Topology and its Applications 156 (2009) 1776–1785

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Centerlines of regions in the sphere

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A R T I C L E   I N F O

Article history:
Received 1 September 2008
Received in revised form 3 March 2009

Keywords:
Center and centerline
Region

A B S T R A C T

Given a region U in the 2-sphere $\mathbb{S}$ such that the boundary of U contains at least two points, let $\mathcal{D}(U)$ be the collection of open circular disks (called maximal disks) in U whose boundary meets the boundary of U in at least two points and let $\mathcal{U}_2$ be the collection of all regions $U \subset \mathbb{S}$ such that for each $D \in \mathcal{D}(U)$, D meets the boundary of U in at most two points. In this paper we study geometric properties of regions $U \in \mathcal{U}_2$. We show for such U that the centerline (i.e., the set of centers of maximal disks) is always a smooth, connected 1-manifold. We also show that the boundary of U has at most two components and, if it has exactly two components, then the boundary is locally connected.

These results are related the set of points $E(X, Y)$ which are equidistant to two disjoint closed sets X and Y. In particular we investigate when the equidistant set is a 1-manifold.

0. Introduction

In this paper we discuss two notions from elementary geometry in a more general setting. Generalizing the notion of a diameter of a circle we define the centerline of certain more general regions of the two-sphere and show that under quite general conditions the centerline is a smooth one-manifold. The preparatory work is the study of the equidistant set of two disjoint closed sets, which is a natural generalization of the (perpendicular) bisector.

Given an region U in the 2-sphere $\mathbb{S}$ such that the boundary of U contains at least two points, let $\mathcal{D}(U) = \{D_\alpha\}_{\alpha \in \mathcal{A}}$ be the collection of open circular disks D_α, called *balls*, in U such that the boundary of D_α meets the boundary ∂U of U in at least two points. Hence each D_α is a maximal ball (with respect to containment) in U. Our work is related to the following results. For each $D_\alpha \in \mathcal{D}(U)$ let F_α be the convex hull of $\overline{D_\alpha} \cap \partial U$. It is known that the collection $\mathcal{F}(U) = \{F_\alpha\}_{\alpha \in \mathcal{A}}$ foliates U. Here one can either use the hyperbolic convex hull (see [9]) or the Euclidian convex hull (see [5]). If we use the hyperbolic convex hull, and use on each F_α the restriction of the hyperbolic metric on the disk D_α, one obtains the so-called K–P metric on U (see [9,8,7]). This metric is invariant under Möbius transformations of the 2-sphere. In [7], it is show that for regions $U \subset \mathbb{S}$ with the property that there is an open circular disk D in the region U such that the boundary of D meets the boundary of U in at least three points, the only isometries of the K–P metric are restrictions of Möbius transformations. For a region $U \subset \mathbb{S}$, let $E_2(U)$ be the set of centers of disks D_α in $\mathcal{D}(U)$. By results in [3], $E_2(U)$ is always locally an $\mathbb{R}$-tree (see also [4] and [6] for related results).

Let $\mathcal{U}_2$ denote the collection of regions $U \subset \mathbb{S}$ with the property that for every open circular disk D in U the boundary of D meets the boundary of U in at most two points. Simple examples of regions in $\mathcal{U}_2$ are an ellipse and a region whose boundary consists of two disjoint round circles. In a subsequent paper a survey of possible regions will be presented.

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[1] The author was supported in part by NSF-DMS-0405774.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.03.007

In this paper we show that for each $U \in \mathcal{U}_2$, the set of centers $E_2(U)$ is a non-empty, connected, smooth one-manifold. This last result is closely related to results about equidistant sets (see Definition 1.3). In [1], Bell proves that the equidistant set of two disjoint continua in the plane is a connected one-manifold. In [3] it is shown that the equidistant set of two disjoint closed sets in $\mathbb{S}$ or $\mathbb{R}^2$ is always locally connected. Under an extra condition of the closed sets being noninterlaced (a condition that is fulfilled in the case of disjoint continua), we show in Section 2 that the equidistant set is a one-manifold. This result is crucial for the results in [10] where it is shown that every isotopy of a plane continuum can be extended to an isotopy of the entire plane. We also study topological properties of the boundary of U. Some of the results in this paper were first obtained in [3] which also contains additional related results.

1. Preliminaries

We think of the two-dimensional sphere $\mathbb{S}$ as the boundary of the unit ball in Euclidian three-space with the origin as its center. We shall use spherical coordinates: a point $p \in \mathbb{S}$ has coordinates (φ, ϑ) where φ is the angle between the position vector $\vec{p}$ and the *xy-plane*, $-\pi/2 \leqslant \varphi \leqslant \pi/2$ and ϑ is the angle from the positive x-axis to the projection of the position vector $\vec{p}$ of p onto the xy-plane, $-\pi \leqslant \vartheta < \pi$.

Let x and y be two points in $\mathbb{S}$. The *great circle* through x and y is the intersection of the sphere and a plane through the points x, y and the origin; this is also called a *straight line* through x and y. By a (*straight*) *line segment* between x and y we mean a part of a straight line between x and y that has shortest length. The length of the segment between x and y is the *spherical distance* between x and y; note that the spherical distance is measured in radians and that its value lies between 0 and π. If $\vec{x}$ and $\vec{y}$ are the position vectors of x and y, then the spherical distance of x and y is equal to $\arccos(\vec{x} \cdot \vec{y})$.

A spherical triangle of which each side has length less than π is called an Euler triangle. For an Euler triangle the usual triangle inequalities hold: if the sides of an Euler triangle have length a, b and c respectively, then $|a-b| \leqslant c \leqslant a+b$. The former inequality entails the continuity of the spherical distance function.

Notation 1.1. Consider two disjoint sets X and Y in the sphere $\mathbb{S}$. Let $Z = X \cup Y$ denote the union of X and Y. For any $w \in \mathbb{C}$ let $B(w, Z)$ be the maximal open ball centered at w that is disjoint from Z. If such a ball does not exist let $B(w, Z) = \emptyset$. Furthermore, in the case that $B(w, Z)$ is non-empty, let $S(w, Z) = \partial B(w, Z)$, the boundary of $B(w, Z)$. If $B(w, Z) = \emptyset$, define $S(w, Z) = \{w\}$. Note that $\overline{Z} \cap S(w, Z) \neq \emptyset$ *for all* $w \in \mathbb{C}$.

Definition 1.2. Suppose X and Y are disjoint closed subsets of $\mathbb{S}$. Let $Z = X \cup Y$. For any point $w \in \mathbb{S} \setminus Z$ we shall say that X and Y are *noninterlaced with respect to* w if there exist disjoint continua J_X and J_Y in $S(w, Z)$ such that $X \cap S(w, Z) \subset J_X$ and $Y \cap S(w, Z) \subset J_Y$. If X and Y are noninterlaced with respect to every point $w \in \mathbb{S} \setminus Z$, we will simply say that X and Y are *noninterlaced*. Note that J_X or J_Y may be empty.

