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Application of the Numerator Formula to k-Rowed Plane Partitions

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1. INTRODUCTION

Let k be a positive integer and n a nonnegative integer. A ***k*-rowed plane partition of *n*** is a two-dimensional array $\{n_{ij}\}_{1 \leqslant i \leqslant k, j \geqslant 1}$ of nonnegative integers whose sum is n and such that $n_{pq} \geqslant n_{ij}$ whenever $p \leqslant i$ and $q \leqslant j$. Thus a one-rowed plane partition of n is an ordinary partition of n. Let $p_k(n)$ denote the number of k-rowed plane partitions of n, and let $\pi_k(q)$ (q an indeterminate) be the corresponding generating function, that is,

$$\pi_k(q) = \sum_{m=0}^{\infty} p_k(m) q^m.$$

A fundamental theorem of MacMahon (see [1, p. 184, Corollary 11.3]) gives a product expansion for $\pi_k(q)$:

$$\pi_k(q) = \prod_{j=1}^{\infty} (1 - q^j)^{-\min(k,j)}. \tag{1.1}$$

The case $k = 1$ was known to Euler, who also found the following famous formula for the reciprocal of $\pi_1(q)$:

$$\prod_{j=1}^{\infty} (1 - q^j) = \sum_{m=-\infty}^{\infty} (-1)^m q^{(1/2)m(3m+1)} \tag{1.2}$$

(see [1, Chap. 1]). This formula leads to Euler's recursion for computing the partition function $p_1(q)$ [1, Corollary 1.8, p. 12].

Our main result is a formula for $\pi_k(q)^{-1}$ which generalizes (1.2); see Theorem 3.4 below. The proof uses (1.1) and (1.2), and is based on results concerning Macdonald's identities [14(a)] and Kac–Moody, or GCM (generalized Cartan matrix), Lie algebras announced in [10(a)] and proved in [10(b)]. (These Lie

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algebras had been introduced by V. G. Kac and R. V. Moody.) We combine these results with the "numerator formula", which we explain presently. We also obtain (Theorem 3.5) a new kind of formula for $\prod_{j=1}^{\infty}(1 - q^j)^k$, of a slightly different sort from the ones discovered in [10(a), (b)].

The numerator formula concerns the numerator in Weyl's character formula for a finite-dimensional irreducible module for a complex semisimple Lie algebra. It is well known that the denominator in Weyl's character formula has a certain product expansion, given by Weyl's "denominator formula." In general, the numerator has no such expansion. However, after a simple rewriting of the character formula (see Corollary 2.3), the character, the numerator, and the denominator become polynomials in certain variables: the exponentials of minus the simple roots. When these variables are all set equal to a single variable, say q, then the numerator becomes a polynomial in q which *does* have a product expansion. This fact is also well known, although it is usually expressed differently. It is a step in one of the classical proofs of Weyl's dimension formula; see, for example, [2; 6, p. 256]. We call the setting of the variables equal to q the "principal specialization" for a reason explained in [10(b), Section 17] (cf. Section 2), and we call the product formula for the principally specialized numerator the "numerator formula"; for the precise statement, see Theorem 2.4.

Before it was noticed that the numerator formula was already known, an analogous formula had been found in [11] for two infinite-dimensional Kac–Moody Lie algebras. The relevant modules are the infinite-dimensional "standard modules", for which Kac [7(b)] had proved Weyl's character formula. The main discovery in [11] was that after principal specialization, the characters of a certain pair of standard modules become power series in q which are closely related to the product sides of the two Rogers–Ramanujan identities in combinatorial analysis. (We get power series instead of polynomials in q because of the infinite dimensionality of the Lie algebras and the modules.)

In this paper, we show that both the classical numerator formula and the result in [11] are generalized by a new numerator formula for all Kac–Moody Lie algebras (see Theorems 2.4 and 2.6 below). (See also [7(c)].) The general proof (Section 2) is straightforward but a little delicate. It uses the fact that the Weyl group is a Coxeter group, a fact proved by Kac[7(a)], Lim [13], and Solomon and Verma [16]. We emphasize that in special cases (for example, when the GCM is classical or symmetric), the proof simplifies considerably. From the present point of view, what was proved (by concrete computation) in [11] was the numerator formula for the two Euclidean Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$. The formula for these two cases is exploited further in [3], and for case $A_1^{(1)}$, also in [12]. The numerator formula for more general Euclidean Lie algebras is used in [7(c), 8].

In [5], Hughes proves that the Gaussian polynomials

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]!\,[n-i]!},$$

where $[j]! = (1-q)(1-q^2)\cdots(1-q^j)$, and the polynomial $\prod_{j=1}^{n}(1+q^j)$ are unimodal, i.e., that their coefficients first increase and then decrease, not necessarily strictly. (For the Gaussian polynomial case, this had been proved long age by means of classical invariant theory (cf. [1, p. 48; 17(b)]). Hughes uses a classical theorem of Dynkin asserting (in the present terminology) that the principally specialized character of a finite-dimensional irreducible module of a finite-dimensional complex semisimple Lie algebra is a unimodal polynomial. Stanley points out that Hughes' proofs are considerably simplified if one quotes the numerator formula for the Lie algebras and modules with which Hughes works [17(b)]. It is also clear that one can prove partition-theoretic results analogous to Hughes' by combining Dynkin's theorem with the numerator formula for general semisimple Lie algebras and finite-dimensional modules.

Common factors can be canceled from or inserted in the principal specializations of the numerator and denominator in the character formula in interesting ways. Stanley has studied what amounts to this for the case $\mathfrak{sl}(n, \mathbb{C})$, and in doing so, he has expressed the principally specialized characters for this Lie algebra in terms of the "contents" and the "hook lengths" of plane partitions [17(a), Theorem 15.3, p. 263]. It would be interesting to study the analogous "cancellation" for general Lie algebras. Theorem 3.4 and Stanley's result (cf. also [17(b)]) can in principle be combined in order to give a formula for the reciprocal of MacMahon's generating function in terms of plane partitions.

