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The C^∞-convergence of SG circle patterns to the Riemann mapping ☆

Shi-Yi Lan [a,b], Dao-Qing Dai [c,*]

[a] *Department of Physics, Sun Yat-Sen (Zhongshan) University, Guangzhou 510275, PR China*

[b] *Department of Mathematics, Guangxi University for Nationalities, Nanning 530006, PR China*

[c] *Department of Mathematics, Sun Yat-Sen (Zhongshan) University, Guangzhou 510275, PR China*

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Abstract

Thurston conjectured that the Riemann mapping function from a simply connected region onto the unit disk can be approximated by regular hexagonal packings. Schramm introduced circle patterns with combinatorics of the square grid (SG) and showed that SG circle patterns converge to meromorphic functions. He and Schramm proved that hexagonal disk packings converge in C^∞ to the Riemann mapping. In this paper we show a similar C^∞-convergence for SG circle patterns.

Keywords: Circle pattern; Riemann mapping; C^∞-convergence; Discrete Schwarzian

1. Introduction

In 1985 Thurston [14] conjectured that the Riemann mapping function f from a simply connected region Ω onto the unit disk $\mathbb{U}$ can be approximated as follows. Almost fill Ω with circles of radius ϵ packed in the regular hexagonal pattern. There is a combinatorially isomorphic packing of circles in $\mathbb{U}$. This correspondence f_ϵ of ϵ-circle in Ω with circles of varying radii in $\mathbb{U}$

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* Corresponding author. Fax: +(86)(20)8403 7978.

E-mail addresses: shiyilan05@sina.com (S.-Y. Lan), stsddq@mail.sysu.edu.cn (D.-Q. Dai).

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should converge to f after suitable normalization. This conjecture was proved in [10]. Refinements and generalizations were given in [4–8,11,13], etc.

For the study of circle configurations, classical circle packings comprised by disjoint open disks are generalized to circle patterns, where the disks may overlap. Schramm [12] introduced circle patterns with the combinatorics of the square grid and showed that SG circle patterns converge to meromorphic functions. In comparison with hexagonal circle packings, the framework of SG circle patterns is more tractable. Doyle has constructed entire immersed hexagonal circle packings analogous to the exponential map and conjectured that these packings (i.e., Doyle spirals) are the only entire immersed hexagonal circle packings (see [3]). For SG circle patterns, this conjecture is false because there is an SG circle pattern analogous to the entire function $erf(\sqrt{iz})$, where $erf(z) = \int^z e^{-w^2}\,dw$ (see [12]). Next, the collection of entire SG circle patterns in the sphere is infinite-dimensional and SG circle patterns can be used to exhibit some explicit finite-dimensional families (see [1,2,12]).

In [7] He and Schramm proved that hexagonal disk packings converge in $C^\infty(\Omega)$ to the Riemann mapping. Then it is natural to ask whether there is similar C^∞-convergence for SG circle patterns. Here, we shall give an affirmative answer to this question.

Our method involves modifications and generalizations of those in [7]. We define two discrete Schwarzians, and their boundedness is proved directly using the formulas of Laplacian and Taylor for the Schwarzians, while in [7] only one discrete Schwarzian is needed and the proof of its boundedness is equivalent to that of the third order derivatives (see [7, Lemma 6.1]). Moreover, we construct Möbius transformations by means of the three intersection points in each circle, whereas in [7] they are defined by the three tangent points of three circles in the hexagonal circle packings (see [7, §9]). We also note that recent work of Matthes [9] proves C^∞-convergence of SG circle patterns to the conformal map. However, our approach is different from that of Matthes. First, Matthes constructs the approximating functions by solving a discrete Cauchy problem. While we define directly the approximating mappings using the relationship between two combinatorially equivalent SG circle patterns. Secondly, any order partial derivatives of the discrete functions in [9] are defined as the respective difference quotients. Here the partial derivatives of the discrete functions are defined by the discrete directional derivatives. Thirdly, it is proved [9] that the discrete solutions converge to the solution of analytic Cauchy problem in C^∞, which lead to C^∞-convergence of the corresponding circle patterns. Whereas here it is shown the Möbius transformations converge in C^∞. Using the relation between the approximating mappings and the Möbius transformations, we get that the approximating functions converge to the Riemann mapping in C^∞.

This paper is organized as follows. We shall introduce briefly some terminology and definitions related to SG circle patterns and discrete differential operators in Section 2. In Section 3, using the Möbius invariants for SG circle patterns, we give the definition of discrete Schwarzians of SG circle patterns. Then we construct suitable Möbius transformations that can be expressed in terms of the discrete Schwarzians. In Section 4, we shall show that the discrete Schwarzians are uniformly bounded in C^∞. We shall show C^∞ convergence in Section 5.

2. SG circle patterns and discrete differential operators

In this section we shall introduce briefly some terminology and definitions related to SG circle patterns and discrete differential operators (see [7,12] for more details).

For each positive number $\epsilon > 0$, let SG^ϵ be the cell complex whose vertices form the square lattice $V^\epsilon = \epsilon\mathbb{Z} + i\epsilon\mathbb{Z} = \{v = \epsilon n + i\epsilon m\colon\ (n, m) \in \mathbb{Z} \times \mathbb{Z}\}$, whose edges are the pair $[v, v']$ such

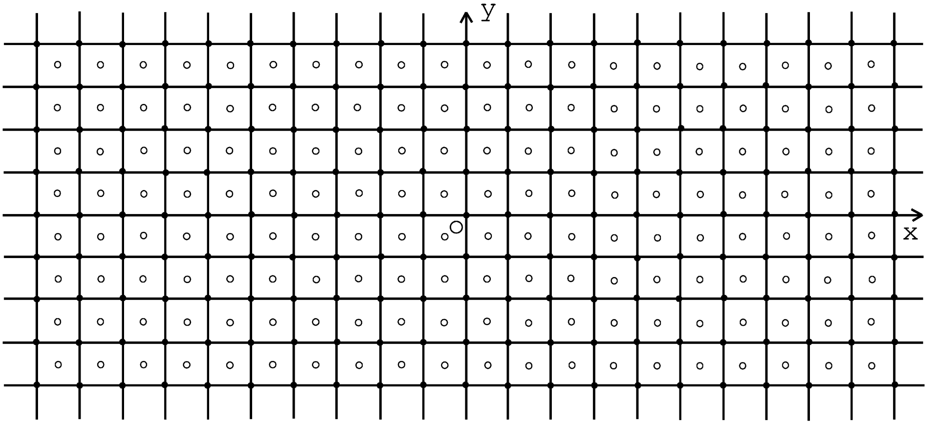

Fig. 1. Cell complex SG^ϵ. V^ϵ and $\hat{V}^\epsilon$ consist of the black points and the white points, respectively.