Definition 1.3. Suppose X and Y are disjoint non-empty subsets of $\mathbb{S}$. Let

$$L(X, Y) = \{w \in \mathbb{S}\colon\ d(w, X) < d(w, Y)\}, \qquad E(X, Y) = \{w \in \mathbb{S}\colon\ d(w, X) = d(w, Y)\}.$$

We call $E(X, Y)$ the equidistant set of X and Y. Note that $L(X, Y)$ is open and $E(X, Y)$ is closed.

Notation 1.4. Let V be a set in $\mathbb{S}$. Then for any $\delta > 0$ let $B_\delta(V) = \{x \in \mathbb{S}\colon\ d(x, V) < \delta\}$. If $V = \{x\}$ we write $B_\delta(x)$ instead of $B_\delta(\{x\})$. Instead of $B_\delta(B(w, Z))$ we will write $B(w, Z + \delta)$.

Lemma 1.5 *(Collar). Let $Z \subset \mathbb{S}$ be the union of two noninterlaced closed sets $X, Y \subset \mathbb{S}$. Pick a point $w \in \mathbb{S} \setminus Z$ and consider the ball $B(w, Z)$. Let J_X and J_Y be as in Definition* 1.2. *Choose $\varepsilon > 0$ such that $B_\varepsilon(J_X)$ and $B_\varepsilon(J_Y)$ are disjoint.*

Then there exists $\delta > 0$ such that

$$B(w, Z + \delta) \cap X \subset B_\varepsilon(J_X) \quad \text{and} \quad B(w, Z + \delta) \cap Y \subset B_\varepsilon(J_Y).$$

Proof. As J_X and J_Y are compact and disjoint there exists an ε as required. By the compactness of X and Y the existence of δ follows. □

Definition 1.6. Let K be a closed subset of the sphere $\mathbb{S}$. For any point $x \in \mathbb{S}$ we define the set of closest points $c_K(x)$ by $c_K(x) = S(x, K) \cap K$.

Lemma 1.7. *Let Z be the disjoint union of two non-empty closed subsets X and Y of the sphere $\mathbb{S}$. Then we can equivalently define $L(X, Y)$ and $L(Y, X)$ as follows*:

$$L(X, Y) = \{w \in \mathbb{S}\colon\ c_Z(w) \subset X\}, \qquad L(Y, X) = \{w \in \mathbb{S}\colon\ c_Z(w) \subset Y\}.$$

Lemma 1.8. *Let $C \subset \mathbb{S}$ be a round circle with center m and let l be a straight line l going through m. Denote the points where this line intersects the circle by b and $-b$. Consider a point a on the straight line segment $(m, b]$. Then for any point $c \in C$ such that $c \neq b$ we have that $d(a, b) < d(a, c)$. Furthermore, the distance $d(a, c)$ strictly increases when the point $c \in C$ moves along the circle from b to $-b$ (in either way).*

Proof. Choose a local coordinate system in three space such that the origin coincides with the center of the sphere, m lies on the positive x-axis, the circle C is parallel to the yz-plane, the y-coordinate of b is zero, and the z-coordinate of b is positive. Then $b = (b_1, 0, b_3)$ and $a = (a_1, 0, a_3)$. Now pick a point $c \in C$ such that $c \neq b$, then $c = (b_1, c_2, c_3)$ with $c_3 < b_3$. Then $d(a, b) = \arccos(a_1 b_1 + a_3 b_3)$ and, similarly, $d(a, c) = \arccos(a_1 b_1 + a_3 c_3)$. Since arccos is a strictly decreasing function and since $b_3 > c_3$ we have that $d(a, b) < d(a, c)$. The second statement of the lemma follows analogously. □

Corollary 1.9. *If, in the above lemma, we choose the point $a \in [-b, m)$, then $d(a, c)$ is strictly decreasing as the point c moves from b to $-b$ along the circle. Clearly, if $a = m$, $d(a, c)$ is constant.*

Corollary 1.10. *Let K be a closed set in the sphere $\mathbb{S}$ and let $w \in \mathbb{S} \setminus K$. Furthermore let $x \in c_K(w)$. Then for every p on the straight line segment $(w, x]$ we have that $B(p, K) \subset B(w, K)$ and $c_K(p) = \{x\}$.*

Lemma 1.11 *(Non-crossing Lemma). Let x and y be two points in $\mathbb{S} \setminus K$ such that $x \neq y$. Furthermore let $c_x \in c_K(x)$ and $c_y \in c_K(y)$. Then one of the following three situations holds:*

(i) $[x, c_x) \cap [y, c_y) = \emptyset$,
(ii) $[y, c_y) \subset [x, c_x)$,
(iii) $[x, c_x) \subset [y, c_y)$.

Proof. Consider two cases $c_x \neq c_y$ and $c_x = c_y$. The case $c_x = c_y$ is trivial. If $c_x \neq c_y$ then the segments $[x, c_x]$ and $[y, c_y]$ are disjoint by Corollary 1.10. □

Lemma 1.12. *Let Z be the union of two disjoint closed sets X and Y in the sphere $\mathbb{S}$, and let $w \in E(X, Y)$. Choose $x \in c_X(w)$ and $y \in c_Y(w)$. Then the straight line segment $(w, x]$ is contained in $L(X, Y)$ and the straight line segment $(w, y]$ is contained in $L(Y, X)$. Secondly, if $w \in L(X, Y)$ and $x \in c_X(w)$, then the line segment $[w, x]$ is contained in $L(X, Y)$. A similar statement holds for points in $L(Y, X)$.*

Proof. Let $p \in (w, x]$; by Lemma 1.10 we have that $B(p, X) \subset B(w, X) = B(w, Z)$ and $c_X(p) = \{x\}$. Therefore $c_Z(p) = \{x\}$ and hence by Lemma 1.7 we have that $p \in L(X, Y)$. Similarly one can show that $(w, y]$ is contained in $L(Y, X)$. The second part of the lemma follows analogously. □

Lemma 1.13. *Let Z be the union of two disjoint closed sets X and Y in the sphere $\mathbb{S}$. Then $L(X, Y)$ is connected if X is connected. Similarly, if Y is connected then $L(Y, X)$ is connected.*

Proof. Let X be connected and let C be a component of $L(X, Y)$. We want to show that $X \cap C \neq \emptyset$. Let $w \in C \subset L(X, Y)$, then there exists an $x \in c_Z(w) \subset X$. By Lemma 1.12 we must have that the straight line segment $L = [w, x]$ (which could be degenerate) is contained in $L(X, Y)$. Hence $L \subset C$ and $x \in C$. So $X \cap C \neq \emptyset$ and since $X \subset L(X, Y)$ it follows that $L(X, Y)$ is connected. The second statement in the lemma follows similarly. □

The following result follows directly from the continuity of the distance function.