The Gaussian polynomial $\left[{n \atop i}\right]$ $(i = 1,..., n-1)$ is the principally specialized character of the ith fundamental module of $\mathfrak{sl}(n, \mathbb{C})$. It is also known that $\left[{n \atop i}\right](q)$ is the number of $(i-1)$-dimensional subspaces of $(n-1)$-dimensional projective space over the field with q elements. This and analogous coincidences are discussed in Section 4, which was written in collaboration with J. Jantzen. In Section 4, certain principally specialized characters for finite-dimensional simple Lie algebras are related to the Poincaré polynomials of the irreducible compact Hermitian symmetric spaces and to the cardinalities of certain finite varieties. We are grateful to W. Kantor for stimulating conversations on this material.

Principal specialization for infinite-dimensional Kac–Moody Lie algebras was introduced in [10(a)], and principal specialization for their standard modules was introduced in [11, 3]. The ideas of studying the standard modules "concretely" and of relating them to power series identities were also introduced in [11, 3]. The general numerator formula presented here was discovered in 1977 as a direct offshoot of this work.

2. The Numerator Formula

Let $A = (A_{ij})_{i,j\in\{0,\ldots,l\}}$ be an $(l+1) \times (l+1)$ symmetrizable GCM and let $\mathfrak{l} = \mathfrak{l}(A)$ be the corresponding Kac–Moody Lie algebra (denoted $\mathfrak{g}(A)$ in [4, 10(b)]), over the field $\mathbb{C}$ of complex numbers. (We could work over any field of characteristic zero.) As usual, denote the canonical generators by h_i, e_i, f_i ($i = 0,..., l$), and let $\mathfrak{h}$ be the span of the h_i. Let $\mathfrak{l}^e$ be the extended Lie algebra defined as in [4] or [10(b)], so that $\mathfrak{l}^e$ is the semidirect product with $\mathfrak{l}$ of a certain finite-dimensional space $\mathfrak{d}$ of derivations of $\mathfrak{l}$. Let $\mathfrak{h}^e$ be the abelian subalgebra $\mathfrak{d} \oplus \mathfrak{h}$ of $\mathfrak{l}^e$. The roots of $\mathfrak{l}$ may now be defined in the natural way as elements of $(\mathfrak{h}^e)^*$ (the dual space of $\mathfrak{h}^e$):

For all $\varphi \in (\mathfrak{h}^e)^*$, let $\mathfrak{l}^\varphi = \{x \in \mathfrak{l} \mid [h, x] = \varphi(h)x$ for all $h \in \mathfrak{h}^e\}$; call φ a *root* if $\varphi \neq 0$ and $\mathfrak{l}^\varphi \neq 0$.

If A is the classical Cartan matrix of a finite-dimensional semisimple Lie algebra, then we may take $\mathfrak{d} = 0$, so that $\mathfrak{l}^e = \mathfrak{l}$ is the semisimple Lie algebra corresponding to A. In this case, the entire discussion in this section reduces to a discussion of ordinary finite-dimensional semisimple Lie algebras.

Let $\Delta \subset (\mathfrak{h}^e)^*$ be the set of roots of $\mathfrak{l}$, $\Delta_+ \subset \Delta$, the set of positive roots, and $\alpha_0, \alpha_1,..., \alpha_l \in \Delta_+$, the simple roots. Then $e_i \in \mathfrak{l}^{\alpha_i}$ for $i = 0,..., l$. Also, $A_{ij} = \alpha_j(h_i)$ for all $i, j = 0,..., l$. For each $i = 0,..., l$, define the *simple reflection* r_i of $(\mathfrak{h}^e)^*$ by the condition

$$r_i\varphi = \varphi - \varphi(h_i)\alpha_i$$

for all $\varphi \in (\mathfrak{h}^e)^*$. Let W be the *Weyl group* of $\mathfrak{l}$, i.e., the group of automorphisms of $(\mathfrak{h}^e)^*$ generated by $r_0,..., r_l$. Then W is a Coxeter group with generators r_i and defining relations $(r_i)^2 = 1$ for all $i = 0,..., l$ and $(r_ir_j)^{m_{ij}} = 1$ for all $i, j = 0,..., l$ with $i \neq j$; here $m_{ij} = 2, 3, 4, 6$, or ∞ according as $A_{ij}A_{ji} = 0, 1, 2, 3$, or $\geqslant 4$, respectively. For all $w \in W$, note that $\det w = \pm 1$ according to whether w can be expressed as a product of an even (resp., odd) number of simple reflections.

Define $\rho \in (\mathfrak{h}^e)^*$ to be any fixed element satisfying the conditions $\rho(h_i) = 1$ for all $i = 0,..., l$. Then $r_i\rho - \rho = -\alpha_i$ for each i, and $w\rho - \rho$ is a nonnegative integral linear combination of $-\alpha_0,..., -\alpha_l$ for each $w \in W$.

