ = 0\) and \(u^- \neq 0\).

We will prove that the above cases do not hold, therefore \(u^\pm \neq 0\). In what follows, we will prove only (I) because the other cases follow with the same type of arguments.

Analysis of (I): Applying Proposition 3.2 to the sequences \(\{u^+_n\}\) and \(\{u^-_n\}\), there exist \(\eta, R > 0\) and sequences \(\{y^1_n\}\) and \(\{y^2_n\}\) in \(\mathbb{R}^2\) with \(|y^1_n|, |y^2_n| \to \infty\) verifying

\[
\liminf_{n \to \infty} \int_{U_{R,y^1_n}} |u^+_n|^2 \geq \eta
\]
and
\[ \liminf_{n \to \infty} \int_{U_{R,y_n}^2} |u_n|^2 \geq \eta. \]

Defining \( w_n(x) = u_n(x + y_n^1) \) and \( z_n(x) = u_n(x + y_n^2) \), there exist \( w, z \in H^1(\mathbb{R}^2 \setminus \{0\}) \) such that \( w_n \to w \) and \( z_n \to z \) in \( H_{\text{loc}}^1(\mathbb{R}^2) \), with \( w^+ \neq 0 \) and \( z^- \neq 0 \). Since \( I'_\infty(w) = I'_\infty(z) = 0 \), we have
\[ I'_\infty(w^+)w^+ = 0 \quad \text{and} \quad I'_\infty(z^-)z^- = 0. \]

In this way,
\[ 2c_\infty \leq I_\infty(w^+) + I_\infty(z^-) = \left[ I_\infty(w^+) - \frac{1}{\partial} I'_\infty(w^+)w^+ \right] + \left[ I_\infty(z^-) - \frac{1}{\partial} I'_\infty(z^-)z^- \right]. \]

By Fatou’s Lemma
\[ \liminf_{n \to \infty} \left[ \int_{\mathbb{R}^2 \setminus \Omega_n^1} (|\nabla w_n^+|^2 + (w_n^+)^2) + \frac{1}{\partial} \int_{\mathbb{R}^2 \setminus \Omega_n^2} (f(w_n^+)w_n^+ - F(w_n^+)) \right] \geq I_1 \]
and
\[ \liminf_{n \to \infty} \left[ \int_{\mathbb{R}^2 \setminus \Omega_n^1} (|\nabla z_n^-|^2 + (z_n^-)^2) + \frac{1}{\partial} \int_{\mathbb{R}^2 \setminus \Omega_n^2} (f(z_n^-)z_n^- - F(z_n^-)) \right] \geq I_2, \]
where \( \Omega_n^1 = \Omega - y_n^1, \Omega_n^2 = \Omega - y_n^2 \), \( I_1 = I_\infty(w^+) - \frac{1}{\partial} I'_\infty(w^+)w^+ \) and \( I_2 = I_\infty(z^-) - \frac{1}{\partial} I'_\infty(z^-)z^- \). Consequently
\[ 2c_\infty \leq \liminf_{n \to \infty} \{ I(u_n^+) + I(u_n^-) \} = \lim_{n \to \infty} I(u_n) = \hat{c} < c_1 + c_\infty \]
which is an absurd. \( \square \)

5. Proof of lemmas and propositions

In this section, we will prove some lemmas and propositions used in Section 3.