Lemma 1.14 *(USC Lemma). Let K be a closed subset of the sphere $\mathbb{S}$ and let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence converging to a point x_∞. Then $\limsup c_K(x_i) \subset c_K(x_\infty)$.*

Proof. Let $p \in \limsup c_K(x_i)$, then there exist a sequence $\{y_j\}_{j \in \mathbb{N}}$ and a subsequence $\{i_j\}_{j \in \mathbb{N}}$ of $\mathbb{N}$ such that $y_j \in c_K(x_{i_j})$ and $\lim y_j = p$. By continuity of the distance function it follows that $\lim_{j \to \infty} d(x_{i_j}, y_j) = d(x_\infty, p)$. Let $r = d(x_\infty, K)$. We want to show that $d(x_\infty, p) = r$ (whence $p \in C_K(x_\infty)$). Clearly $d(x_\infty, p) \geqslant r$ since $p \in K$. Now suppose, by way of contradiction, that $d(x_\infty, p) = r + \varepsilon$ for some $\varepsilon > 0$. There exists an $N \in \mathbb{N}$ such that $d(x_{i_N}, y_N) > r + 2\varepsilon/3$ and $d(x_{i_N}, x_\infty) < \varepsilon/3$. Then

$$d(x_{i_N}, K) = d(x_{i_N}, y_N) > r + \frac{2\varepsilon}{3}$$

as well as

$$d(x_{i_N}, K) \leqslant d(x_{i_N}, x_\infty) + d(x_\infty, K) < r + \frac{\varepsilon}{3}$$

a contradiction. □

2. Results

Theorem 2.1. *Let X and Y be disjoint closed subsets of $\mathbb{S}$ and let $w \in E(X,Y)$ be such that X and Y are noninterlaced with respect to w. Then there exists an $\eta > 0$ such that $E(X,Y) \cap B_\eta(w)$ is a one-manifold.*

Proof. Let $w \in E(X,Y)$ and let $Z = X \cup Y$. Let J_X and J_Y be as in Definition 1.2. Without loss of generality, we may assume that J_X and J_Y are both minimal with respect to length and that J_X is the shorter arc. Choose a spherical coordinate system (φ, ϑ) such that the origin coincides with w and such that the positive ϑ-axis passes through the midpoint of J_X. Define the northern hemisphere by $\{(\varphi,\vartheta) \mid \varphi > 0\}$. Let a_X and a_Y be the endpoints of J_X and J_Y respectively such that $d(a_X, a_Y)$ is equal to the distance of J_X and J_Y. We may assume that both a_X and a_Y are located in the northern hemisphere (after a coordinate change $\varphi \to -\varphi$ if necessary). Note that the distance between J_X and J_Y is equal to $d(a_X, a_Y)$. Now, choose ε with $0 < \varepsilon < d(a_X, a_Y)/4$. Then by Lemma 1.5 there is a $\delta > 0$ such that

$$B(w, Z+\delta) \cap X \subset B_\varepsilon(J_X) \quad \text{and} \quad B(w, Z+\delta) \cap Y \subset B_\varepsilon(J_Y).$$

Then for any point in $B_\delta(w)$ the closest point in Z to that point is in $B_\varepsilon(J_X) \cup B_\varepsilon(J_Y)$.

Suppose that $w_1 = (0, \vartheta_1)$ with $0 < \vartheta_1 < \delta/2$. Let B_1 be the closed ball with center w_1 and radius $d(w_1, a_X)$. Then we have $B_1 \cap Z \subset B_\varepsilon(J_X)$ in view of Lemma 1.8, hence $w_1 \in L(X,Y)$. In a similar fashion, suppose that $w_2 = (0, \vartheta_2)$ with $-\delta/2 < \vartheta_2 < 0$. Let B_2 be the closed ball with center w_2 and radius $d(w_2, a_Y)$. Then $B_2 \cap Z \subset B_\varepsilon(J_Y)$ and $w_2 \in L(Y,X)$. As has been noted before $L(X,Y)$ and $L(Y,X)$ are open. Hence there exists σ with $0 < \sigma < \varepsilon$ such that for all $\varphi \in (-\sigma, \sigma)$ we have $(\varphi, \vartheta_1) \in B_\delta(w) \cap L(X,Y)$ and $(\varphi, \vartheta_2) \in B_\delta(w) \cap L(Y,X)$.

For every $\varphi \in (-\sigma, \sigma)$ define

$$l_\varphi = \{(\varphi, \vartheta) \in \mathbb{C}\colon\ \vartheta_2 \leqslant \vartheta \leqslant \vartheta_1\}.$$

Claim. $|l_\varphi \cap E(X,Y)| = 1$.

Proof of Claim. Let $p = (\varphi, \vartheta_p)$ and $q = (\varphi, \vartheta_q)$ be two points in $\ell_\varphi \cap E(X,Y)$ with $\vartheta_q < \vartheta_p$. Furthermore, let $p_X \in c_X(p) \subset B_\varepsilon(J_X)$ and $p_Y \in c_Y(p) \in B_\varepsilon(J_Y)$. Then we have $d(p, p_X) = d(p, p_Y)$. In a similar fashion we define q_X and q_Y with $d(q, q_X) = d(q, q_Y)$.

Now consider the sphere $S(p, Z)$. The line pq is a diameter of $S(p, Z)$. Let p^* be the intersection of $S(p, Z)$ and the line pq so that p is on the line segment qp^*. We choose a new spherical coordinate system by translating the origin to $(\varphi, 0)$. In the new system the diameter pq falls along the ϑ axis. The set J_X need not be symmetric with respect to the new ϑ-axis, but has been shifted up or down over $\leqslant \sigma$. Let q^* be the intersection of qq_X and $S(p, Z)$ and let q^{**} be the reflection of q^* in the diameter pq. Note that $d(q, q^*) = d(q, q^{**})$. Moving counterclockwise around the circle $S(p, Z)$ from p^* we first meet q^* or q^{**} and next p_Y. In view of Lemma 1.8 we may conclude $d(q, p_Y) < d(q, q^{**})$. It follows that

$$d(q, q_X) \geqslant d\big(q, q^*\big) = d\big(q, q^{**}\big) > d(q, p_Y) \geqslant d(q, q_Y),$$

a contradiction. □

Let $K = \{(x, 0) \in \mathbb{S}\colon\ -\sigma' < x < \sigma'\}$. Then we can define a function $f : K \to E(X,Y)$ by letting $f((x,0)) = l_x \cap E(X,Y)$.