Let $P = \{\lambda \in (\mathfrak{h}^e)^* \mid \lambda(h_i) \in \mathbb{Z}_+$ for all $i = 0,..., l\}$. ($\mathbb{Z}_+$ is the set of nonnegative integers.) For each $\lambda \in P$, there is a unique (up to equivalence) irreducible $\mathfrak{l}^e$ module X^λ such that X^λ is generated by a nonzero vector x_0 with the properties that $\mathfrak{l}^\varphi \cdot x_0 = 0$ for all $\varphi \in \Delta_+$ and $h \cdot x_0 = \lambda(h)x_0$ for all $h \in \mathfrak{h}^e$. X^λ is called the *standard* $\mathfrak{l}^e$-module with *highest weight* λ. (In case $\mathfrak{l}$ is finite-dimensional semisimple, the standard modules are exactly the finite-dimensional irreducible modules.) X^λ is the direct sum of its *weight spaces* X_μ^λ for $\mu \in (\mathfrak{h}^e)^*$; here

$$X_\mu^\lambda = \{x \in X^\lambda \mid h \cdot x = \mu(h)x \text{ for all } h \in \mathfrak{h}^e\}.$$

Each weight space is finite dimensional, and if $X_\mu^\lambda \neq 0$, then $\lambda - \mu$ is a nonnegative integral linear combination of simple roots. The *character* of X^λ is defined to be the formal (possibly infinite) sum

$$\chi(X^\lambda) = \sum_{\mu \in (\mathfrak{h}^e)^*} (\dim X_\mu^\lambda)\, e(\mu),$$

where $e(\cdot)$ is understood as a formal exponential. Thus the character is the generating function of the dimensions of the weight spaces. In [7(b)], where Kac introduces ρ and the standard modules, he proves the following formal identities, which are generalizations of Weyl's character and denominator formulas:

PROPOSITION 2.1. *For all* $\lambda \in P$,

$$\chi(X^\lambda) = \frac{\sum_{w\in W} (\det w)e(w(\lambda+\rho)-\rho)}{\sum_{w\in W} (\det w)e(w\rho-\rho)}.$$

PROPOSITION 2.2. *We have*

$$\sum_{w\in W} (\det w)e(w\rho-\rho) = \prod_{\varphi\in\Delta_+} (1-e(-\varphi))^{\dim \mathfrak{l}^\varphi}.$$

As is pointed out in [7(b), 15], Proposition 2.2 is also a generalization of Macdonald's identities [14].

We reformulate Proposition 2.1 as follows:

COROLLARY 2.3. *Let*

$$N(\lambda) = \sum_{w\in W} (\det w)e(w(\lambda+\rho)-(\lambda+\rho))$$

and

$$D = \sum_{w\in W} (\det w)e(w\rho-\rho),$$

so that $D = N(0)$. *Then*

$$\chi(X^\lambda)/e(\lambda) = N(\lambda)/D.$$

Remark. The point of this rewriting is that $N(\lambda)$, D, and $\chi(X^\lambda)/e(\lambda)$ are all elements of the formal power series ring $\mathbb{Z}[[e(-\alpha_0),..., e(-\alpha_l)]]$, $\mathbb{Z}$ being the ring of integers. This is because $w(\lambda+\rho)-(\lambda+\rho)$ is a nonnegative integral linear combination of $-\alpha_0,..., -\alpha_l$.

Unlike D, $N(\lambda)$ does not in general have a product expansion. (For an interesting special case in which it does, see [3, Sect. 4]). However, in [11] it was discovered for a certain pair of infinite-dimensional Kac–Moody Lie algebras that $N(\lambda)$ *does* have a product expansion *after* all the power series variables $e(-\alpha_0),..., e(-\alpha_l)$ are set equal, and the resulting specialization of $N(\lambda)$ equals a suitable specialization of D. This result was used in [11] to place the product sides of the Rogers–Ramanujan identities in a natural Lie-algebraic context. Here we formulate and prove this "numerator formula" in the present much more general setting.

Let A^t be the generalized Cartan matrix which is the transpose of A. Then A^t is again symmetrizable. Let $\mathfrak{l}'$ be the Kac–Moody Lie algebra $\mathfrak{l}(A^t)$. We call $\mathfrak{l}'$ the *transpose Lie algebra*. Let the analogs for $\mathfrak{l}'$ of $\mathfrak{l}^e$, $\mathfrak{h}$, $\mathfrak{h}^e$, $h_0,\dots,h_l$, Δ, Δ_+, $\alpha_0,\dots,\alpha_l$, $r_0,\dots,r_l$, W, ρ, P, $e(\cdot)$, $\mathfrak{l}^\varphi (\varphi \in \Delta)$, and D be denoted with primes: $\mathfrak{l}'^e$, $\mathfrak{h}'$, $\mathfrak{h}'^e$, $h_0',\dots,h_l'$, Δ', Δ_+', $\alpha_0',\dots,\alpha_l'$, $r_0',\dots,r_l'$, W', ρ', P', $e'(\cdot)$, $\mathfrak{l}'^\varphi$ $(\varphi \in \Delta')$, and D', respectively. Then

$$D' = \sum_{w\in W'} (\det w) e'(w\rho' - \rho').$$

Note that we do not need to assume that $\dim \mathfrak{h}^e = \dim \mathfrak{h}'^e$, i.e., we may adjoin spaces of derivations of unequal dimensions to $\mathfrak{l}$ and $\mathfrak{l}'$ to construct $\mathfrak{l}^e$ and $\mathfrak{l}'^e$. We do, however, make the natural assumption that the indices are ordered so that $A_{ji} = \alpha_j'(h_i')$ for all $i, j = 0,\dots, l$. (Recall that $A_{ij} = \alpha_j(h_i)$ for all $i, j = 0,\dots, l$).

DEFINITION. Let q be an indeterminate. The *principal q-specializations* are the two homomorphisms of power series rings

$$\mathbb{Z}[[e(-\alpha_0),\dots, e(-\alpha_l)]] \to \mathbb{Z}[[q]]$$

and

$$\mathbb{Z}[[e'(-\alpha_0'),\dots, e'(-\alpha_l')]] \to \mathbb{Z}[[q]]$$

which take $e(-\alpha_i)$ (resp., $e'(-\alpha_i')$) to q for all $i = 0,\dots, l$.

Remark. The reason why these specializations are called "principal" is explained in [10(b), Section 17]; for an "affine" Kac–Moody Lie algebra, principal specialization is related in a certain natural way to Kostant's "principal" automorphism [9] of a finite-dimensional simple Lie algebra.