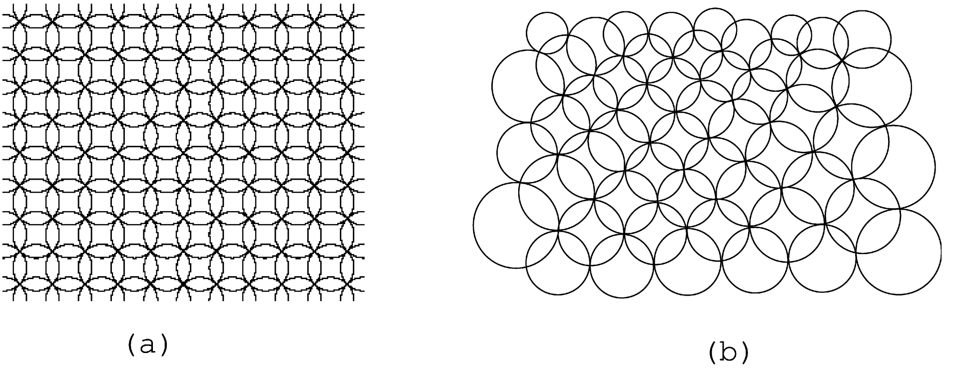

Fig. 2. (a) Regular SG circle pattern and (b) general SG circle pattern.

that $|v - v'| = \epsilon$ and $v, v' \in V^\epsilon$, and whose 2-cells are the squares $\{v + x + iy\colon\ x, y \in [0, \epsilon]\}$, $v \in V^\epsilon$ (see Fig. 1).

An indexed collection $P^\epsilon = \{P_v\colon\ v \in V^\epsilon\}$ of oriented circles in the Riemann sphere $\hat{\mathbb{C}}$ is said to be a circle pattern for SG^ϵ or a SG circle pattern (see Fig. 2) if the following three conditions hold.

(a) Whenever v, v' are neighbors in SG^ϵ, the corresponding circles $P_v, P_{v'}$ intersect orthogonally.

(b) If v_1, v_2 are neighbors of a vertex v in SG^ϵ, and they belong to the same square of SG^ϵ, then the circles P_{v_1}, P_{v_2} are distinct and tangent.

(c) Whenever the situation is as in (b) and v_2 is neighbor of v, which is one step counterclockwise from v_1, the circular order of the triplet of points $P_v \cap P_{v_1} - P_{v_2}$, $P_{v_1} \cap P_{v_2}$, $P_v \cap P_{v_2} - P_{v_1}$ agrees with the orientation of P_v.

Clearly, the 1-skeleton of SG^ϵ is the graph of regular SG circle pattern Q^ϵ whose each circle has radius equal to $\epsilon/\sqrt{2}$.

Let W be a subset of V^ϵ. A vertex $v \in W$ is said to be an interior vertex of W if all its neighboring vertices are contained in W. Let *int* W denote the set of the interior vertices of W. Given a function $\rho: W \to \mathbb{R}$, we define the discrete directional derivative $\partial\rho_j : \mathit{int}\, W \to \mathbb{R}$ by

$$\partial_j^\epsilon \rho(v) = \bigl(\rho\bigl(v + i^j\epsilon\bigr) - \rho(v)\bigr)/\epsilon,$$

for each $j \in \mathbb{Z}_4$. The discrete Laplacian of a function $\rho: W \to \mathbb{R}$ is a function in *int* W defined by the formula

$$\Delta^\epsilon \rho(v) = 1/\bigl(4\epsilon^2\bigr) \sum_{j=0}^{3} \bigl(\rho\bigl(v + i^j\epsilon\bigr) - \rho(v)\bigr).$$

Suppose that Ω is a simply connected bounded region in the complex plane. For any differentiable function $F: \Omega \to \mathbb{R}$, let $\partial_j F$ denote the directional derivative

$$\partial_j F(z) = \lim_{s\to 0} \frac{F(z + i^j s) - F(z)}{z},$$

where $j = 0, 1, 2, 3$. Let $f: \Omega \to \mathbb{C}$ be some function defined in Ω. For each $\epsilon > 0$, let f^ϵ be some function defined on some set of vertices $\widetilde{V}^\epsilon \subset V^\epsilon$, with values in $\mathbb{C}^d$. Assume that for each $z \in \Omega$, there are some $\delta_1, \delta_2 > 0$ such that $\{v \in V^\epsilon\colon |v - z| < \delta_2\} \subset \widetilde{V}^\epsilon$ whenever $\epsilon \in (0, \delta_1)$.

If for every $z \in \Omega$ and every $\delta > 0$, there are some $\delta_1, \delta_2 > 0$ such that $|f(z) - f^\epsilon(v)| < \delta$, for every $\epsilon \in (0, \delta_1)$ and every $v \in V^\epsilon$ with $|v - z| < \delta_2$, then we say f^ϵ converges to f, locally uniformly in Ω.

Let $n \in \mathbb{N}$, and suppose that f is C^n-smooth. If for every sequence $j_1, j_2, \ldots, j_k \in \mathbb{Z}_4$ with $k \leqslant n$ we have $\partial_{j_k}^\epsilon \partial_{j_{k-1}}^\epsilon \cdots \partial_{j_1}^\epsilon f^\epsilon \to \partial_{j_k}\partial_{j_{k-1}} \cdots \partial_{j_1} f$ locally uniformly in Ω, then we say that f^ϵ converges to f in $C^n(\Omega)$. The functions f^ϵ are said to be uniformly bounded in $C^n(\Omega)$ provided that for every compact $K \subset \Omega$ there is some constant $C(K, n)$ such that

$$\bigl\|\partial_{j_k}^\epsilon \partial_{j_{k-1}}^\epsilon \cdots \partial_{j_1}^\epsilon f^\epsilon\bigr\|_{K\cap V^\epsilon} < C(K, n)$$

whenever $k \leqslant n$, and ϵ is sufficiently small, where $\|\cdot\|$ denotes L^∞-norm. The functions f^ϵ are uniformly bounded in $C^\infty(\Omega)$, if they are uniformly bounded in $C^n(\Omega)$ for every $n \in \mathbb{N}$.

3. The Möbius invariants and discrete Schwarzians

In this section, using the Möbius invariants for SG circle patterns (also see [12]), we give the definition of discrete Schwarzians of SG circle patterns. Then we construct suitable Möbius transformations that can be expressed in terms of the discrete Schwarzians.