**Proof of Proposition 3.1.** Let \( \bar{u} \) be a positive ground state solution of problem \((P_\infty)\) and define \( u_n(x) = \bar{u}(x - x_n), x_n = (0, \ldots, n) \). By the characterization of \( c_1 \)
given in (3.1),
\[ c_1 \leq \max_{t \geq 0} I(tu_n). \]
Let \( \gamma_n \in (0, \infty) \) such that
\[ I(\gamma_n u_n) = \max_{t \geq 0} I(tu_n), \]
then
\[ c_1 \leq I(\gamma_n u_n) \]
\[ = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} (|\gamma_n \nabla u_n|^2 + |\gamma_n u_n|^2) - \int_{\mathbb{R}^2 \setminus \Omega} Q(x) F(\gamma_n u_n) \]
\[ = I_\infty(\gamma_n u_n) - \frac{1}{2} t_n \gamma_n^2 + \int_\Omega \tilde{Q} F(\gamma_n u_n) + \int_{\mathbb{R}^2 \setminus \Omega} (\tilde{Q} - Q) F(\gamma_n u_n), \quad (5.1) \]
where
\[ t_n = \int_\Omega (|\nabla u_n|^2 + |u_n|^2). \]
Now, notice that \( I(\gamma_n u_n) = \max_{t \geq 0} I(tu_n) \) if and only if
\[ \int_{\mathbb{R}^2 \setminus \Omega} (|\nabla u_n|^2 + |u_n|^2) = \int_{\mathbb{R}^2 \setminus \Omega} Q(x) \frac{f(\gamma_n u_n)}{(\gamma_n u_n)^2} u_n^2. \quad (5.2) \]
It is not difficult to see that the sequence \((\gamma_n)\) is bounded and that \( \gamma_n \to 1 \), for some subsequence still denoted by \((\gamma_n)\). By (2.1) and (5.1)
\[ c_1 \leq I_\infty(\bar{u}) - t_n \left( \frac{\gamma_n^2}{2} - \mathcal{O}(\varepsilon) \right) + C_\varepsilon \int_\Omega \gamma_n u_n \left( e^{4\pi\gamma_n \tau_n^2} - 1 \right) \tilde{Q} \, dx \]
\[ + \int_{\mathbb{R}^2 \setminus \Omega} (\tilde{Q} - Q) F(\gamma_n u_n) \, dx \]
thus,
\[ c_1 \leq I_\infty(\bar{u}) - C t_n + s_n, \]
where $C$ is a positive constant and

$$s_n = C \int_{\Omega} u_n(e^{4\pi\gamma_n^\tau u_n^2} - 1) \bar{Q} \, dx + \int_{\mathbb{R}^2 \setminus \Omega} (\bar{Q} - Q) F(\gamma_n^\tau u_n).$$

We claim that

$$\frac{s_n}{t_n} \to 0. \quad (5.3)$$

Indeed, by Theorem 2.1

$$t_n = \int_{\Omega} (|\nabla u_n|^p + |u_n|^p) \geq \int_{\Omega} |u_n|^p \geq Ce^{-2an}.$$

Estimate of $s_n$:

Fix $R > 0$ such that $\Omega \subset B_R(0)$ and observe that

$$\int_{\Omega} u_n(e^{4\pi\gamma_n^\tau u_n^2} - 1) \, dx = \int_{\Omega_n} \bar{u}(e^{4\pi\gamma_n^\tau u_n^2} - 1) \, dx,$$

where $\Omega_n = \Omega + x_n$. Consequently,

$$\int_{\Omega} u_n(e^{4\pi\gamma_n^\tau u_n^2} - 1) \, dx \leq Ce^{-bn}(e^{4\pi\tau_1 e^{-2nb}} - 1),$$

where $\gamma_n^\tau \leq \tau_1 \forall n \in \mathbb{N}$. Note that,

$$\int_{\mathbb{R}^2 \setminus \Omega} (\bar{Q} - Q) F(\gamma_n^\tau u_n) = \int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| > r_n\}} (\bar{Q} - Q) F(\gamma_n^\tau u_n)$$

$$+ \int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| \leq r_n\}} (\bar{Q} - Q) F(\gamma_n^\tau u_n),$$

where $r_n = (1 - r)n$ with $r > 0$ and $r$ near 0. From $(Q_2)$, it follows that

$$\int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| > r_n\}} (\bar{Q} - Q) F(\gamma_n^\tau u_n) \leq Ce^{-mr_n}.$$

From conditions $(f_1)$ and $(f_3)$, it follows that

$$|F(s)| \leq c|s|^{q+1} + Ce^{e^{4\pi\tau s^2} - 1}|s| \quad \forall s \in \mathbb{R},$$
\[ \int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| \leq r_n\}} (\bar{Q} - Q) F(\gamma_n u_n) \leq C_1 e \int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| \leq r_n\}} |u_n|^{q+1} \]
\[ + C_2 \int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| \leq r_n\}} u_n (e^{4\pi \tau_1 u_n^2} - 1) \, dx. \]

Therefore,
\[ \int_{(\mathbb{R}^2 \setminus \Omega) \cap \{|x| \leq r_n\}} (\bar{Q} - Q) F(\gamma_n u_n) \leq C e^{b(q+1)r_n n^2} + C e^{-br_n (e^{4\pi \tau_1 e^{-2bn}} - 1)n^2}. \]