DEFINITION. Let $(s_0,\dots, s_l)$ be a sequence of positive integers. The *q-specializations of type* $(s_0,\dots, s_l)$ are the two homomorphisms

$$\mathbb{Z}[[e(-\alpha_0),\dots, e(-\alpha_l)]] \to \mathbb{Z}[[q]]$$

and

$$\mathbb{Z}[[e'(-\alpha_0'),\dots, e'(-\alpha_l')]] \to \mathbb{Z}[[q]]$$

which take $e(-\alpha_i)$ (resp., $e'(-\alpha_i')$) to q^{s_i} for all $i = 0,\dots, l$. Note that the principal q-specializations are the q-specializations of type $(1,\dots, 1)$.

We can now state the numerator formula and two immediate consequences.

THEOREM 2.4. *Let $\lambda \in P$. Then the principal q-specialization of the numerator $N(\lambda)$ (see Corollary 2.3) equals the q-specialization of type $((\lambda+\rho)(h_0),\dots, (\lambda+\rho)(h_l))$ of the denominator D' for the transpose Lie algebra $\mathfrak{l}'$. In particular (using Proposition 2.2), the principal q-specialization of the numerator has a product expansion.*

Taking $\lambda = 0$ gives:

COROLLARY 2.5. *The principal q-specialization of the denominator D equals the principal q-specialization of the denominator D' for the transpose Lie algebra.*

From Theorem 2.4, Corollary 2.5, and the character formula (Corollary 2.3), we get:

THEOREM 2.6. *Let $\lambda \in P$. Then the principal q-specialization of $\chi(X^\lambda)/e(\lambda)$ has a product expansion, and in fact equals D'_λ/D'_0, where D'_λ is the q-specialization of type $((\lambda+\rho)(h_0),..., (\lambda+\rho)(h_l))$ of D', and D'_0 is the principal q-specialization of D'.*

Remark. If the GCM A is symmetric (which means exactly that all roots have equal length if $\mathfrak{l}$ is finite-dimensional simple), then of course it is unnecessary to deal with the transpose Lie algebra, and the proofs and statements of the results simplify correspondingly. They also simplify, and in fact become classical, if $\mathfrak{l}$ is finite-dimensional semisimple.

We now prove Theorem 2.4.

W acts on $\mathfrak{h}^e$ by the contragredient of its action on $(\mathfrak{h}^e)^*$.

LEMMA 2.7. *For all $h \in \mathfrak{h}^e$ and $i = 0,..., l$,*

$$r_i h = h - \alpha_i(h)h_i .$$

In particular, W preserves $\mathfrak{h}$.

Proof. For all $\mu \in (\mathfrak{h}^e)^*$,

$$\begin{aligned}\mu(r_i h) &= (r_i\mu)(h)\\ &= (\mu - \mu(h_i)\,\alpha_i)(h)\\ &= \mu(h - \alpha_i(h)h_i).\end{aligned} \qquad \text{Q.E.D.}$$

Notation. Let $R \subset (\mathfrak{h}^e)^*$ be the span of Δ, and let $R' \subset (\mathfrak{h}'^e)^*$ be the span of Δ'. Note that W preserves R and that W' preserves R'. Define $\delta \in \mathfrak{h}^e$ to be any fixed element such that $\alpha_i(\delta) = 1$ for all $i = 0,..., l$, and similarly, let $\delta' \in \mathfrak{h}'^e$ be any fixed element such that $\alpha'_i(\delta') = 1$ for all $i = 0,..., l$.

LEMMA 2.8. *For all $w \in W$,*

$$w\rho - \rho \in R$$

and

$$w\delta - \delta \in \mathfrak{h}.$$

For all $w' \in W'$,

$$w'\rho' - \rho' \in R'$$

and

$$w'\delta' - \delta' \in \mathfrak{h}'.$$

Proof. It is of course sufficient to prove only the two assertions about w, and the first of these was already observed above. To prove that $w\delta - \delta \in \mathfrak{h}$, we use induction on the length of w (the minimal number of simple reflections needed to express w). Lemma 2.7 proves our assertion for w of length 1. Suppose that $w\delta - \delta \in \mathfrak{h}$ for all w of lenth n. Any Weyl group element of length $n + 1$ can be written $r_i w$, for some $i = 0,..., l$ and some w of length n. We have

$$r_i w\delta - \delta = r_i(w\delta - \delta) + r_i\delta - \delta,$$

completing the proof. Q.E.D.

Define a linear isomorphism $\iota: \mathfrak{h} \to R'$ by the conditions $\iota(h_i) = \alpha_i'$ for all $i = 0,..., l$, and a linear isomorphism, also denoted $\iota: R \to \mathfrak{h}'$, by the conditions $\iota(\alpha_i) = h_i'$.

LEMMA 2.9. *For all* $\varphi \in R$ *and* $h \in \mathfrak{h}$,

$$\iota(h)(\iota(\varphi)) = \varphi(h).$$

Proof. We have

$$\alpha_i'(h_j) = A_{ij} = \alpha_j(h_i),$$

i.e.,

$$\iota(h_i)(\iota(\alpha_j)) = \alpha_j(h_i)$$

for all $i, j = 0,..., l$. Now just use linearity. Q.E.D.

W and W' are isomorphic under the homomorphism $\iota: W \to W'$ determined by the conditions $\iota(r_i) = r_i'$ for all $i = 0,..., l$. In fact, as mentioned above, the defining relations for the Coxeter groups W and W' depend only on the products $A_{ij}A_{ji}$.