For the cell complex SG^ϵ, let $\hat{V}^\epsilon$ denote the set of centers of squares of SG^ϵ, i.e., $\hat{V}^\epsilon = \{v + \epsilon/2 + i\epsilon/2\colon v \in V^\epsilon\}$ (see Fig. 1). For the sake of convenience, let $\omega_j = i^j(\epsilon/2 + i\epsilon/2)$ for any $j = 0, 1, 2, 3$. Let $P^\epsilon = \{P_v^\epsilon\colon v \in V^\epsilon\}$ be a circle pattern for SG^ϵ on the Riemann sphere $\hat{\mathbb{C}}$. Note that the intersection of the four circles $P_{u+\omega_j}^\epsilon$ $(j = 0, 1, 2, 3)$ is a single point for any $u \in \hat{V}^\epsilon$. We denote the point by $q(u)$. For any $v \in V^\epsilon$, note that the four points $q(v + \omega_j)$ $(j = 0, 1, 2, 3)$ are all on the circle P_v^ϵ, so the first Möbius invariant function $\nu_\epsilon^{(1)}: V^\epsilon \to \mathbb{R}$ of P^ϵ is defined by

$$\begin{aligned}\nu_\epsilon^{(1)}(v) &= -\operatorname{cr}\bigl[q(v + \omega_0), q(v + \omega_2); q(v + \omega_3), q(v + \omega_1)\bigr]\\ &= -\frac{(q(v + \omega_0) - q(v + \omega_3))(q(v + \omega_2) - q(v + \omega_1))}{(q(v + \omega_0) - q(v + \omega_1))(q(v + \omega_2) - q(v + \omega_3))}.\end{aligned} \tag{1}$$

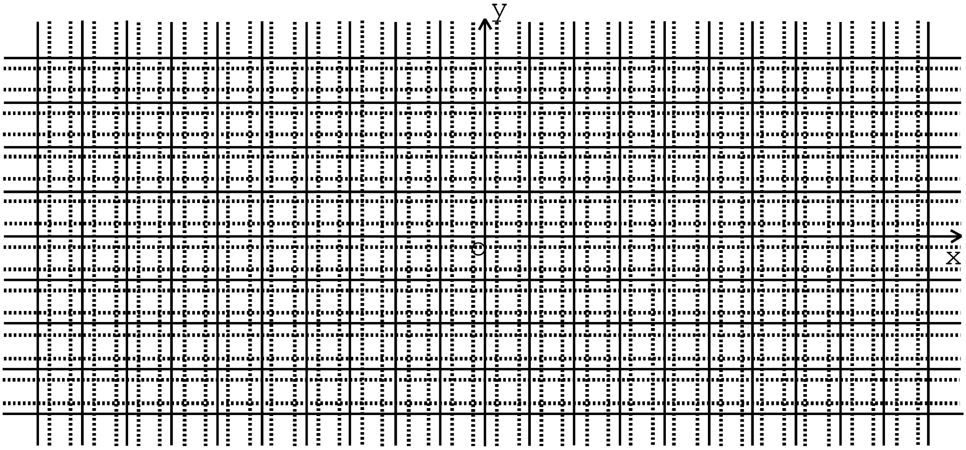

Fig. 3. The cell complex SG_h^ϵ consists of these small squares with dotted edges.

It is clear that $\nu_\epsilon^{(1)}(v)$ does not change when a Möbius transformation is applied to all four points and that $\nu_\epsilon^{(1)}(v) > 0$ because $q(v+\omega_0), q(v+\omega_2)$ separates $q(v+\omega_3)$ and $q(v+\omega_1)$. The second Möbius invariant function $\nu_\epsilon^{(2)} : \hat{V}^\epsilon \to \mathbb{C}$ of P^ϵ is defined by

$$\nu_\epsilon^{(2)}(u) = -\operatorname{cr}\bigl[q(u+\epsilon), q(u-\epsilon); q(u-i\epsilon), q(u+i\epsilon)\bigr]$$

for each $u \in \hat{V}^\epsilon$. Note that $\nu_\epsilon^{(2)}$ does not change if we apply a Möbius transformation to P^ϵ. Let m be a Möbius transformation that takes $q(u)$ to ∞. Then the four circles $m(P^\epsilon_{u+\omega_j})$ $(j = 0, 1, 2, 3)$ are mapped to lines, and together form a rectangle with vertices $m(q(u+\epsilon))$, $m(q(u+i\epsilon))$, $m(q(u-\epsilon))$ and $m(q(u-i\epsilon))$. Hence we deduce easily that

$$\nu_\epsilon^{(2)}(u) = -\left(\frac{m(q(u+\epsilon)) - m(q(u-i\epsilon))}{m(q(u+\epsilon)) - m(q(u+i\epsilon))}\right)^2 = \frac{|m(q(u+\epsilon)) - m(q(u-i\epsilon))|^2}{|m(q(u+\epsilon)) - m(q(u+i\epsilon))|^2}. \tag{2}$$

So $\nu_\epsilon^{(2)}$ is real and positive. Here it is pointed out that $\nu_\epsilon^{(j)}$ $(j = 1, 2)$ is similar to s in [7, §4].

Now we define the two discrete Schwarzians of P^ϵ as follows: let

$$h_\epsilon^{(1)}(v) = \epsilon^{-2}\bigl(\nu_\epsilon^{(1)}(v) - 1\bigr) \tag{3}$$

for each vertex $v \in V^\epsilon$ and let

$$h_\epsilon^{(2)}(u) = \epsilon^{-2}\bigl(\nu_\epsilon^{(2)}(u) - 1\bigr) \tag{4}$$

for every $u \in \hat{V}^\epsilon$. It is easy to see that $h_\epsilon^{(1)} = h_\epsilon^{(2)} \equiv 0$ if P^ϵ is a regular SG circle pattern, because $\nu_\epsilon^{(1)} = \nu_\epsilon^{(2)} \equiv 1$ for regular SG circle patterns. And the same, $h_\epsilon^{(j)}$ $(j = 1, 2)$ is also similar to h in [7, §4].