By the estimates obtained above
\[ \frac{s_n}{t_n} \leq \frac{C e^{-bn (e^{4\pi \tau_1 e^{-2nb}} - 1)}}{e^{-2na}} + \frac{C e^{-mr_n}}{e^{-2an}} + \frac{C e^{-b(q+1)r_n n^2}}{e^{-2na}} + \frac{C e^{-br_n (e^{4\pi \tau_1 e^{-2bn}} - 1)n^2}}{e^{-2na}}, \]
and since \( \frac{a}{b} \to 1 \) as \( \delta \to 0 \) (see Theorem 2.1), we obtain
\[ \frac{s_n}{t_n} \to 0. \]

From (5.3), it follows that \( c_1 < c_\infty \).

**Proof of Proposition 3.2.** By hypotheses
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{U_{R,y}} |u_n|^2 = 0, \]

and by an inequality of Trudinger–Moser type found in [7], there exist \( \tau, q > 1 \), sufficiently close to 1 such that the sequence
\[ f_n(x) = e^{4\pi \tau v_n^2} - 1 \quad \forall x \in \mathbb{R}^2 \]
belongs to $L^q(\mathbb{R}^2)$ and there exists $C > 0$ such that $|f_n|_q \leq C$ for all $n \in \mathbb{N}$. Therefore, the sequence

$$h_n(x) = e^{4\pi^2u_n^2 - 1} \quad x \in \mathbb{R}^2 \setminus \Omega$$

belongs to $L^q(\mathbb{R}^2 \setminus \Omega)$ and there exists $C > 0$ such that $|h_n|_q \leq C$ for all $n \in \mathbb{N}$. On the other hand, we have

$$\int_{\mathbb{R}^2 \setminus \Omega} f(u_n)u_n \leq e \int_{\mathbb{R}^2 \setminus \Omega} u_n^2 + C e \int_{\mathbb{R}^2 \setminus \Omega} u_n(e^{4\pi^2u_n^2} - 1)$$

which implies that

$$\int_{\mathbb{R}^2 \setminus \Omega} f(u_n)u_n \leq eC + C \left\{ \int_{\mathbb{R}^2 \setminus \Omega} |u_n|^{q'} \right\}^{q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$ 

From this, we infer that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2 \setminus \Omega} f(u_n)u_n = 0.$$ 

By similar arguments,

$$\lim_{n \to \infty} \int_{\mathbb{R}^2 \setminus \Omega} F(u_n) = 0. \quad \square$$

**Proof of Proposition 3.3.** Using analogous arguments explored in [4], we get

$$c_\infty < \frac{(\theta - 2)}{4\xi^2 \theta}.$$

On the other hand, using the hypotheses involving the sequence $\{u_n\}$, we have

$$\frac{(\theta - 2)}{2\theta} \limsup_{n \to \infty} \|u_n\|^2 \leq c_1.$$

Thus, there exists $n_\alpha \in \mathbb{N}$ such that

$$\|u_n\| < \frac{1}{\sqrt{2} \xi} \quad \forall n \geq n_\alpha.$$
Denoting $v_n = E u_n$, we have that

$$
\| v_n \|_{H^1(\mathbb{R}^2)} \leq \xi \| u_n \|
$$

and then,

$$
\| v_n \|_{H^1(\mathbb{R}^2)} < \frac{1}{\sqrt{2}} < 1 \quad \forall n \geq n_0.
$$

Using similar arguments explored in [4], it follows that

$$
\int_{\mathbb{R}^2 \setminus \Omega} f(u_n) v \to \int_{\mathbb{R}^2 \setminus \Omega} f(u) v \quad \forall v \in H^1(\mathbb{R}^2 \setminus \Omega),
$$

where $u$ is the weak limit of $\{u_n\}$. The last limit implies that $u$ is a critical point of $I$. Now, let us show that $u$ is nonzero. Assuming by contradiction that $u = 0$, we have two situations to consider:

(I) \quad \lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{U_{R,y}} |u_n|^2 = 0

or

(II) \quad \text{There exist } \eta > 0 \text{ and } y_n \in \mathbb{R}^2 \text{ such that } \liminf_{n \to \infty} \int_{U_{R,y_n}} |u_n|^2 \geq \eta.