Claim. *$f : K \to E(X,Y)$ is continuous.*

Proof of Claim. Suppose, by way of contradiction, that f is not continuous. Then there exists a sequence $\{(z_i, 0)\}_{i \in \mathbb{N}}$ in K converging to a point $(z, 0) \in K$ such that $y_\infty = \lim f((z_i, 0)) \neq f((z, 0))$. But since $E(X,Y)$ is closed we must have that $y_\infty \in E(X,Y)$. This clearly contradicts the previous claim. □

This last claim establishes the fact that locally $E(X,Y)$ is the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. We can now pick $\eta = \sigma'$ so that $E(X,Y) \cap B_\eta(w)$ is a one-manifold. □

It follows from results in [3] that if X and Y are disjoint and closed, then $E(X,Y)$ is locally connected and 1-dimensional. Hence, $E(X,Y)$ is a 1-manifold if and only if it does not contain a triod. By the proof of Theorem 2.1 this is the case if and only if X and Y are noninterlaced. Hence we have the following corollary (see also [2] for related results).

Corollary 2.2. *Suppose that X and Y are disjoint closed subsets of the sphere. Then $E(X,Y)$ is a 1-manifold if and only if X and Y are noninterlaced.*

Proof. By Theorem 2.1 and the remark above, it only remains to be shown that if X and Y fail to be noninterlaced with respect to some point, then $E(X,Y)$ contains a triod. Hence suppose that X and Y fail to be noninterlaced with respect to the point w. Then there exist at least three disjoint intervals $(x_i, y_i) \subset S(w, X \cup Y)$ with end points $x_i \in X$ and $y_i \in Y$. Then for all points on the segments $[x_i, w)$, the closest point in $Z = X \cup Y$ is x_i and for all points on the segments $[y_i, w)$ the closest point is y_i. Hence the set $E(X,Y)$ must separate $B(w,Z)$ between $[y_i, w)$ and $[x_i, w$ for each i. This implies that $E(X,Y)$ must contain a triod with vertex w as required. □

Theorem 2.3 *(Bell). If X and Y are disjoint continua in $\mathbb{S}$ then $E(X,Y)$ is a connected one-manifold.*

Proof. Since X and Y are disjoint continua it follows from the Θ-Curve Theorem that they are noninterlaced. Therefore, by Theorem 2.1, $E(X,Y)$ is a one-manifold. The connectedness of $E(X,Y)$ follows from Lemma 1.13. □

We will show next that for every $U \in \mathcal{U}_2$ the boundary of U consists of at most two components. This result is also obtained independently in [8]. Since the argument used in the proof will be used later in the paper, we have included a complete proof. We will assume, by way of contradiction that the boundary of U contains more than two components. Notice that in this case $\mathbb{S} \setminus U$ has at least three components and we can divide the complement of U into three disjoint, non-empty compact sets K_1, K_2, and K_3. Each of these sets is noninterlaced with respect to the union of the other two, because each point in U has at most two closest points. So by Theorem 2.1 it follows that the three corresponding equidistant sets are one-manifolds. We then consider components of these equidistant sets, which will be simple closed curves. We will show that for every simple closed curve one of the complementary domains consists of closest points to a set K_i. After that we will show that these simple closed curves cannot intersect each other; this in turn shows that the sphere $\mathbb{S}$ is a finite union of disjoint closed disks, which clearly is a contradiction.

Theorem 2.4. *Let U be an open connected set in $\mathbb{S}$ with boundary ∂U. If for every $x \in U$ we have that $|c_{\partial U}(x)| \leqslant 2$, then ∂U consists of at most two components.*

Proof. Let $K = \mathbb{S} \setminus U$. Suppose, by way of contradiction, that ∂U consists of more than two components. Then we can write $K = K_0 \cup K_1 \cup K_2$, where K_0, K_1, K_2 are mutually disjoint compact sets. Notice that since $K = \mathbb{S} \setminus U$ and U is connected, no component of K separates the sphere. Hence K_i has exactly one complementary domain which we will call K_i^c. Also note that the condition $|c_{\partial U}(x)| \leqslant 2$ for every $x \in U$ implies that for each i K_i is noninterlaced with the sum of $K_{i\oplus 1}$ and $K_{i\oplus 2}$, where $\oplus$ denotes addition modulo 3. By Corollary 2.2 we have that $E_i = E(K_i, K_{i\oplus 1} \cup K_{i\oplus 2})$ is a 1-manifold for $i = 0, 1, 2$. Note that since $\mathbb{S}$ is compact this implies that each E_i has finitely many components. Furthermore we have that each of these components is a simple closed curve. For $i = 0, 1, 2$, denote the components of E_i by C_i^n where $n \in N_i = \{1, \ldots, k_i\}$. For every $i = 0, 1, 2$ and every $n \in N_i$ let D_i^n be the complementary domain of C_i^n that contains a point $p \in K_i$ such that $d(p, C_i^n) = d(K_i, C_i^n)$. Clearly each D_i^n is an open disk. We claim that each D_i^n consist of points closest to K_i. To see that this is true note that every point in C_i^n cannot have two closest points in $K_i^n = K_i \cap D_i^n$, since it already has a closest point in $K_{i\oplus 1} \cup K_{i\oplus 2}$ by definition of the C_i^n. Hence for each point $p \in C_i^n$ there exists a unique point $c_p \in K_i^n$ such that $d(p, c_p) = d(p, K_i)$.

For every point $p \in C_i^n$ we can consider the straight line segment to its unique closest point c_p in ∂K_i^n. It follows from Lemma 1.11 that for two distinct points p and q in C_i^n we have that $[p, c_p] \cap [q, c_q] = \emptyset$. We now claim the following

$$D_i^n \cap K_i^c \subset P = \bigcup_{p \in C_i^n} [p, c_p].$$

To see why this is true assume, by way of contradiction, that there exists an $x \in D_i^n \cap K_i^c$ such that $x \notin P$. Note that $D_i^n \cap K_i^c$ is an open connected set. Since x lies in the complementary domain of K_i^n there exists an arc A from x to a point $q \in C_i^n$ such that $A \cap K_i^n = \emptyset$. Since $q \in P$ and P is closed by Lemma 1.14, there exists a last point r, going along A from q to x, such that $r \in P$. Let $p \in C_i^n$ be such that $r \in [p, c_p]$ and let A' be the part of the arc A going from r to x. Now part of the arc $[p, c_p]$ is shielded from one side by A'. This is a contradiction since there should be arcs $[t, c_t]$ converging to the arc $[p, c_p]$ from both sides. This shows that D_i^n consist of points that are closest to K_i^n.

Since the disks D_i^n consist of closest points it is clear that if $i \neq j$ then $D_i^n \cap D_j^m = \emptyset$ for all $n \in N_i$ and $m \in N_j$.

Let $i, j \in \{0, 1, 2\}$ and let $n \in N_i$, $m \in N_j$. We claim that if $\{i, m\} \neq \{j, n\}$ then $C_i^n \cap C_j^m = \emptyset$.