LEMMA 2.10. *For all* $w \in W$,

$$\iota(w\rho - \rho) = \iota(w)\,\delta' - \delta'$$

and

$$\iota(w\delta - \delta) = \iota(w)\,\rho' - \rho'.$$

Proof. By Lemma 2.8, $\iota(w\rho - \rho)$ and $\iota(w\delta - \delta)$ are defined. We use induction on the length of w. For all $i = 0,\ldots, l$,

$$\iota(r_i\rho - \rho) = \iota(-\alpha_i) = -h_i' = \iota(r_i)\delta' - \delta',$$

by Lemma 2.7 applied to $\mathfrak{l}'$. Assume that $\iota(w\rho - \rho) = \iota(w)\,\delta' - \delta'$. Then

$$\begin{aligned}\iota(r_iw\rho - \rho) &= \iota(r_i(w\rho - \rho) + r_i\rho - \rho)\\ &= \iota(w\rho - \rho - (w\rho - \rho)(h_i)\alpha_i) + \iota(r_i)\delta' - \delta'\\ &= \iota(w)\delta' - \delta' - (w\rho - \rho)(h_i)h_i' + \iota(r_i)\delta' - \delta'.\end{aligned}$$

But by Lemma 2.9,

$$\begin{aligned}(w\rho - \rho)(h_i) &= \iota(h_i)(\iota(w\rho - \rho))\\ &= \alpha_i'(\iota(w)\delta' - \delta'),\end{aligned}$$

and so, using Lemma 2.7 for $\mathfrak{l}'$,

$$\begin{aligned}\iota(r_iw\rho - \rho) &= \iota(r_i)(\iota(w)\,\delta' - \delta') + \iota(r_i)\,\delta' - \delta'\\ &= \iota(r_iw)\,\delta' - \delta',\end{aligned}$$

proving the first assertion. The second follows by reversing the roles of $\mathfrak{l}$ and $\mathfrak{l}'$.

Q.E.D.

LEMMA 2.11. *For all $\lambda \in (\mathfrak{h}^e)^*$, define $\delta_\lambda' \in \mathfrak{h}'^e$ to be any element such that*

$$\alpha_i'(\delta_\lambda') = (\lambda + \rho)(h_i) = (\lambda + \rho)(\iota^{-1}(\alpha_i'))$$

for all $i = 0,\ldots, l$. Then for all $w \in W$,

$$(w(\lambda + \rho) - (\lambda + \rho))(\delta) = (\iota(w)^{-1}\rho' - \rho')(\delta_\lambda').$$

Proof. We have

$$\begin{aligned}(w(\lambda + \rho) - (\lambda + \rho))(\delta) &= (\lambda + \rho)(w^{-1}\delta) - (\lambda + \rho)(\delta)\\ &= (\lambda + \rho)(w^{-1}\delta - \delta)\\ &= (\lambda + \rho)(\iota^{-1}(\iota(w)^{-1}\rho' - \rho')) \qquad \text{(Lemma 2.10)}\\ &= (\iota(w)^{-1}\rho' - \rho')(\delta_\lambda'). \qquad \text{Q.E.D.}\end{aligned}$$

For all $\lambda \in P$, the principal q-specialization of $N(\lambda)$ is clearly

$$\sum_{w \in W} (\det w) q^{-(w(\lambda+\rho)-(\lambda+\rho))(\delta)} = \sum_{w \in W} (\det w) q^{-(\iota(w)^{-1}\rho'-\rho')(\delta'_\lambda)} \qquad \text{(Lemma 2.11)},$$

$$= \sum_{w \in W'} (\det w) q^{-(w^{-1}\rho'-\rho')(\delta'_\lambda)}$$

$$= \sum_{w \in W'} (\det w) q^{-(w\rho'-\rho')(\delta'_\lambda)},$$

and this is exactly the q-specialization of type $((\lambda + \rho)(h_0), ..., (\lambda + \rho)(h_l))$ of D'. This completes the proof of Theorem 2.4. Q.E.D.

3. Application to Plane Partitions

We now restrict our attention to the Kac–Moody Lie algebras $\mathfrak{l}$ which are *affine*, i.e., by definition, the quotient of $\mathfrak{l}$ by its center is of the form $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}$ is finite-dimensional simple and t is an indeterminate. Our formula for the reciprocal of the generating function for k-rowed plane partitions will come from the special case $\mathfrak{g} = \mathfrak{sl}(k, \mathbb{C})$.

Let W_0 and Δ_0 be the Weyl group and root system, respectively, of $\mathfrak{g}$, with $\Delta_{0,+}$ the set of positive roots of $\mathfrak{g}$. Then $\mathfrak{g}$ embeds naturally into $\mathfrak{l}$ in such a way that W_0 becomes identified with the subgroup of W generated by $r_1, ..., r_l$, Δ_0 with the intersection of Δ and the integral span of $\alpha_1, ..., \alpha_l$, and $\Delta_{0,+}$ with $\Delta_0 \cap \Delta_+$. Let Δ'_+ be the complement of $\Delta_{0,+}$ in Δ_+. If both sides in Proposition 2.2 are divided by the ordinary Weyl denominator $\prod_{\varphi \in \Delta_{0,+}} (1 - e(-\varphi))$ for $\mathfrak{g}$ and if Weyl's character formula for $\mathfrak{g}$ is used, then Proposition 2.2 yields a formula for $\prod_{\varphi \in \Delta'_+} (1 - e(-\varphi))^{\dim \mathfrak{l}^\varphi}$. This formula is essentially formula (0.4) in Macdonald [14] (cf. also Theorem 13.4 in [10(b)]).

In [14], this division was performed in order to allow the specialization $e(-\alpha_i) \mapsto 1$ $(i = 1, ..., l)$, $e(-\alpha_0) \mapsto q$ in Proposition 2.2; if the division were not performed, the result of this specialization would only be $0 = 0$. After the division, however, this specialization, together with Weyl's dimension formula for $\mathfrak{g}$, gave Macdonald his formula for $\eta(q)^{\dim \mathfrak{g}}$ [14, formula (0.5)]. But in [10(a), (b)], it was discovered that the *principal* specialization of Proposition 2.2 is interesting, and we remark now that the principal specialization of Proposition 2.2 *after the division by* $\prod_{\varphi \in \Delta_{0,+}} (1 - e(-\varphi))$ is *also* interesting. In place of Weyl's dimension formula for $\mathfrak{g}$, we now use the numerator formula (Section 2 above) for $\mathfrak{g}$, which is of course a classical formula since $\mathfrak{g}$ is finite dimensional. Our method, then is essentially to apply principal specialization to [14, formula (0.4)], and to combine with the numerator formula for $\mathfrak{g}$ and the results in [10(a), (b)]. It turns out to be technically convenient in our argument below to bypass the numerator formula for $\mathfrak{g}$ by starting from the appropriate results in [10(b)].