We will show that the aforementioned cases (I) and (II) do not hold, thus we can conclude that $u \neq 0$.

**Analysis of (I):** If (I) holds, by Proposition 2.2, we get

$$
\lim_{n \to \infty} \int_{\mathbb{R}^2 \setminus \Omega} f(u_n) u_n = 0.
$$

This fact implies that $\| u_n \| \to 0$, which is an absurd, because $I(u_n) \to c_1 > 0$. Therefore, (I) does not hold.

**Analysis of (II):** Let $w_n(x) = u_n(x + y_n)$ for $x \in \mathbb{R}^2 \setminus \Omega_n$ where $\Omega_n = \Omega - y_n$. From Sobolev imbedding, we have that $|y_n| \to \infty$. Hence the limit set related to $\mathbb{R}^2 \setminus \Omega_n$ as $n$ goes to infinity is $\mathbb{R}^2$. Notice also that $\{w_n\}$ is bounded in $H^1_{\text{loc}}(\mathbb{R}^2)$ and its weak limit $w$ is different from zero. Denoting $\tilde{w}_n = E w_n$, it follows that

$$
\| \tilde{w}_n \|_{H^1(\mathbb{R}^2)} \leq \xi \| w_n \| \| \Theta_n \|.
$$
where $\Theta_n = \mathbb{R}^2 \setminus \Omega_n$. Then,

$$\|\widehat{w}_n\|_{H^1(\mathbb{R}^2)} \leq \xi \|u_n\| < \frac{1}{\sqrt{2}} \quad \forall n \in \mathbb{N}. $$

Using similar arguments explored in the previous results, we conclude that $w$ is a critical point of the functional $I_\infty$ and $w_n \to w$ in $H^1_{\text{loc}}(\mathbb{R}^2)$. Thus, by Fatou’s lemma

$$c_\infty \leq I_\infty(w) = I_\infty(w) - \frac{1}{\theta} I'_\infty(w)w \leq \liminf_{n \to \infty} I(u_n) = c_1 < c_\infty$$

which is an absurd, and (II) also does not hold.

The equality $I(u_1) = c_1$ follows from definition of $c_1$ and of limit

$$\liminf_{n \to \infty} I(u_n) \leq c_1. \quad \square$$

**Proof of Proposition 3.4.** Let $\bar{u}$ be a ground state solution of $(P_\infty)$ and $u_1$ is a positive ground state of $(P)$. Let us define $\bar{u}_n(x) = \bar{u}(x - x_n)$, where $x_n = (0, \ldots, 0, n)$ and for $\alpha, \beta > 0$

$$h^\pm(\alpha, \beta, n) = \int_{\mathbb{R}^2 \setminus \Omega} |\nabla (\alpha u_1 - \beta \bar{u}_n)^\pm|^2 + |(\alpha u_1 - \beta \bar{u}_n)^\pm|^2$$

$$- \int_{\mathbb{R}^2 \setminus \Omega} Qf((\alpha u_1 - \beta \bar{u}_n)^\pm)(\alpha u_1 - \beta \bar{u}_n)^\pm.$$}

Since

$$\int_{\mathbb{R}^2 \setminus \Omega} (|\nabla u_1|^2 + u_1^2) - \int_{\mathbb{R}^2 \setminus \Omega} Qf(u_1)u_1 = 0,$$

by $(f_3)$ it yields that

$$\int_{\mathbb{R}^2 \setminus \Omega} \left(\frac{1}{2} |\nabla u_1|^2 + \frac{1}{2} u_1^2 \right) - \int_{\mathbb{R}^2 \setminus \Omega} Qf \left(\frac{1}{2} u_1 \right) \frac{1}{2} u_1$$

$$= \int_{\mathbb{R}^2 \setminus \Omega} Q \left( \frac{f(u_1)}{u_1} - \frac{f(\frac{1}{2} u_1)}{\frac{1}{2} u_1} \right) \left(\frac{u_1}{2}\right) > 0,$$
and

\[
\int_{\mathbb{R}^2 \setminus \Omega} (|2\nabla u_1|^2 + |2u_1|^2) - \int_{\mathbb{R}^2 \setminus \Omega} Q f(2u_1)2u_1
\]

\[
= \int_{\mathbb{R}^2 \setminus \Omega} Q \left( \frac{f(u_1)}{u_1} - \frac{f(2u_1)}{2u_1} \right) (2u_1)^2 < 0.
\]