To see this, first consider the case that $i = j$. In this case $m \neq n$ and it is clear that $C_i^n \cap C_j^m = \emptyset$ since E_i is a one manifold. Now consider the case that $i \neq j$ and $C_i^n \neq C_j^m$. Assume, by way of contradiction, that if $C_i^n \cap C_j^m \neq \emptyset$, then $C_i^n \not\subseteq C_j^m$. Pick a point w in $C_i^n \cap C_j^m \subset U$, such that $w \in \overline{C_i^n \setminus C_j^m} \cap C_j^m$. This point will have a closest point a in K_i^n and a closest point b in K_j^m. Now choose a sequence $\{w_k\}_k$ in $C_i^n \setminus C_j^m$ converging to w. Note that each w_k has a closest point c_k in $K_{i\oplus 1} \cup K_{i\oplus 2} \setminus D_j^m$. Let $c = \lim_{k\to\infty} c_k$, then $c \in K_{i\oplus 1} \cup K_{i\oplus 2} \setminus D_j^m$ and therefore $c \neq b$. But by Lemma 1.14 c is closest point of w, a contradiction to the assumption that $|c_{\partial U}(x)| \leqslant 2$ for all $x \in U$. The last case to consider is if $C_i^n = C_j^m$. But in this case it would follow that $\mathbb{S}$ is the disjoint union of D_i^n, $C_i^n = C_j^m$ and D_j^m. This means that for $i \neq k \neq j$ there are no points in $\mathbb{S}$ that are closest to K_k,

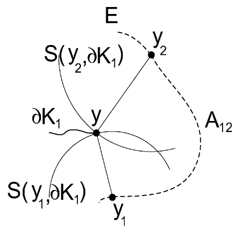

Fig. 1.

a contradiction. Hence it follows that $\mathbb{S}$ is the finite disjoint union of closed disks $\overline{D_i^n} = D_i^n \cup C_i^n$, which is a contradiction since $\mathbb{S}$ is connected. □

A natural question to ask is: *If U is an open connected subset of* $\mathbb{S}$ *with the property that each point in U has at most n closest points, does it follow that U can have at most n boundary components?* The following result answers that question negatively.

Example 2.5. We present an example of an open connected subset U of $\mathbb{S}$ with boundary ∂U with the property that $|c_{\partial U}(x)| \leqslant 3$ for all x, but yet ∂U has infinitely many components.

Proof. First we will construct a nested sequence of open sets U_n inductively such that each U_n has the following properties. Each U_n is open, for every $x \in U_n$ we have that $|c_{\partial U_n}(x)| \leqslant 3$ and ∂U_n has $n+3$ components. Furthermore, any given round circle hits U_n in at most three points. Start by choosing three points z_1, z_2, z_3 that lie on one round circle S_1. Let $U_0 = \mathbb{S} \setminus \{z_1, z_2, z_3\}$. Choose a point z_4 such that $d(z_4, z_3) < 1/2$ and such that $z_4 \notin S_1$. Let $U_1 = U_0 \setminus \{z_4\}$, clearly U_1 satisfies all mentioned properties. Suppose we have constructed U_n. Then the points $z_1, \ldots, z_{n+3}$ determine $\binom{n+3}{3}$ circles S_i. Now choose a point z_{n+4} such that $d(z_{n+4}, z_3) < 1/2^{n+1}$ and such that z_{n+4} does not lie on any of the circles S_i. Then define $U_{n+1} = U_n \setminus \{z_{n+4}\}$. We can now define a set U by $U = \bigcap_i U_i$, this set is open because it is the complement of a closed set and it has infinitely many boundary components $z_1, z_2, \ldots$. For every $x \in U$ we have that $|c_{\partial U_n}(x)| \leqslant 3$, since if this was not the case there would be a circle in $\mathbb{S}$ containing 4 points, but that would contradict the construction of the U_i's. □

Definition 2.6. Let K be a set in the sphere $\mathbb{S}$ and let $x \in K$. Furthermore let N be the cardinality of the set of components of $K \setminus \{x\}$. The point $x \in K$ is called a cutpoint of order N if $N \geqslant 2$.

Theorem 2.7. *Let U be an open connected set in* $\mathbb{S}$ *with boundary* ∂U *and suppose that for every* $x \in U$ *we have* $|c_{\partial U}(x)| \leqslant 2$ *and that* ∂U *consists of exactly two components. Let* K_1 *and* K_2 *be the components of* $\mathbb{S} \setminus U$*. Then* $\partial U = \partial K_1 \cup \partial K_2$ *is locally connected and* ∂K_i *has the property that any cutpoint is of order two. Moreover, if* $\partial K_i = K_i$ *then* K_i *is a point or an arc.*

Proof. By Theorem 2.3 $E = E(K_1, K_2)$ is a compact connected one-manifold, hence E is a simple closed curve. We show that ∂K_1 has the required properties. The proof for ∂K_2 will be analogous. Let D_1 be the complementary domain of E that contains K_1. Since every point in $\mathbb{S}$ has at most two closest points it follows that each point $x \in E$ has a unique closest point c_x in ∂K_1. This allows us to define a continuous function $f : E \to \partial K_1$ by letting $f(x) = c_x$. In a similar way as was done in the proof of Theorem 2.4 we can show:

$$\overline{D_1} = K_1 \cup X, \quad \text{where } X = \bigcup_{x \in E} [x, c_x].$$

Note that by the USC Lemma 1.14, X is closed. The straight line segments (x, c_x) foliate the complement of K_1 in D_1. This shows that the function $f : E \to \partial K_1$ is onto. Since if there would exist a $y \in \partial K_1$ such that $y \notin f(E)$ then $y \notin X$ and hence there would be an open neighborhood V of y inside D_1 such that $V \cap X = \emptyset$. But $V \cap \mathbb{S} \setminus K_1 \neq \emptyset$ contradicts the fact that $\overline{D_1} = K_1 \cup X$. Therefore ∂K_1, being the continuous image of a locally connected continuum, is a locally connected continuum. From now on we will assume that ∂K_1 is non-degenerate. For any $y \in \partial K_1$ define N_y to be the *cardinality of the set of components of* $f^{-1}(y)$.

Claim. *For every* $y \in \partial K_1$ *we have that* $N_y \leqslant 2$*. Furthermore, if* $N_y = 2$ *then* $f^{-1}(y) = \{y_1, y_2\}$ *for some* $y_1, y_2 \in E$ *and* $[y_1, y] \cup [y_2, y]$ *is a straight line segment.*

Proof of Claim. Let $y \in \partial K_1$. Without loss of generality we may assume that $f^{-1}(y)$ consists of more than one point. Let $y_1 \neq y_2 \in f^{-1}(y)$ and let A_{12} be the open arc in E between y_1 and y_2. Suppose that $[y_1, y] \cup [y_2, y]$ is not a straight line segment (see Fig. 1).