Let SG_h^ϵ denote $(1/2)SG^\epsilon + (1+i)\epsilon/4$, and the set of vertices of SG_h^ϵ is denoted by V_h^ϵ (see Fig. 3). For any $w \in V_h^\epsilon$, let $v \in V^\epsilon$ be the unique vertex of SG^ϵ that is closest to w. Set $u_1 = v + 2(w - v)$, $u_2 = v + 2i(w - v)$, $u_3 = v - 2i(w - v)$, then $u_1, u_2, u_3 \in \hat{V}^\epsilon$. Let M_w be the Möbius transformation that takes $\infty, 0, 1$ to $q(u_1), q(u_2), q(u_3)$, respectively. Set

$$M_{[w_1,w_2]} = M_{w_1}^{-1} \circ M_{w_2}$$

for each directed edge $[w_1, w_2]$ of SG_h^ϵ. Let $e_j(\tilde{v}) = [\tilde{v} + \omega_j/2, \tilde{v} + \omega_{j+1}/2]$ for any $\tilde{v} \in V^\epsilon \cup \hat{V}^\epsilon$ and for any $j \in \mathbb{Z}_4$, then we have

Lemma 1. *There hold for the following equalities*:

$$M_{e_2(u)}(z) = M_{e_0(u)}(z) = 1 - i\epsilon\big(h_\epsilon^{(2)}(u) + 1\big)^{1/2} z, \tag{5}$$

$$M_{e_3(u)}(z) = M_{e_1(u)}(z) = 1 - i(1/\epsilon)\big(h_\epsilon^{(2)}(u) + 1\big)^{-1/2} z \tag{6}$$

for each $u \in \hat{V}^\epsilon$ *and*

$$M_{e_2(v)}(z) = M_{e_0(v)}(z) = \big(\epsilon^{-2}\big(h_\epsilon^{(1)}(v) + 1\big)^{-1} + 1\big)^{-1}/(1-z), \tag{7}$$

$$M_{e_3(v)}(z) = M_{e_1(v)}(z) = \big(\epsilon^{2}\big(h_\epsilon^{(1)}(v) + 1\big) + 1\big)^{-1}/(1-z) \tag{8}$$

for every $v \in V^\epsilon$.

Proof. For any $u \in \hat{V}^\epsilon$, we first consider the directed edge $e_2(u) = [u + \omega_2/2, u + \omega_3/2] \in SG_h^\epsilon$. By the definition of $M_{[w_1,w_2]}$, we get that $M_{[w_1,w_2]}$ does not change if we apply a Möbius transformation to P^ϵ. So we may assume that $M_{u+\omega_2/2}$ is the identity. It follows from the definition of M_w that

$$q(u) = \infty, \qquad q(u-\epsilon) = 0, \qquad q(u - i\epsilon) = 1.$$

Hence we deduce that the four points $q(u \pm \epsilon), q(u \pm i\epsilon)$ form the vertices of a rectangle. From (2), we get that

$$\nu_\epsilon^{(2)}(u) = -\left(\frac{q(u+\epsilon) - q(u - i\epsilon)}{q(u - i\epsilon) - q(u-\epsilon)}\right)^2,$$

which implies

$$q(u+\epsilon) = q(u - i\epsilon) - i\big(\nu_\epsilon^{(2)}\big)^{1/2}\big(q(u-i\epsilon) - q(u-\epsilon)\big) = 1 - i\big(\nu_\epsilon^{(2)}\big)^{1/2}.$$

Since that $M_{[u+\omega_2/2, u+\omega_3/2]}$ takes $\infty, 0, 1$ to $q(u), q(u - i\epsilon), q(u+i\epsilon)$, respectively, we conclude that

$$M_{e_2(u)}(z) = 1 - i\big(\nu_\epsilon^{(2)}\big)^{1/2} z.$$

It follows from (4) that $M_{e_2(u)}(z) = 1 - i\epsilon(h_\epsilon^{(2)}(u) + 1)^{1/2} z$. Similarly, we deduce that $M_{e_0(u)}(z) = 1 - i\epsilon(h_\epsilon^{(2)}(u) + 1)^{1/2} z$ and that (6) holds.

Next, we consider the directed edge $e_0(v) = [v + \omega_0/2, v + \omega_1/2] \in SG_h^\epsilon$ for any $v \in V^\epsilon$. We assume without loss of generality that $M_{v+\omega_0/2}$ is the identity. Then we obtain

$$q(v + \omega_0) = \infty, \qquad q(v + \omega_1) = 0, \qquad q(v + \omega_3) = 1.$$

By (1), we get

$$q(v + \omega_2) = 1/\big(\nu_\epsilon^{(1)}(v) + 1\big).$$

Note that $M_{[v+\omega_0/2, v+\omega_1/2]}$ takes $\infty, 0, 1$ to $q(v+\omega_1), q(v+\omega_2), q(v+\omega_0)$, respectively, combining with (3), we deduce that

$$M_{e_0(v)} = \big(\big(\nu_\epsilon^{(1)}(v)\big)^{-1} + 1\big)^{-1}/(1-z) = \big(\epsilon^{-2}\big(h_\epsilon^{(1)}(v) + 1\big)^{-1} + 1\big)^{-1}/(1-z),$$

with the same arguments as above, we deduce that

$$M_{e_2(v)}(z) = \big(\epsilon^{-2}\big(h_\epsilon^{(1)}(v) + 1\big)^{-1} + 1\big)^{-1}/(1-z)$$

and that (8) holds. □

4. The C^∞-boundedness of discrete Schwarzians

We shall show that the discrete Schwarzians are uniformly bounded in $C^\infty(\Omega)$ in this section.

Without loss of generality, we assume $0 \in \Omega$. For any $\epsilon > 0$, let $\widetilde{SG}^\epsilon_\Omega$ be the union of squares of SG^ϵ that are contained in Ω and whose distance to $\partial\Omega$ is greater than or equal to $\sqrt{2}\epsilon/2$. Let SG^ϵ_Ω be the closure of the connected component of the interior of $\widetilde{SG}^\epsilon_\Omega$ that contains 0 (see Fig. 4). The set of vertices of SG^ϵ_Ω is denoted by V^ϵ_Ω, and the set of centers of squares of SG^ϵ_Ω by $\hat{V}^\epsilon_\Omega$. It is clear that the 1-skeleton of SG^ϵ_Ω is the graph of some sub-pattern of Q^ϵ contained in Ω.

Let $f : \Omega \to \mathbb{U}$ be a Riemann function, where $\mathbb{U}$ is the unit disk. Set

$$\nu^{(1)}_\epsilon(v) = 1 + \epsilon^2 \operatorname{Re} S_f(v)$$

for each boundary vertex $v \in \partial V^\epsilon_\Omega$, where S_f denotes the Schwarzian derivative of f. Then SG^ϵ Dirichlet principle [12, Theorem 6.2] implies that $\nu^{(1)}_\epsilon$ is extended to a solution of the SG^ϵ-Dirichlet problem on SG^ϵ_Ω. Let $\nu^{(2)}_\epsilon$ be the companion of $\nu^{(1)}_\epsilon$ in the solution of the $SG^\epsilon - CR$ equation [12, §5], i.e.,

$$\frac{\nu^{(2)}_\epsilon(v+\omega_0)}{\nu^{(2)}_\epsilon(v+\omega_1)} = \left(\frac{\nu^{(1)}_\epsilon(v+i\epsilon)^{-1}+1}{\nu^{(1)}_\epsilon(v)^{-1}+1}\right)^2 \tag{9}$$

and

$$\frac{\nu^{(2)}_\epsilon(v+\omega_0)}{\nu^{(2)}_\epsilon(v+\omega_3)} = \left(\frac{\nu^{(1)}_\epsilon(v+\epsilon)+1}{\nu^{(1)}_\epsilon(v)+1}\right)^2 \tag{10}$$

for any $v \in V^\epsilon_\Omega$, such that

$$\nu^{(2)}_\epsilon(\omega_j) = 1 + \epsilon^2 \operatorname{Im} S_f(\omega_j), \tag{11}$$

where $\omega_j = i^j(\epsilon/2 + i\epsilon/2)$ $(j = 0, 1, 2, 3)$. Then we have

Lemma 2.