Thus, for \(n\) large enough we get

\[
\int_{\mathbb{R}^2 \setminus \Omega} \left( \left| \frac{1}{2} \nabla \tilde{u}_n \right|^2 + \left| \frac{1}{2} \tilde{u}_n \right|^2 \right) - \int_{\mathbb{R}^2 \setminus \Omega} Q(x) f \left( \frac{1}{2} \tilde{u}_n \right) \frac{1}{2} \tilde{u}_n > 0,
\]

and

\[
\int_{\mathbb{R}^2 \setminus \Omega} (|2\nabla \tilde{u}_n|^2 + |2\tilde{u}_n|^2) - \int_{\mathbb{R}^2 \setminus \Omega} Q(x) f(2\tilde{u}_n)2\tilde{u}_n < 0.
\]

Since, \(\tilde{u}(x) \to 0\) as \(|x| \to \infty\), there exists \(n_\alpha > 0\) such that

\[
\begin{align*}
& \left\{ \begin{array}{l}
 h^+(\frac{1}{2}, \beta, n) > 0, \\
 h^+(2, \beta, n) < 0,
\end{array} \right.
\end{align*}
\]

(5.4)

for \(n \geq n_\alpha\) and \(\beta \in \left[ \frac{1}{2}, 2 \right]\). Now, for all \(\alpha \in \left[ \frac{1}{2}, 2 \right]\) we have

\[
\begin{align*}
& \left\{ \begin{array}{l}
 h^+(\alpha, \frac{1}{2}, n) > 0, \\
 h^+(\alpha, 2, n) < 0.
\end{array} \right.
\end{align*}
\]

(5.5)

By the Mean Value Theorem (see [13]), there exist \(\alpha^*, \beta^*\) such that \(\frac{1}{2} \leq \alpha^*, \beta^* \leq 2\) and

\[ h^\pm(\alpha^*, \beta^*, n) = 0 \quad \text{for } n \geq n_\alpha, \]

that is

\[
\alpha^* u_1 - \beta^* \tilde{u}_n \in \mathcal{B} \quad \text{for } n \geq n_\alpha.
\]

Hence, we only need to verify that

\[
\sup_{\frac{1}{2} \leq \alpha, \beta \leq 2} I(\alpha u_1 - \beta \tilde{u}_n) < c_1 + c_\infty \quad \text{for } n \geq n_\alpha.
\]
Indeed, since
\[
I(\varepsilon u_1 - \beta \tilde{u}_n) = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \varepsilon u_1 - \beta \nabla \tilde{u}_n|^2 + |\varepsilon u_1 - \beta \tilde{u}_n|^2 \\
- \int_{\mathbb{R}^2 \setminus \Omega} Q(x) F(\varepsilon u_1 - \beta \tilde{u}_n),
\]
using Lemma 3.1, we get
\[
I(\varepsilon u_1 - \beta \tilde{u}_n) \leq \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\nabla (\varepsilon u_1) - \nabla (\beta \tilde{u}_n)|^2 + \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\varepsilon u_1 - \beta \tilde{u}_n|^2 - I_1,
\]
where
\[
I_1 = \int_{\mathbb{R}^2 \setminus \Omega} Q F(\varepsilon u_1) + \int_{\mathbb{R}^2 \setminus \Omega} Q F(\beta \tilde{u}_n) - 2 \int_{\mathbb{R}^2 \setminus \Omega} f(\varepsilon u_1) \beta \tilde{u}_n + \varepsilon u_1 f(\beta \tilde{u}_n).
\]
Since $u_1$ is a solution of $(P)$ and $\tilde{u}_n$ depends of a ground state of $(P_\infty)$, we have
\[
I(\varepsilon u_1 - \beta \tilde{u}_n) \leq I(\varepsilon u_1) + I_\infty(\beta \tilde{u}_n) - \int_{\mathbb{R}^2 \setminus \Omega} (Q - \tilde{Q}) F(\beta \tilde{u}_n) \\
+ C_1 \int_{\mathbb{R}^2 \setminus \Omega} (f(u_1) \beta \tilde{u}_n + \varepsilon u_1 f(\beta \tilde{u}_n)) + \int_{\Omega} \tilde{Q} F(\beta \tilde{u}_n).
\]
Therefore, we conclude that
\[
\sup_{\frac{1}{2} \leq \varepsilon, \beta \leq 2} I(\varepsilon u_1 - \beta \tilde{u}_n) \leq \sup_{\varepsilon \geq 0} I(\varepsilon u_1) + \sup_{\beta \geq 0} I_\infty(\beta \tilde{u}_n) \\
- \int_{\mathbb{R}^2 \setminus \Omega} (Q - \tilde{Q}) F\left(\frac{1}{2} \tilde{u}_n\right) \\
+ C_1 \int_{\mathbb{R}^2 \setminus \Omega} (f(\varepsilon u_1) \beta \tilde{u}_n + \varepsilon u_1 f(\beta \tilde{u}_n)) \\
+ \int_{\Omega} \tilde{Q} F(2\tilde{u}_n). \tag{5.6}
\]
Now, by $(Q_3)$, we obtain
\[
\int_{\mathbb{R}^2 \setminus \Omega} (Q - \tilde{Q}) F\left(\frac{1}{2} \tilde{u}_n\right) \geq C e^{-\mu n}, \tag{5.7}
\]
and by \((f_1)\) we get
\[
\int_{\Omega} \tilde{Q} F(\bar{u}_n) \leq C e^{-n b} \left( e^{4 \pi n e^{-2n b}} - 1 \right) + e^{-b(q+1)n}.
\] (5.8)