In this case we see that the boundary of K_1 has to be disjoint from the open area bounded by the arcs $[y_1, y]$, $[y_2, y]$ and A_{12}. For every point $x \in A_{12}$ we have that $(x, c_x) \cap A_{12} = \emptyset$ and $(x, c_x) \cap (y_i, y) = \emptyset$ for $i = 1, 2$. Therefore, every point in A_{12} has to have y as a closest point. So in this case $f^{-1}(y)$ is connected. This shows that $N_y \leqslant 2$ for all $y \in \partial K_1$. □

Now assume, by way of contradiction, that there exists a point $y \in \partial K_1$ such that $\partial K_1 \setminus \{y\}$ has at least three components. Label three of these components by C_1, C_2 and C_3. Each of these components is open in ∂K_1 since ∂K_1 is locally connected, therefore the components of $f^{-1}(C_i)$ are open intervals for each i. Choose three such open intervals U_1, U_2 and U_3. Then $E \setminus \{U_1 \cup U_2 \cup U_3\}$ consists of three components and since the endpoints of U_i map to y, each of these components contains a point mapping to y. So $N_y \geqslant 3$, a contradiction. Hence every cutpoint of ∂K_1 has order two.

Now assume that $\partial K_1 = K_1$. This implies that ∂K_1 cannot contain a simple closed curve and therefore that ∂K_1 is a dendrite. By the result above every cutpoint of this dendrite has order two. Hence ∂K_1 is an arc or a point. □

Definition 2.8. Let U be an open set in the sphere $\mathbb{S}$. We define $E_2(U)$ as follows:

$$E_2(U) = \{x \in U: \ |c_{\partial U}(x)| = 2\}.$$

Theorem 2.9. *Let $U \in \mathcal{U}_2$. Then $E_2(U)$ is a connected smooth non-empty one-manifold.*

Proof. The proof consists of several parts. First we show that $E_2(U)$ is non-empty. Next we prove that $E_2(U)$ is a manifold (not necessarily connected). Then we present an analysis of the foliation of U which is generated by $E_2(U)$. This analysis is used to prove the connectedness of $E_2(U)$. Finally it is shown that $E_2(U)$ is smooth.

Let $m \in \overline{U}$ such that $d(m, \partial U)$ is maximal.

Claim 1. *$m \in E_2(U)$.*

Proof of Claim 1. Suppose, by way of contradiction, that $|c_{\partial U}(m)| = 1$. Let $c_m \in c_{\partial U}(m)$ be the unique closest point of m. Consider a local spherical coordinate system such that $m = (0, 0)$ and $c_m = (0, \vartheta)$ with $\vartheta > 0$. Since ∂U is non-degenerate, the coordinates of c_m are well defined. The spherical coordinates of a point p are denoted by p_φ and p_ϑ, thus $p = (p_\varphi, p_\vartheta)$. By Lemma 1.5 there exist δ and ε such that for all $x \in B_{\delta/2}(m)$ we have that $c_{\partial U}(x) \subset B_\varepsilon(c_m)$. But then for any point $y = (0, \vartheta_y)$ with $-\delta/2 < \vartheta_y < 0$ we have by Lemma 1.8 that $d(y, \partial U) > d(m, \partial U)$, a contradiction. □

This claim shows that $E_2(U)$ is non-empty. For every point $w \in E_2(U)$ let c_w^1 and c_w^2 denote its two closest points in ∂U. For a point $w \notin E_2(U)$ let $c_w = c_w^1 = c_w^2$ denote its unique closest point.

Claim 2. *For every $x \in U \setminus \overline{E_2(U)}$ there exists a $w \in \overline{E_2(U)}$ such that $x \in [w, c_w^1) \cup [w, c_w^2)$.*

Proof of Claim 2. Let c_x denote the unique closest point of x in ∂U. Choose a local spherical coordinate system such that $x = (0, 0)$ and $c_x = (0, -\vartheta)$, where $\vartheta > 0$. Consider the set $D = \{y \geqslant 0: \ c_{\partial U}((0, y)) = \{c_x\}\}$ and let m be the supremum of D. Let $w = (0, m)$; clearly $x \in [w, c_x)$. If $w \in E_2(U)$ the claim is proved. Now assume, by way of contradiction, that $w \notin \overline{E_2(U)}$, and that w has c_x as unique closest point. Then there exists a $\xi > 0$ such that for all $y \in B_\xi(w)$ we have $|c_{\partial U}(y)| = 1$. By Lemma 1.5 there exist $0 < \varepsilon < d(x, \partial U)/10$ and a $0 < \delta < \xi$ such that for all $y \in B_\delta(w)$ we have that $c_{\partial U}(y) \subset B_\varepsilon(c_x)$. For every y in $B_\delta(w)$ let c_y denote its unique closest point. Define the following subsets of $S(w, \partial U)$:

$$R = \{p \in S(w, \partial U): \ p_\varphi > 0, \ p_\vartheta < m\}, \qquad L = \{q \in S(w, \partial U): \ q_\varphi < 0, \ q_\vartheta < m\}.$$

Clearly $c_x \in \overline{R} \cap \overline{L}$. We can define a function $f : B_\delta(w) \to \overline{R} \cup \overline{L}$ by letting $f(y) = S(w, \partial U) \cap [y, c_y]$. This function is continuous by Lemma 1.14. Let $A = \{(0, y_\vartheta) \in B_\delta(w): \ m < y_\vartheta < m + \delta\}$. Then for all $y \in A$ we must have that $f(y) \in L$ or $f(y) \in R$. In fact, since $f^{-1}(R) \cap A$ and $f^{-1}(L) \cap A$ are open in A and $A = f^{-1}(R) \cup f^{-1}(L)$ we must have that either $f(A) \subset L$ or $f(A) \subset R$. Without loss of generality we may assume that $f(A) \subset L$. Choose a point $p \in B_\delta(w)$ and a point $q \in A$ such that $p_\varphi > 0$, $p_\vartheta = m$, and the straight line segment $B = [p, q]$ is contained in $B_\delta(w)$. Then $f(p) \in \overline{R}$ and every point in $B \setminus \{q\}$ has positive φ-coordinate. Going from p to q along the straight line segment B let r be the last point such that $f(r) \in \overline{R}$. Let l be the straight line through the points r and w. Now let $s \in l \cap S(w, \partial U)$ be such that $s_\vartheta < m$. Note that $s_\varphi < 0$ since $r_\varphi > 0$. Then for every point $y \in (r, q)$ we have that the point $f(y)$ must lie to the left of s by Lemma 1.11. But this contradicts continuity of f at r. □

Claim 3. *$E_2(U)$ is a one-manifold.*

Proof of Claim 3. Let $w \in E_2(U)$ and choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(c_w^1)} \cap \overline{B_\varepsilon(c_w^2)} = \emptyset$ and $B_\varepsilon(c_w^1) \cap S(w, \partial U)$, $B_\varepsilon(c_w^2) \cap S(w, \partial U)$ are both arcs. Define closed sets X and Y as follows:

$$X = \overline{B_\varepsilon(c_w^1)} \cap \partial U, \qquad Y = \overline{B_\varepsilon(c_w^2)} \cap \partial U.$$