THEOREM 3.1. *Let* $\varphi(q) = \prod_{j\geq 1}(1 - q^j)$. *Let h be the Coxeter number of the complex simple Lie algebra* $\mathfrak{g}$; $m_1, \ldots, m_l$, *the exponents of* $\mathfrak{g}$; $\mathfrak{h}_0 \subset \mathfrak{g}$ *a Cartan subalgebra*; $\Delta_0 \subset \mathfrak{h}_0^*$, *the set of roots*; $\mathfrak{h}_{\mathbf{R}}^* \subset \mathfrak{h}_0^*$, *the real span of* Δ_0; $(\cdot, \cdot)$, *the canonical bilinear form on* $\mathfrak{h}_{\mathbf{R}}^*$ *induced by the Killing form of* $\mathfrak{g}$; $\|\cdot\|^2$, *the corresponding quadratic form*; $\Delta_{0,+} \subset \Delta_0$, *the set of positive roots*; $\rho_0 = \frac{1}{2}\sum_{\beta\in\Delta_{0,+}}\beta$; $\rho_0' = \frac{1}{2}\sum_{\beta\in\Delta_{0,+}} 2\beta/\|\beta\|^2$ (*the half-sum of positive roots for the root system dual to* Δ); M, *the lattice in* $\mathfrak{h}_0^*$ *generated by* $\{\beta/\|\beta\|^2 \mid \beta \in \Delta_0\}$. *Then*

$$\varphi(q)^l \prod_{\substack{n\in\mathbf{Z}_+\\ i=1,\ldots,l}} (1 - q^{m_i+nh})$$

$$= \sum_{\mu\in M} q^{h\|-\mu+\rho_0-(\rho_0'/2h)\|^2 - h\|\rho_0-(\rho_0'/2h)\|^2} \prod_{\varphi\in\Delta_{0,+}} (1 - q^{(\varphi, 2h\mu+\rho_0')}).$$

Proof. By Theorem 17.3 and Proposition 17.7 of [10(b)], the left-hand side in the statement of the present theorem equals

$$\sum_{\sigma\in W_0} \det\sigma \sum_{\mu\in M} q^{h\|\mu+\sigma\rho_0-(\rho_0'/2h)\|^2 - h\|\rho_0-(\rho_0'/2h)\|^2}. \qquad (*)$$

But

$$\|\mu + \sigma\rho_0 - (\rho_0'/2h)\|^2$$

$$= \|\mu + \rho_0 - (\rho_0'/2h)\|^2 + 2(\sigma\rho_0 - \rho_0, \mu + \rho_0 - (\rho_0'/2h)) + (\sigma\rho_0 - \rho_0, \sigma\rho_0 - \rho_0)$$

$$= \|\mu + \rho_0 - (\rho_0'/2h)\|^2 + 2(\sigma\rho_0 - \rho_0, \mu - (\rho_0'/2h)) + (\sigma\rho_0 - \rho_0, \sigma\rho_0 + \rho_0).$$

Since

$$(\sigma\rho_0 - \rho_0, \sigma\rho_0 + \rho_0) = \|\sigma\rho_0\|^2 - \|\rho_0\|^2 = 0,$$

(*) becomes

$$\sum_{\mu\in M} q^{h\|\mu+\rho_0-(\rho_0'/2h)\|^2 - h\|\rho_0-(\rho_0'/2h)\|^2} \sum_{\sigma\in W_0} (\det\sigma) q^{2h(\sigma\rho_0-\rho_0, \mu-(\rho_0'/2h))}.$$

But the sum over W_0 is

$$\sum_{\sigma\in W_0} (\det\sigma) q^{(\sigma\rho_0-\rho_0, 2h\mu-\rho_0')} = \prod_{\varphi\in\Delta_{0,+}} (1 - q^{-(\varphi, 2h\mu-\rho_0')}),$$

by Weyl's denominator formula for $\mathfrak{g}$. Now just change μ to $-\mu$. Q.E.D.

COROLLARY 3.2. *In the notation of Theorem* 3.1, *suppose that* $\mathfrak{g}$ *has only one root length.*

Then

$$\varphi(q)^l \prod_{\substack{n\in\mathbb{Z}_+\\ i=1,\ldots,l}} (1-q^{m_i+nh}) = \sum_{\mu\in M} q^{h\|\mu\|^2} \prod_{\varphi\in\Delta_{0,+}} (1-q^{2h(\varphi,\mu+\rho_0)}).$$

Moreover, M is the lattice generated by $\{h\beta \mid \beta\in\Delta_0\}$.

Proof. It is well known that in this case, $\|\beta\|^2 = 1/h$ for $\beta\in\Delta_0$, and so $\rho_0 = \rho_0'/2h$. Q.E.D.