(i) *The equality*

$$h^{(1)}_\epsilon(v) = \operatorname{Re} S_f(v) + \epsilon^2 \cdot O(1) \tag{12}$$

holds for each $v \in V^\epsilon_\Omega$.

(ii) *The equality*

$$h^{(2)}_\epsilon(u) = \operatorname{Im} S_f(u) + \delta_\epsilon(u)\epsilon^2 \cdot O(1) \tag{13}$$

holds for each $v \in \hat{V}^\epsilon_\Omega$, where $\delta_\epsilon(u)$ denotes the combinatorial distance in SG^ϵ_Ω from u to ω_0.

Proof. (i) First, for any $v \in int\, V^\epsilon_\Omega$, expand S_f in power series about v and note that $\sum_{k=0}^3 i^{jk} = 0$ for $j = 1, 2, 3$, we obtain

$$\Delta^\epsilon \operatorname{Re} S_f = \operatorname{Re}\left(1/\left(4\epsilon^2\right)\sum_{j=0}^{3}\left[S_f\left(v+i^j\epsilon\right) - S_f(v)\right]\right) = O\left(\epsilon^2\right).$$

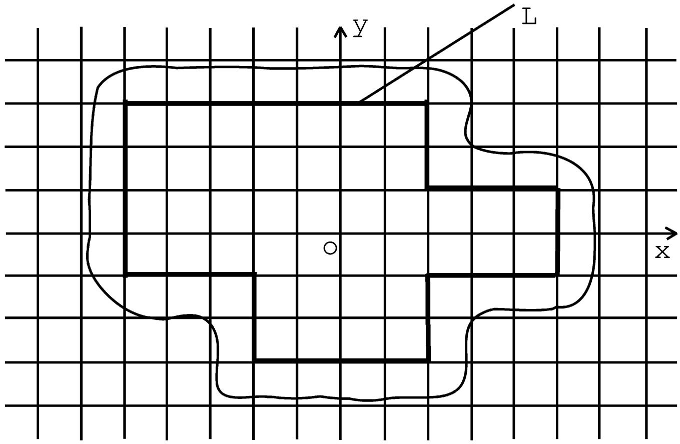

Fig. 4. The cell complex SG_Ω^ϵ is the closure of domain bounded by closed curve L.

Consider the function

$$g_1(v) = h_\epsilon^{(1)}(v) - \operatorname{Re} S_f + \beta|v|^2,$$

where $\beta \in (0, \epsilon^2)$ is some function of ϵ. Similar to the proof of [12, Lemma 9.2], we deduce from the Taylor's formula and the properties of Möbius invariant $h_\epsilon^{(1)}$ that g_1 has no maxima in *int* V_Ω^ϵ if β is chosen as $\beta = C\epsilon^2$ with $C > 0$ a sufficiently large constant. Since that $g_1(v) = \beta|v|^2 = O(\epsilon^4)$ on $\partial V_\Omega^\epsilon$, we get

$$h_\epsilon^{(1)}(v) \leqslant \operatorname{Re} S_f(v) + O\big(\epsilon^2\big).$$

On the other hand, if we let $g_2(v) = h_\epsilon^{(1)}(v) - \operatorname{Re} S_f(v) - \beta|v|^2$, then similar to the above arguments, we deduce that

$$h_\epsilon^{(1)}(v) \geqslant \operatorname{Re} S_f(v) + O\big(\epsilon^2\big).$$

So it follows that

$$h_\epsilon^{(1)}(v) = \operatorname{Re} S_f(v) + O\big(\epsilon^2\big).$$

(ii) It follows from (9) that

$$\log \nu_\epsilon^{(2)}(v + \omega_0) - \log \nu_\epsilon^{(2)}(v + \omega_1) = 2\log\big(\nu_\epsilon^{(1)}(v + i\epsilon)^{-1} + 1\big) - 2\log\big(\nu_\epsilon^{(1)}(v)^{-1} + 1\big).$$

By (3) and Taylor formula, we deduce that

$$\begin{aligned}
&\log \nu_\epsilon^{(2)}(v + \omega_0) - \log \nu_\epsilon^{(2)}(v + \omega_1) \\
&\quad = 2\log\big(\big(1 + \epsilon^2 h_\epsilon^{(1)}(v + i\epsilon)\big)^{-1} + 1\big) - 2\log\big(\big(1 + \epsilon^2 h_\epsilon^{(1)}(v)\big)^{-1} + 1\big) \\
&\quad = 2\log\big(2 - \epsilon^2 h_\epsilon^{(1)}(v + i\epsilon)\big) - 2\log\big(2 - \epsilon^2 h_\epsilon^{(1)}(v)\big) + O\big(\epsilon^4\big) \\
&\quad = \epsilon^2 h_\epsilon^{(1)}(v) - \epsilon^2 h_\epsilon^{(1)}(v + i\epsilon) + O\big(\epsilon^4\big).
\end{aligned}$$

Hence it follows from (12) that

$$\begin{aligned}\log \nu_\epsilon^{(2)}(v+\omega_0) - \log \nu_\epsilon^{(2)}(v+\omega_1) &= \epsilon^2 \operatorname{Re}\bigl(S_f(v) - S_f(v+i\epsilon)\bigr) + O\bigl(\epsilon^4\bigr) \\ &= \epsilon^2 \operatorname{Re}\bigl(i\epsilon S_f'(v)\bigr) + O\bigl(\epsilon^4\bigr) \\ &= \epsilon^3 \operatorname{Re} S_f'(v) + O\bigl(\epsilon^4\bigr) \\ &= \epsilon^2 \operatorname{Im} S_f(v+i\epsilon) - \epsilon^2 \operatorname{Im} S_f(v) + O\bigl(\epsilon^4\bigr). \qquad (14)\end{aligned}$$

Similarly, we deduce from (10) that

$$\log \nu_\epsilon^{(2)}(v+\omega_0) - \log \nu_\epsilon^{(2)}(v+\omega_3) = \epsilon^2 \operatorname{Im} S_f(v+\epsilon) - \epsilon^2 \operatorname{Im} S_f(v) + O\bigl(\epsilon^4\bigr). \qquad (15)$$

By (11), we deduce that

$$\log \nu_\epsilon^{(2)}(\epsilon/2 + i\epsilon/2) = \epsilon^2 \operatorname{Im} S_f(\epsilon/2 + i\epsilon/2) + O\bigl(\epsilon^4\bigr). \qquad (16)$$

So we conclude from (14)–(16) that

$$\log \nu_\epsilon^{(2)}(u) = \epsilon^2 \operatorname{Im} S_f(u) + \delta_\epsilon(u) O\bigl(\epsilon^2\bigr),$$

which, combining with (4), implies that the equality (13) holds. □

Lemma 3. *Let $v \in V_\Omega^\epsilon$, $u \in \hat{V}_\Omega^\epsilon$, and suppose that the distance δ from v (respectively, u) to $\partial\Omega$ is greater than 2ϵ. Then*

$$h_\epsilon^{(1)}(v) \leqslant C, \qquad h_\epsilon^{(2)}(u) \leqslant C$$

for some constant $C = C(\delta, f)$ which depends only on δ and f.