Clearly X and Y are noninterlaced with respect to w and $w \in E(X, Y)$, so by Theorem 2.1 there exists a $0 < \delta < d(w, X)/10$ such that $E(X, Y) \cap B_\delta(w)$ is a one manifold. Furthermore, by Lemma 1.5, we can choose δ so small that for all $z \in B_\delta(w)$ we have $c_{\partial U}(z) \subset X \cup Y$. Clearly $E(X, Y) \cap B_\eta(w) \subset E_2(U) \cap B_\eta(w)$ for all $0 < \eta < \delta$. Hence $E_2(U)$ is a 1-manifold. □

Let E be the component of $E(X, Y) \cap \overline{B_\delta(w)}$ containing w. We can define two functions $f_X : E \to \partial U$ and $f_Y : E \to \partial U$ by for each $x \in E$:

$$f_X(x) = c_{\partial U}(x) \cap X, \qquad f_Y(x) = c_{\partial U}(x) \cap Y.$$

Both functions are continuous by Lemma 1.14. We claim that both f_X and f_Y are monotone. To see why this is true for f_X, let $y \in X$ be such that y has at least two preimages p and q in E. Let A be the arc in E going from p to q and let T be the open region enclosed by $[p, y]$, $[q, y]$ and A. Since $\delta < d(w, X)/10$ it follows that $T \subset B(p, X) \cup B(q, X)$ and therefore $\partial U \cap T = \emptyset$. By the Non-crossing Lemma 1.11 and the fact that for every $x \in E$ we have that

$$\big(\big[x, f_X(x)\big) \cup \big[x, f_Y(x)\big)\big) \cap E_2(U) = \{x\}$$

it follows that each point in A has y as its unique closest point in X, hence f_X is monotone. In a similar fashion it can be shown that f_Y is monotone. Hence each of $f_X(E)$ and $f_Y(E)$ is either an arc or a point. We claim that the following set

$$CE = \bigcup_{y \in E} \big[y, f_X(y)\big) \cup \big[y, f_Y(y)\big) \tag{1}$$

is a closed neighborhood of w in U. The fact that CE is closed follows from the USC Lemma 1.14, to see why CE is a neighborhood let s and t denote the endpoints of the arc E. Let S_X be the simple closed curve formed by the union of the arcs E, $f_X(E)$ (this set could be a point), $[s, f_X(s)]$ and $[t, f_X(t)]$. Define the set R_X to be the closed disk bounded by S_X that contains $[w, f_X(w)]$. Similarly define S_Y and R_Y. We will show that $CE = R_X \cup R_Y$. Clearly $CE \subset R_X \cup R_Y$, so assume, by way of contradiction, that there exists a point y in $R = R_X \cup R_Y$ such that $y \notin CE$. Since CE is closed there exists an open set $W \subset R_X \cup R_Y$ containing y such that $W \cap CE = \emptyset$. Let V be a component of W containing y, then $V \subset R_X$ or $V \subset R_Y$. Without loss of generality, we may assume that $V \subset R_X$. Note that for any $z \in E$ we have that $[z, f_X(z)] \cap V = \emptyset$. For every $z \in E$ define $R_X(z)$ to be the closed region enclosed by the arc between s and z, the arc between $f_X(s)$ and $f_X(z)$, $[s, f_X(s)]$ and $[z, f_X(z)]$. Define two sets E_l and E_r as follows:

$$E_l = \big\{z \in E\colon\ V \not\subset R_X(z)\big\}, \qquad E_r = \big\{z \in E\colon\ V \subset R_X(z)\big\}.$$

Note that $E = E_l \cup E_r$ and $E_r \cap E_l = \emptyset$. Furthermore both E_l and E_r are closed since the set $\{[z, f_x(z)]\}_{z \in E}$ forms a continuous family of arcs. Since $s \in E_l$ and $t \in E_r$ this contradicts the connectedness of E. Hence CE is a neighborhood of w. Note that we can use the same method of proof to show that CE is a closed neighborhood for each point $z \in [w, f_X(w))$.

For each $y \in CE \setminus E$ we have that $y \notin E_2(U)$. Hence $E(X, Y) \cap B_\eta(w) = E_2(U) \cap B_\eta(w)$ for some $\eta > 0$, which proves that $E_2(U)$ is a one-manifold.

Let C be a component of $E_2(U)$. Define the set P as follows:

$$P = \bigcup_{w \in \overline{C}} \big[w, c_w^1\big) \cup \big[w, c_w^2\big).$$

Note that in the case that $w \in \partial U$, $[w, c_w^1) \cup [w, c_w^2) = \emptyset$.

Claim 4. *P is closed in U.*

Proof of Claim 4. Let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence in P converging to a point $z_\infty \in U$. By the definition of P there exists a sequence w_i in $\overline{C}$ such that $z_i \in A_i = [w_i, c_{w_i}^1) \cup [w_i, c_{w_i}^2)$. Without loss of generality we may assume that $w_\infty = \lim w_i$ exists. Clearly $w_\infty \in \overline{C}$ and if $w_\infty \in \partial U$ then this would imply that $z_\infty = \lim z_i \in \partial U$. So $w_\infty \notin \partial U$. By Lemmas 1.11 and 1.14 we must have that the arcs A_i converge into the arc $A_\infty = [w_\infty, c_{w_\infty}^1) \cup [w_\infty, c_{w_\infty}^2)$, i.e. $\lim A_i \subset \overline{A_\infty}$. Hence $z_\infty = \lim z_i \in \overline{A_\infty}$. This shows that $z_\infty \in A_\infty \subset P$. □

Claim 5. *P is open in U.*

Proof of Claim 5. Let $z \in P$, this means there exists a $w \in \overline{C} \setminus \partial U$ such that $z \in A = [w, c_w^1) \cup [w, c_w^2)$. If $w \in C$, then the arc A divides U into two components. We will refer to these components as the left and right sides of A. If $w \in C$ then choose a $p \in C$ on the left side of A and a $q \in C$ on the right side such that $d(p, q) < d(w, \partial U)/10$. Let C' be the closed arc in C going from p to q, then the set P' defined by

$$P' = \bigcup_{y \in C'} \big[y, c_y^1\big) \cup \big[y, c_y^2\big)$$

is neighborhood of z contained in P. If $w \in \overline{C} \setminus C$ then w has a unique closest point c_w, let $A = [w, c_w)$. For each x in C let $A_x = [x, c_x^1) \cup [x, c_x^2)$. We claim that $w \notin A_x$ for each $x \in C$. To see why this is true suppose, by way of contradiction, that there exists a $y \in C$ such that $w \in A_y$. The arc A_y divides U into two components. We will refer to these components as the left and right sides of A_y. Now pick a point $p \in C$ on the left side of A_y and a point $q \in C$ on the right side of A_y. Note that $A_p \cap A_y = \emptyset = A_q \cap A_y$ and that the arcs A_p and A_q both divide U into two components as well. Define F_p to be the component of $U \setminus A_p$ that does not contain y and define F_q to be the component of $U \setminus A_q$ that does not contain y. Since $C \cap A_x = \{x\}$ for every $x \in C$ and C is homeomorphic to the interval $(0,1) \subset \mathbb{R}$ it follows that once C enters F_p or F_q it cannot leave F_p or F_q anymore. But then $w \notin \overline{C}$, a contradiction. Hence $w \notin A_x$ for each $x \in C$. For each $x \in C$ let D_x be the component of $U \setminus A_x$ such that $w \in D_x$, note that $A \subset D_x$ for every $x \in C$. Since C cannot leave a component D_x once it has entered it, it follows that the components D_x are nested. We can now define an order on C by letting