One can now divide both sides in Theorem 3.1 or Corollary 3.2 by the principal specialization of Weyl's denominator for $\mathfrak{g}$, i.e., by $\prod_{\varphi\in\Delta_{0,+}}(1-q^{(\varphi,\rho_0')})$. (Note that (φ,ρ_0') is just the height of the root φ.) Since we are interested only in the equal-root-length case, we state:

COROLLARY 3.3. *In the setting of Corollary* 3.2,

$$\varphi(q)^l \prod_{\substack{n\in\mathbb{Z}_+\\ i=1,\ldots,l}} (1-q^{m_i+nh}) \Big/ \prod_{\varphi\in\Delta_{0,+}} (1-q^{2h(\varphi,\rho_0)})$$

$$= \sum_{\mu\in M} q^{h\|\mu\|^2} \prod_{\varphi\in\Delta_{0,+}} (1-q^{2h(\varphi,\mu+\rho_0)})/(1-q^{2h(\varphi,\rho_0)}).$$

For $\mathfrak{g} = \mathfrak{sl}(k,\mathbb{C})$, $k\geqslant 2$, this asserts that

$$\varphi(q)^k\varphi(q^k)^{-1} \prod_{1\leqslant i<j\leqslant k} (1-q^{j-i})^{-1}$$

$$= \sum_{\substack{\mu_1,\ldots,\mu_k\in k\mathbb{Z}\\ \Sigma\mu_i=0}} q^{(1/2)\Sigma_{i=1}^k \mu_i^2} \prod_{1\leqslant i<j\leqslant k} (1-q^{\mu_i-\mu_j+j-i})/(1-q^{j-i}).$$

When $k=1$, this formula becomes trivial. Multiplying through by $\varphi(q^k)$ and using Euler's formula for φ, we conclude:

THEOREM 3.4. *For all $k\geqslant 1$,*

$$\prod_{j\geqslant 1} (1-q^j)^{\min(j,k)}$$

$$= \left\{\sum_{\nu\in\mathbb{Z}} (-1)^\nu q^{(1/2)k\nu(3\nu+1)}\right\}\left\{\sum_{\substack{\mu_1,\ldots,\mu_k\in k\mathbb{Z}\\ \Sigma\mu_i=0}} q^{(1/2)\Sigma_{i=1}^k\mu_i^2} \prod_{1\leqslant i<j\leqslant k} \frac{1-q^{\mu_i-\mu_j+j-i}}{1-q^{j-i}}\right\}.$$

Remarks. (1) By MacMahon's theorem (1.1) on the generating function of k-rowed plane partitions [1, Corollary 11.3, p. 184], Theorem 3.4 is a formula for the reciprocal of this generating function. Note that for $k=1$, it reduces to Euler's recursion for ordinary partitions. (But recall also that we have used Euler's formula in the proof.)

(2) Theorem 3.4 uses the principal specialization, i.e., the specialization of type $(1,\dots,1)$. Note that our method is flexible enough to give other formulas for the reciprocal of the generating function of k-rowed plane partitions, by using for example the specialization of Proposition 2.2 of type $(2,1,\dots,1)$ for $\mathfrak{g} = \mathfrak{sl}(k+1, \mathbb{C})$. (For general complex simple $\mathfrak{g}$, this specialization was used in [10(a), (b)] to obtain a formula for $\eta(q)^{\operatorname{rank}\mathfrak{g}}$.)

By looking at Remark (2) in [10(a)], which lists the left-hand sides in Corollary 3.2 for general $\mathfrak{g}$, we see that Corollary 3.2 gives a new kind of formula for an arbitrary power of $\varphi(q)$. For example, for $\mathfrak{g} = \mathfrak{sl}(k, \mathbb{C})$:

THEOREM 3.5. *We have*

$$\varphi(q)^k = \Big\{\sum_{\nu\in\mathbb{Z}} (-1)^\nu q^{(1/2)k\nu(3\nu+1)}\Big\}\Big\{\sum_{\substack{\mu_1,\dots,\mu_k\in k\mathbb{Z}\\ \sum\mu_i=0}} q^{(1/2)\sum_{i=1}^k \mu_i^2} \prod_{1\leq i<j\leq k} (1-q^{\mu_i-\mu_j+j-i})\Big\}.$$

Remark. In view of Remark (2) after Theorem 3.4, we also get a new formula for the arbitrary power $\eta(q)^{\operatorname{rank}\mathfrak{g}}$ of $\eta(q)$ by using the $(2,1,\dots,1)$ specialization. This formula looks somewhat like the one in Theorem 3.1.

4. PRINCIPALLY SPECIALIZED CHARACTERS, POINCARÉ POLYNOMIALS OF HERMITIAN SYMMETRIC SPACES, AND CARDINALITIES OF FINITE VARIETIES

This section was written jointly with J. Jantzen.

Let $\mathfrak{l}$ be a complex simple Lie algebra of rank l. We use the notation of Section 2, with $\mathfrak{d} = 0$, and the index set taken to be $\{1,\dots,l\}$ in place of $\{0,\dots,l\}$.

Fix a value of $i = 1,\dots,l$ such that the corresponding simple root α_i' of the transpose Lie algebra $\mathfrak{l}'$ has coefficient 1 in the expansion of the highest root of $\mathfrak{l}'$ in terms of simple roots. Let V be the corresponding fundamental $\mathfrak{l}$-module, so that the highest weight λ of V is defined by the conditions $\lambda(h_j) = \delta_{ij}$ $(j = 1,\dots,l)$. Denote by χ_V the principally specialized character of V, so that $\chi_V(q)$ is the principal q-specialization of $\chi(V)/e(\lambda)$. For $w \in W$, denote by $l(w)$ the length of W, and for a subset W^* of W, define the polynomial

$$p(W^*)(q) = \sum_{w\in W^*} q^{l(w)}.$$

Let W_1 be the subgroup of W generated by the reflections r_j with $j \neq i$, and let W^1 be the set of those $w \in W$ such that $w^{-1}\alpha_j \in \Delta_+$ for all $j \neq i$. It is well known that $W = W_1 W^1$ and that

$$p(W^1) = p(W)/p(W_1).$$

Note that $p(W) = p(W')$, W' being the Weyl group of $\mathfrak{l}'$.

PROPOSITION 4.1. *We have*

$$\chi_V(q) = p(W^1)(q).$$

In particular, dim V *is the number of elements in* W^1.