On the other hand, one has
\[
\int_{\mathbb{R}^2 \setminus \Omega} f(u_1) \bar{u}_n \leq e \int_{\mathbb{R}^2 \setminus \Omega} |u_1|^q |\bar{u}_n| + C \int_{\mathbb{R}^2 \setminus \Omega} \left( e^{4 \pi n u_1^2} - 1 \right) \bar{u}_n.
\]

Notice that
\[
\int_{\mathbb{R}^2 \setminus \Omega} |u_1|^q |\bar{u}_n| = \int_{\mathcal{O}_n^1} |u_1|^q |\bar{u}_n| + \int_{\mathcal{O}_n^2} |u_1|^q |\bar{u}_n|
\]
and
\[
\int_{\mathbb{R}^2 \setminus \Omega} \left( e^{4 \pi n u_1^2} - 1 \right) \bar{u}_n = \left( \int_{\mathcal{O}_n^1} + \int_{\mathcal{O}_n^2} \right) \left( e^{4 \pi n u_1^2} - 1 \right) \bar{u}_n,
\]
where \(\mathcal{O}_n^1 = (\mathbb{R}^2 \setminus \Omega) \cap \{|x| < \frac{1}{q+1} n\}\) and \(\mathcal{O}_n^2 = (\mathbb{R}^2 \setminus \Omega) \cap \{|x| \geq \frac{1}{q+1} n\}\). Thus
\[
\int_{\mathbb{R}^2 \setminus \Omega} |u_1|^q |\bar{u}_n| \leq C_1 e^{-n \frac{q}{q+1} b n}
\] (5.9)
and
\[
\int_{\mathbb{R}^2 \setminus \Omega} \left( e^{4 \pi n u_1^2} - 1 \right) \bar{u}_n \leq C e^{-n \frac{q}{q+1} b n} + C \left( e^{4 \pi n e^{-2n b}} - 1 \right),
\]
hence
\[
\int_{\mathbb{R}^2 \setminus \Omega} f(u_1) \bar{u}_n \leq C_1 e^{-n \frac{q}{q+1} b n} + C \left( e^{4 \pi n e^{-2n b}} - 1 \right).
\] (5.10)

Using similar arguments, we get
\[
\int_{\mathbb{R}^2 \setminus \Omega} u_1 f(\bar{u}_n) \leq C e^{-n \frac{b n}{q+1}} + C \left( e^{4 \pi n e^{-2n b}} - 1 \right).
\] (5.11)
From (5.6)–(5.11), we have for $n$ large enough

$$
\sup_{\frac{1}{2} \leq x, \beta \leq 2} I(zu_1 - \beta \bar{u}_n) < \sup_{x \geq 0} I(zu_1) + \sup_{\beta \geq 0} I_\infty(\beta \bar{u}_n)
$$

$$
= c_1 + c_\infty.
$$

Consequently

$$
\hat{c} < c_1 + c_\infty,
$$

which proves the proposition. □

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References