$$x < y \quad \Leftrightarrow \quad D_x \subsetneq D_y.$$

Now pick a point $y \in C$ and consider the set

$$B = (w, y)_C = \{x \in C\colon\ x < y\}.$$

Choose an strictly decreasing sequence $\{w_i\}_{i\in\mathbb{N}}$ in B such that $\lim w_i = w$ and $w_i \in B_{d(w,\partial U)}(w)$ for each $i \in \mathbb{N}$. The open arc B locally separates the plane, so for every point $x \in B$ we can find a small open neighborhood N_x such that B separates N_x into two components L_x and R_x. We will call L_x the left side of B at x and R_x the right side of B at x. By a local compactness argument this allows us to consistently define a left side L_B and a right side R_B of B. Choose an strictly decreasing sequence $\{w_i\}_{i\in\mathbb{N}}$ in B such that $\lim w_i = w$ and $w_i \in B_{d(w,\partial U)/10}(w)$ for each $i \in \mathbb{N}$. We can now define a function $f_L : (w, w_1]_C \to \partial U$ by letting

$$f_L(x) = \begin{cases} c_x^1 & \text{if } [x, c_x^1) \cap L_B \neq \emptyset, \\ c_x^2 & \text{if } [x, c_x^2) \cap L_B \neq \emptyset. \end{cases}$$

Similarly we can define a function $f_R : (w, w_1]_C \to \partial U$. Using arguments similar to the arguments used to prove that f_X and f_Y are monotone and continuous, we can show that both f_L and f_R are monotone and continuous. For each $i \in \mathbb{N}$ define the following sets:

$$A_i = [w_i, c_{w_i}^1) \cup [w_i, c_{w_i}^2),$$
$$L_i = f_L([w_i, w_1]_C),$$
$$R_i = f_R([w_i, w_1]_C).$$

By Lemmas 1.11 and 1.14 we must have that the arcs A_i converge to the arc A. Notice that for each $i \geqslant 2$ the union of the arcs A_1, A_i, and the sets L_i and R_i (both of which are either a point or an arc) form a simple closed curve S_i. For $i \geqslant 2$ let D_i be the open disk enclosed by S_i, containing $(w_i, w_1)_C$. Then by results from Claim 3 we have

$$D_i = \bigcup_{x \in (w_i, w_1)_C} A_x.$$

Clearly each $D_i \subset P$ and $D_{i+1} \supset D_i$ for each i. Define D as follows:

$$D = \mathrm{cl}_U\left(\bigcup_{i=2}^{\infty} D_i\right).$$

We claim that D is a closed neighborhood of z that is contained in P. The fact that D is contained in P follows from the fact that each D_i is contained in P and P is closed in U by Claim 4. Since $z \in \overline{A} = \lim A_i$ we have that $z \in D$. Let $\delta = \min\{d(c_w, z)/10, d(z, w)\}$. Now assume, by way of contradiction, that there exist a point x with $d(x,z) < \delta$ such that $x \notin D$. Since D is closed in U there exists an open set $W \subset U$ containing x such that $W \cap D = \emptyset$. Let V be an open ball around x inside W such that the radius of V is less than δ. Note that $V \cap A = \emptyset$, since $A \subset D$. Since the arcs A_i converge to the arc A it follows that $V \subset D_k$ for some $k \in \mathbb{N}$, hence $V \subset D$, a contradiction. □

Claims 4 and 5 show that $P = U$ and hence that $E_2(U)$ is connected.

Claim 6. *$E_2(U)$ is smooth.*

Proof of Claim 6. Pick a point w in $E_2(U)$ and choose a local spherical coordinate system such that $w = (0,0)$, and c_w^1 and c_w^2 lie symmetric with respect to the negative φ-axis, the ϑ-coordinate of c_w^1 being positive. For each $x \neq w$ let $\arg x$ be the angle (in $[0, 2\pi)$) between the positive φ-axis and the line segment $[w, x]$. We claim that the φ-axis tangent is to $E_2(U)$ at w. Suppose that this is not the case. Then we may assume, without loss of generality, that there is a sequence

$w_i = (\varphi_i, \vartheta_i)$, with $\varphi_i > 0$ and $\vartheta_i > 0$, of points in $E_2(U)$ converging to w such that for some $\alpha > 0$ we have $\arg w_i > \alpha$. Let l_α be the line that makes an angle α with the positive φ-axis and let c_α be the point on $S(w, \partial U)$ that is symmetric with c_w^1 with respect to the line l_α. Let B be the subarc of $S(w, \partial U)$ that joins the points c_w^1 and c_α but is disjoint from c_w^2. Choose $\epsilon > 0$ such that:

(1) the ϵ-neighborhoods of c_w^1 and c_w^2 are disjoint,
(2) the ϵ-neighborhoods of c_w^2 and B are disjoint, and
(3) the intersection of $B(w, \partial U + \delta)$ and ∂U is contained in the union of the said neighborhoods (cf. Collar Lemma).

Now by the USC Lemma, if one of the closest points of w_i to ∂U is in $B_\epsilon(c_w^1)$ (and this holds true if i is large enough), the other must be in $B_\epsilon(c_w^2)$. But inspection of the intersection of $S(w_i, \partial U)$ and ∂U shows that the other closest point must be in the ϵ-neighborhood of B. This is a contradiction □

This completes the proof of the theorem. □

We have shown that if $U \in \mathcal{U}_2$ such that the boundary of U has two components, then the boundary of U is locally connected and any cut point in the boundary is of order two. The case when the boundary of U is connected (or equivalently when U is simply connected) is more complicated. It can be shown that in this case the boundary of U is not necessarily locally connected. However, it can be shown that the set of accessible points in ∂U consists of at most two arc components A_1 and A_2 which are associated with the two sides of the centerline $E_2(U)$. Hence U can be roughly pictured as a long thin tube with two ends and $E_2(U)$ as its center line. The closure of the center line is no longer necessarily an arc: the remainder of $E_2(U)$ may be non-degenerate (i.e., the centerline can behave like a $\sin(1/x)$ function). In particular, the boundary of U can be an indecomposable continuum (i.e., cannot be written as the union of two of its proper sub-continua; see [3] for additional information).

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