Proof. Note that $\operatorname{Re} S_f$ and $\operatorname{Im} S_f$ are bounded on compact subset $K \subset \Omega$ and $\delta_\epsilon = O(1/\epsilon)$ on K. By Lemma 2, we conclude that the lemma holds. □

Similar to the case of regular hexagonal lattice (see [7, Lemma 7.1]), for regular square lattice we have

Lemma 4. *Suppose* (i) *W is a subset of V^ϵ (or $\hat{V}^\epsilon$)*; (ii) *$u \in int\, W$*; (iii) *δ is the distance from u to $V^\epsilon - W$ (or $\hat{V}^\epsilon - W$). If $\psi : W \to \mathbb{R}$, then the inequality*

$$\delta\bigl|\partial_j^\epsilon \psi(u)\bigr| < 5\|\psi\| + (1/2)\delta^2 \bigl\|\Delta^\epsilon \psi\bigr\|_{int\,W} \qquad (17)$$

holds for $j = 0, 1, 2, 3$.

Theorem 1. *Let n be an integer, and let V_δ^ϵ (respectively, $\hat{V}_\delta^\epsilon$) be the set of vertices of V_Ω^ϵ (respectively, $\hat{V}_\Omega^\epsilon$) whose distance to $\partial\Omega$ is at least δ for each $\delta > 0$. Then there are constants $C = C(n, \delta)$ and $\mu = \mu(n, \delta) > 0$ such that*

$$\bigl\|\partial_{j_n}^\epsilon \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon h_\epsilon^{(1)}\bigr\|_{V_\delta^\epsilon} < C, \qquad \bigl\|\partial_{j_n}^\epsilon \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon h_\epsilon^{(2)}\bigr\|_{\hat{V}_\delta^\epsilon} < C \qquad (18)$$

holds for each $\epsilon < \mu$, and $j_0, j_1, \ldots, j_n \in \mathbb{Z}_4$.

Proof. The proof proceeds by induction on n. The case $n = 0$, by Lemma 3, we get that (18) holds. So we suppose that $n > 0$, and that (18) holds for $0, 1, 2, \ldots, n-1$. Let

$$\varphi_1 = \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon h_\epsilon^{(1)}, \qquad \varphi_2 = \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon h_\epsilon^{(2)}.$$

Note that the operators Δ^ϵ and ∂_j^ϵ commute with each other, we have

$$\Delta^\epsilon \varphi_1 = \Delta^\epsilon \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon h_\epsilon^{(1)} = \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon \Delta^\epsilon h_\epsilon^{(1)},$$

and

$$\Delta^\epsilon \varphi_2 = \Delta^\epsilon \partial_{j_{n-1}} \cdots \partial_{j_1}^\epsilon h_\epsilon^{(2)} = \partial_{j_{n-1}}^\epsilon \cdots \partial_{j_1}^\epsilon \Delta^\epsilon h_\epsilon^{(2)}.$$

By Lemma 2, it follows that

$$\Delta^\epsilon h_\epsilon^{(1)} = \Delta^\epsilon \operatorname{Re} S_f + \Delta^\epsilon\big(\epsilon^2 O(1)\big), \qquad \Delta^\epsilon h_\epsilon^{(2)} = \Delta^\epsilon \operatorname{Im} S_f + \Delta^\epsilon\big(\delta_\epsilon \epsilon^2 O(1)\big).$$

Since that $\Delta^\epsilon S_f = O(\epsilon^2)$ and $\delta_\epsilon = O(1/\epsilon)$ on compact subset $K \subset \Omega$, we have

$$\Delta^\epsilon h_\epsilon^{(1)} = O\big(\epsilon^2\big) + O(1), \qquad \Delta^\epsilon h_\epsilon^{(2)} = O\big(\epsilon^2\big) + \Delta^\epsilon\big(\epsilon O(1)\big)$$

on compact subset $K \subset \Omega$. Note that $O(1) \in C^\infty(\Omega)$, we deduce that there exists a constant $C_1 = C_1(\delta, n)$ such that

$$\big\|\Delta^\epsilon \varphi_1(v)\big\|_{V_\delta^\epsilon} \leqslant C_1, \qquad \big\|\Delta^\epsilon \varphi_2(v)\big\|_{\hat{V}_\delta^\epsilon} \leqslant C_1.$$

Since that $|\varphi_1|$ and $|\varphi_2|$ are bounded on V_δ^ϵ and $\hat{V}_\delta^\epsilon$, respectively, it follows from Lemma 4 that $|\partial_{j_n}^\epsilon \varphi_1|$ and $|\partial_{j_n}^\epsilon \varphi_2|$ are bounded on V_δ^ϵ and $\hat{V}_\delta^\epsilon$, respectively, which completes the induction step. So (18) holds for any integer n and any $j_0, j_1, \ldots, j_n \in \mathbb{Z}_4$. □

5. The C^∞-convergence of SG circle patterns

In this section we shall show that SG circle patterns converge in $C^\infty(\Omega)$ to the Riemann mapping function.