Proof. Define the *height* ht ϕ of a root ϕ of $\mathfrak{l}'$ to be the sum of the expansion coefficients of ϕ in terms of the α_j'. Let $\Delta_+{}^1$ be the set of those positive roots of $\mathfrak{l}'$ involving α_i' in their expansion. Then the coefficient of α_i' in any root in $\Delta_+{}^1$ is exactly 1. Hence by Theorem 2.6,

$$\chi_V(q) = \prod_{\phi\in\Delta_+^1} \frac{1 - q^{1+ht\phi}}{1 - q^{ht\phi}}.$$

But by [14(b), Corollary 2.5] (see also [9]),

$$p(W)(q) = p(W')(q) = \prod_{\phi\in\Delta'_+} \frac{1 - q^{1+ht\phi}}{1 - q^{ht\phi}},$$

Δ'_+ being the set of positive roots for $\mathfrak{l}'$. Using the analogous formula for $p(W_1)(q)$ as well, we get

$$\chi_V(q) = p(W)(q)/p(W_1)(q) = p(W^1)(q). \qquad \text{Q.E.D.}$$

Remarks. (1) The following alternate proof of Proposition 4.1 gives a direct connection between the coefficients of $\chi_V(q)$ and those of $p(W^1)(q)$: The hypothesis on λ implies that λ is the only dominant weight of V, and that the weights of V are exactly the elements $w^{-1}\lambda$, $w \in W^1$, these elements being distinct. But it is easy to see that the height of $\lambda - w^{-1}\lambda$ is $l(w^{-1}) = l(w)$, the height of an element of $\mathfrak{h}^*$ being defined as the sum of its expansion coefficients in terms of $\alpha_1,\ldots,\alpha_l$. This completes the argument.

(2) It is easy to write down $\chi_V(q)$ in all the special cases, by using the well-known formula

$$p(W)(q) = \prod_{i=1}^{l} \frac{1 - q^{m_i+1}}{1 - q},$$

where $m_1,\ldots, m_l$ are the exponents of $\mathfrak{l}$. The results are as follows: For $\mathfrak{l}$ of type A_l, all l fundamental modules satisfy our condition, and we get the Gaussian polynomials $\left[{l+1 \atop i}\right](q)$, $i = 1,\ldots, l$. For $\mathfrak{l}$ of type B_l, C_l, E_6, E_7, respectively, there is only one relevant principally specialized character, and it equals $\prod_{j=1}^{l}(1 + q^j)$ (for the spin module), $(1 - q^{2l})/(1 - q)$, $(1 - q^9)(1 - q^{12})/(1 - q)(1 - q^4)$ (for two modules), $(1 + q^5)(1 + q^9)(1 - q^{14})/(1 - q)$, respectively. For $\mathfrak{l}$ of type D_l, the standard $2l$-dimensional module gives $(1 + q^{l-1})(1 - q^l)/(1 - q)$, and the two half-spin modules each give $\prod_{j=1}^{l-1}(1 + q^j)$.

Let G (respectively, G') be a complex connected Lie group with Lie algebra $\mathfrak{l}$ (respectively, $\mathfrak{l}'$), and let P (respectively, P') be the maximal parabolic subgroup corresponding to the index i. It is well known that the generalized flag manifolds G/P and G'/P' have only even-dimensional cells, and that the Poincaré polynomials of G/P and of G'/P' are both equal to $p(W^1)(q^2)$. Hence we have:

COROLLARY 4.2. (1) *The Poincaré polynomial of* G'/P' *is* $\chi_V(q^2)$.

(2) *For all* $j \geqslant 0$, *the* $2j$th *Betti number of* G'/P' *is* $\dim V_{-j}$, *where* V_{-j} *is the sum of the weight spaces of* V *for which the weight* μ *satisfies the condition* $\mathrm{ht}(\lambda - u) = j$.

(3) *The Euler characteristic of* G'/P' *is* $\dim V$.

Remark. The spaces G'/P' in Corollary 4.2 are precisely the irreducible compact Hermitian symmetric spaces. Corollary 4.2 also holds with G'/P' replaced by G/P.

For $\mathfrak{l} = \mathfrak{sl}(n, \mathbb{C})$ the ith flag manifold under consideration is the Grassmann variety of i-planes in $\mathbb{C}^n$. For $\mathfrak{l} = \mathfrak{so}(2n + 1, \mathbb{C})$, G/P is the variety of maximal totally isotropic subspaces of the orthogonal vector space $\mathbb{C}^{2n+1}$. For $\mathfrak{sp}(2n, \mathbb{C})$, G/P consists of the (isotropic) lines in the symplectic space $\mathbb{C}^{2n}$. For $\mathfrak{so}(2n, \mathbb{C})$ (standard $2n$-dimensional representation), we have the variety of isotropic lines in $\mathbb{C}^{2n}$ viewed as an orthogonal space, and for $\mathfrak{so}(2n, \mathbb{C})$ (the two half-spin representations), we have the varieties consisting of the two types of maximal totally isotropic subspaces of the orthogonal space $\mathbb{C}^{2n}$.

Now let q be a prime power, and let K be the field with q elements. Let $G[K]$ be a Chevalley group associated with $\mathfrak{l}$ and K, and let $P[K]$ be the obvious parabolic subgroup. By the standard formulas for the orders of finite Chevalley groups and parabolic subgroups (see, e.g., [18, 19]), we have

$$| G[K]|/| P[K]| = p(W^1)(q).$$

We conclude:

COROLLARY 4.3. *The number of points in the variety* $G[K]/P[K]$ *is* $\chi_V(q)$ (q *having its new meaning*).

The case $\mathfrak{l} = \mathfrak{sl}(n, \mathbb{C})$ gives the Grassmann variety of i-planes in n-space K^n. Thus the number $\left[{n \atop i}\right](q)$ of such i-planes is the principally specialized character χ_V evaluated at the number q. Analogous comments hold for the other classical Lie algebras and suitable finite varieties analogous to the complex examples above.

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