For $\nu_\epsilon^{(1)}$ and $\nu_\epsilon^{(2)}$ given in Section 4, based on the local theory of SG circle patterns [12, Theorem 6.1], there exists an SG circle pattern $P_\Omega^\epsilon = \{P_v^\epsilon\colon v \in V_\Omega^\epsilon\}$ for SG_Ω^ϵ in the unit disk $\mathbb{U}$ that has $\nu_\epsilon^{(1)}$ and $\nu_\epsilon^{(2)}$ as its Möbius invariants. Suppose that P_Ω^ϵ is normalized by Möbius transformation so that

$$q(\omega_0) = f(\omega_0), \qquad q(\omega_1) = f(\omega_1), \qquad q(\omega_3) = f(\omega_3), \tag{19}$$

where $\omega_j = i^j(1+i)\epsilon/2$ ($j = 0, 1, 3$). For each $v \in V_\Omega^\epsilon$, let $g^\epsilon(v)$ be the point $q(v + (1+i)\epsilon/2)$, and let $f^\epsilon(v)$ denote the center of the disk P_v^ϵ, and $r^\epsilon(v)$ the radius of P_v^ϵ. Then we have

Theorem 2. *The discrete functions $f^\epsilon(v)$ and $g^\epsilon(v)$ converge to the conformal mapping $f(z)\colon \Omega \to \mathbb{U}$ in $C^\infty(\Omega)$, and $r^\epsilon(v)/\epsilon$ converges in $C^\infty(\Omega)$ to $|f'|$, where $\mathbb{U}$ denotes the unit disk and $|f'|$ is the module of the derivative of f.*

In order to prove the theorem, we need the following lemma.

Lemma 5. *For each $v \in int\, V_\Omega^\epsilon$, let $T^\epsilon = T^\epsilon(v)$ be the Möbius transformation that takes the three points $\omega_0, \omega_1, \omega_3$ to the points $q(v+\omega_0)$, $q(v+\omega_1)$, $q(v+\omega_3)$, respectively. Then*

(i) *the limit*

$$T(z) = \lim_{\epsilon \to 0} T^\epsilon(v) \tag{20}$$

exists for any $\epsilon \to 0$, and the convergence is $C^\infty(\Omega)$,

(ii) $T(z)(0) = f(z)$.

Proof. (i) Let $B = B^\epsilon$ be the Möbius transformation that takes $\infty, 0, 1$ to $\omega_0, \omega_1, \omega_3$, respectively. Then we have

$$B(z) = \frac{\epsilon z - (1+i)\epsilon/2}{(1-i)z + i}. \tag{21}$$

Recalling the definition of M_w and $M_{[w_1,w_2]}$ as in Section 3, we deduce that

$$T^\epsilon(v) \circ B = M_{v+(1+i)\epsilon/4}$$

and

$$T^\epsilon(v)^{-1} \circ T^\epsilon(v+\epsilon) = B \circ M_{[v+\omega_0/2, v+\omega_0+\epsilon/2]} \circ M_{[v+\omega_0+\epsilon/2, v+\omega_0+\epsilon]} \circ B^{-1}.$$

Using the usual matrix notation for Möbius transformation and the fact that $M_{[w_1,w_2]} = M^{-1}_{[w_2,w_1]}$, we obtain from (5), (7) and (21) that

$$\begin{aligned} T^\epsilon(v)^{-1} \circ T^\epsilon(v+\epsilon) = {} & \begin{pmatrix} \epsilon & -(1+i)\epsilon/2 \\ 1-i & i \end{pmatrix} \circ \begin{pmatrix} -i\epsilon(h_\epsilon^{(2)}(v+\omega_0)+1)^{1/2} & 1 \\ 0 & 1 \end{pmatrix} \\ & \circ \begin{pmatrix} 0 & (\epsilon^{-2}(h_\epsilon^{(1)}(v+\epsilon)+1)^{-1}+1)^{-1} \\ -1 & -1 \end{pmatrix}^{-1} \\ & \circ \begin{pmatrix} \epsilon & -(1+i)\epsilon/2 \\ 1-i & 1 \end{pmatrix}^{-1}. \end{aligned}$$

By Lemma 2 and noting that S_f is Lipschitz, we deduce that

$$T^\epsilon(v)^{-1} \circ T^\epsilon(v+\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon S_f(v)/2 & 1 \end{pmatrix} + O\big(\epsilon^2\big) = I + \epsilon C(v) + O\big(\epsilon^2\big),$$

where I denotes the identity matrix and

$$C(v) = \begin{pmatrix} 0 & 1 \\ -S_f(v)/2 & 0 \end{pmatrix}.$$

This implies

$$T^\epsilon(v+\epsilon) = T^\epsilon(v) + \epsilon T^\epsilon(v)C(v) + T^\epsilon(v)O\big(\epsilon^2\big). \tag{22}$$

Similarly, we have

$$T^\epsilon\big(v + i^j\epsilon\big) = T^\epsilon(v) + i^j\epsilon T^\epsilon(v)C(v) + T^\epsilon(v)O\big(\epsilon^2\big) \tag{23}$$

for $j = 1, 2, 3$. We assume, with no loss of generality, that

$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0,$$

because the statement of Theorem 2 is Möbius invariant. Thus we get from Taylor's formula that

$$f\big(i^j(1+i)\epsilon/2\big) = i^j(1+i)\epsilon/2 + O\big(\epsilon^3\big)$$

for $j = 0, 1, 2, 3$. From (19) and the definition of $T^\epsilon(v)$, it follows that

$$T^\epsilon(0)\big(i^j(1+i)\epsilon/2\big) = i^j(1+i)\epsilon/2 + O\big(\epsilon^3\big)$$

for $j = 0, 1, 2, 3$, which implies

$$T^\epsilon(0) = I + O(\epsilon).$$

By (22) and (23), we deduce that the matrices $T^\epsilon(v)$ ($v \in int\, V_\Omega^\epsilon$) are bounded in compact subsets of Ω, independently of ϵ. On the other hand, (22) and (23) imply

$$\partial_j^\epsilon T^\epsilon(v) = i^j T^\epsilon(v) \cdot C(v) + T^\epsilon(v) \cdot O(\epsilon) = T^\epsilon(v) \cdot O(1) \tag{24}$$

for $j = 0, 1, 2, 3$, where $O(1) = i^j C(v) + O(\epsilon)$. It follows from Theorem 1 that $h_\epsilon^{(1)}$ and $h_\epsilon^{(2)}$ are C^∞-bounded, so $O(1)$ is bounded in $C^\infty(\Omega)$. By repeating differentiation of (24), we conclude that $T^\epsilon(v)$ is bounded in $C^\infty(\Omega)$ uniformly in ϵ. By the properties of C^∞-convergence of functions (see [7, Lemma 2.1]), we obtain that (20) holds for some sequence of $\epsilon \to 0$, and the convergence is $C^\infty(\Omega)$.

In the following we shall show that (20) is also valid for every sequence of $\epsilon \to 0$. Indeed, let $D(z)$ be the matrix solution of the differential equation

$$D'(z) = D(z)C(z) \tag{25}$$

with initial condition $D(0) = I$, then we have

$$D\bigl(z + i^j\epsilon\bigr) = D(z) + i^j\epsilon D(z)C(z) + O\bigl(\epsilon^2\bigr)$$

for $j = 0, 1, 2, 3$. From (22) and (23), we obtain that

$$\begin{aligned}
&\bigl|T^\epsilon\bigl(v + i^j\epsilon\bigr) - D\bigl(v + i^j\epsilon\bigr)\bigr| \\
&\quad\leqslant \bigl|T^\epsilon(v) - D(v)\bigr| + \epsilon\bigl|\bigl(T^\epsilon(v) - D(v)\bigr)C(v)\bigr| + \bigl(1 + \bigl|T^\epsilon(v)\bigr|\bigr)O\bigl(\epsilon^2\bigr) \\
&\quad\leqslant \bigl|T^\epsilon(v) - D(v)\bigr|\bigl(1 + O(\epsilon)\bigr) + \bigl(1 + \bigl|T^\epsilon(v)\bigr|\bigr)O\bigl(\epsilon^2\bigr).
\end{aligned}$$

Note that $T^\epsilon(0) = I + O(\epsilon) = C(0) + O(\epsilon)$, we deduce that

$$\bigl|T^\epsilon(v) - D(z)\bigr| = \bigl(\delta(v)O\bigl(\epsilon^2\bigr) + O(\epsilon)\bigr)\bigl(1 + O(\epsilon)\bigr)^{\delta(v)},$$

where $\delta(v)$ is the combinatorial distance from v to 0 in SG^ϵ. Hence we deduce that

$$\bigl|T^\epsilon(v) - D(v)\bigr| \leqslant O(\epsilon)e^{O(1)} = O(\epsilon)$$

on a compact subset $K \Subset \Omega$, because $\delta(v) = O(1/\epsilon)$ on K. So $T^\epsilon(v) = D(v) + O(\epsilon)$, which implies that (20) holds for every $\epsilon \to 0$.

In Eq. (24), taking limit as $\epsilon \to 0$, we obtain

$$\partial_j T(z) = i^j T(z)C(z). \tag{26}$$

Hence, we get

$$\partial_j T(z) = -\partial_{j+2} T(z),$$

which implies that $T(z)$ is a matrix-value analytic function of z. In addition, it follows from (26) that the determinant of $T(z)$ is constant in Ω. Note that at $z = 0$ this determinant is 1. So $T(z)$ is a Möbius transformation for each $z \in \Omega$.

(ii) Let

$$T(z) = \begin{pmatrix} a(v) & b(v) \\ c(v) & d(v) \end{pmatrix}.$$

Then $T(z)$ satisfies the differential equation (25). By the definitions of Schwarzian derivative and $C(v)$, we deduce that

$$S_{b/d} = S_f.$$

Note that $b/d(0) = f(0) = 0$, $(b/d)'(0) = f'(0) = 1$, $(b/d)''(0) = f''(0) = 0$. So we conclude that $T(z)(0) = b(z)/d(z) = f(z)$. $\square$

Proof of Theorem 2. First, we write $T^\epsilon(v)$ and $T(z)$ as matrices

$$T^\epsilon(v) = \begin{pmatrix} a^\epsilon(v) & b^\epsilon(v) \\ c^\epsilon(v) & d^\epsilon(v) \end{pmatrix}, \qquad T(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Then it follows from Lemma 5(i) that $\epsilon c^\epsilon + d^\epsilon$ and $\epsilon a^\epsilon + b^\epsilon$ converge to d and b in $C^\infty(\Omega)$, respectively. By Lemma 5(ii), we obtain that $b(z)/d(z) = f(z)$. Since the determinant of $T(z)$ is nonzero, d is nonzero in Ω. By the properties of C^∞-convergence of functions (see [7, Lemma 2.2]), we get that $(\epsilon a^\epsilon + b^\epsilon)/(\epsilon c^\epsilon + d^\epsilon) \to b/d$ in $C^\infty(\Omega)$. Note that $g^\epsilon(v) = T^\epsilon(\epsilon)$, so we conclude that the discrete function $g^\epsilon \to f$.

Next, let c be the circle that contains the three points $(1+i)\epsilon/2$, $(-1+i)\epsilon/2$, $(1-i)\epsilon/2$. Since that $T^\epsilon(v)$ maps the three points $(1+i)\epsilon/2$, $(-1+i)\epsilon/2$, $(1-i)\epsilon/2$ to $q(v+(1+i)\epsilon/2)$, $q(v+(-1+i)\epsilon/2)$, $q(v+(1-i)\epsilon/2)$, respectively, and note that the three points $q(v+(1+i)\epsilon/2)$, $q(v+(-1+i)\epsilon/2)$, and $q(v+(1-i)\epsilon/2)$ lie in the circle $P_\epsilon(v)$, so $T^\epsilon(v)$ maps the circle c onto $P^\epsilon(v)$. Hence it follows that $T^\epsilon(v)(0) = f^\epsilon(v)$. Similar to the above arguments, we conclude that $f^\epsilon(v) = T^\epsilon(v)(0)$ converges in $C^\infty(\Omega)$ to f as $\epsilon \to 0$.

Finally, by the definition of SG circle pattern, we easily get

$$\sqrt{r^\epsilon(v) + r^\epsilon(v+\epsilon)} = \left|f^\epsilon(v) - f^\epsilon(v+\epsilon)\right| = \epsilon\left|\partial_0^\epsilon f^\epsilon(v)\right|, \tag{27}$$

$$r^\epsilon(v) + r^\epsilon(v+\epsilon+i\epsilon) = \left|f^\epsilon(v) - f^\epsilon(v+\epsilon+i\epsilon)\right| = \epsilon\left|\partial_0^\epsilon f^\epsilon(v) + \partial_1^\epsilon f^\epsilon(v+\epsilon)\right| \tag{28}$$

and

$$\sqrt{r^\epsilon(v+\epsilon) + r^\epsilon(v+\epsilon+i\epsilon)} = \left|f^\epsilon(v+\epsilon) - f^\epsilon(v+\epsilon+i\epsilon)\right| = \epsilon\left|\partial_1^\epsilon f^\epsilon(v+\epsilon)\right|. \tag{29}$$

From (27)–(29), we get that

$$2r^\epsilon(v)/\epsilon = \left|\partial_0^\epsilon f^\epsilon(v) + \partial_1^\epsilon f^\epsilon(v+\epsilon)\right| + \epsilon\left(\left|\partial_0^\epsilon f^\epsilon(v)\right|^2 - \left|\partial_1^\epsilon f^\epsilon(v+\epsilon)\right|^2\right). \tag{30}$$

Since that $\partial_j^\epsilon f^\epsilon$ $(j = 0, 1, 2, 3)$ converges in $C^\infty(\Omega)$ to f' as $\epsilon \to 0$, we obtain from [7, Lemma 2.2] that $|\partial_j^\epsilon f^\epsilon| \to |f'|$. Thus it follows from (30) that $r(v)/\epsilon$ converges in $C^\infty(\Omega)$ to $|f'|$. This completes the proof of Theorem 2. $\square$

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