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Positive Solutions of Singular and Nonsingular Fredholm Integral Equations

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The existence of positive solutions of the Fredholm nonlinear equation $y(t) = h(t) + \int_0^T k(t, s)[f(y(s)) + g(y(s))]\, ds$ is discussed. It is assumed that f is a continuous, nondecreasing function and g is continuous, nonincreasing, and possibly singular.

1. INTRODUCTION

In this paper we discuss the existence of nonnegative continuous solutions of the nonlinear Fredholm integral equation

$$y(t) = h(t) + \int_0^T k(t, s)\left[f(y(s)) + g(y(s))\right] ds, \qquad t \in [0, T]. \quad (1.1)$$

Using in most cases Krasnoselskii's fixed point theorem, we present several existence results for (1.1). Initially we are generous with our choice of f and g, assuming that f: $[0, \infty) \to [0, \infty)$ is continuous and nondecreasing, while g: $(0, \infty) \to [0, \infty)$ is continuous and nonincreasing. We are also allowing for the eventuality that g may be singular, that is, we consider functions g which may not be defined at zero. Applying Krasnoselskii's fixed point theorem, we observe that the (integral) operator must map a relevant cone intersected with an annulus back into the cone. Careful

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selection of the cone and annulus provides us with an easy treatment of such integral equations with singular nonlinearities. In addition, the solution will inherit properties from both the cone and annulus, providing us in many cases with a more detailed description of the solution than perhaps was initially anticipated. The existence of multiple positive solutions of (1.1) under such conditions is also discussed.

Considering (1.1) with g identically zero, that is,

$$y(t) = h(t) + \int_0^T k(t,s) f(y(s))\, ds, \qquad t \in [0,T], \tag{1.2}$$

allows us a little more scope in our choice of kernel k. Such equations are considered in [2–5] and we present some results which guarantee the existence of at least one nonnegative and multiple nonnegative solutions of (1.2). A result which combines a nonlinear alternative and Krasnoselskii's fixed point theorem to provide at least two nonnegative solutions of (1.2) is also presented.

In an attempt to consider a large class of functions f and g, we compromise with our choice of kernel k. Our final result considers what one requires on f and g, while keeping the conditions on k minimal, in order for

$$y(t) = \int_0^T k(t,s)[f(y(s)) + g(y(s))]\, ds, \qquad t \in [0,T], \tag{1.3}$$

to have at least one positive solution $y \in C[0,T]$. We illustrate the ideas involved here by first letting $f(y) = y^m$, $0 \le m < 1$, and $g \equiv 0$, and second by setting $f(y) \equiv 0$ and assuming $g(y) = y^{-m}$, $0 \le m < 1$. The paper is concluded with a general result of this nature for (1.3).

Before proceeding to the results we state the two fixed point theorems which will be used in the next section.

THEOREM 1.1 (Krasnoselskii's Fixed Point Theorem). *Let E be a Banach space and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$K: C \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to C$$

be a completely continuous operator such that either

(i) $\|Ku\| \le \|u\|$, $u \in C \cap \partial\Omega_1$ *and*

$\|Ku\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$

or

(ii) $\|Ku\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$ *and*

$\|Ku\| \le \|u\|$, $u \in C \cap \partial\Omega_2$

is true. Then K has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

THEOREM 1.2 (Nonlinear Alternative). *Let $\tilde{C}$ be a convex subset of a normed linear space E and let U be an open subset of $\tilde{C}$ with $p^* \in U$. Then every compact, continuous map $N: \overline{U} \to \tilde{C}$ has at least one of the following two properties*:

(i) *N has a fixed point*;

(ii) *there is an $x \in \partial U$ with $x = (1 - \lambda)p^* + \lambda Nx$ for some $0 < \lambda < 1$.*

2. EXISTENCE RESULTS

The objective in this paper is to show the existence of positive solutions of the nonlinear Fredholm integral equation

$$y(t) = h(t) + \int_0^T k(t,s)[f(y(s)) + g(y(s))]\,ds, \qquad t \in [0,T], \quad (2.1)$$

under certain conditions. Initially we will be ambitious with our choice of nonlinearity $f + g$ in that we will assume that f: $[0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function, while g: $(0,\infty) \to (0,\infty)$ is a nonincreasing function and possibly singular. We use Krasnoselskii's fixed point theorem to obtain our first result for (2.1).

THEOREM 2.1. *Suppose that*

$$\left\{\begin{array}{l} \text{there exists } 0 < M < 1 \text{ and } \kappa \in L^1[0,T] \text{ such that} \\ k(t,s) \geq M\kappa(s) \geq 0 \text{ for all } t \in [0,T], \text{ a.e. } s \in [0,T] \\ \text{and } \kappa_1 := \int_0^T \kappa(s)\,ds > 0 \end{array}\right\}, \quad (2.2)$$

$$k_t(s) := k(t,s) \leq \kappa(s) \qquad \text{for all } t \in [0,T], \text{ a.e. } s \in [0,T], \quad (2.3)$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0,T] \text{ to } L^1[0,T], \quad (2.4)$$

$$h \in C[0,T] \text{ and } h(t) \geq M|h|_0 \text{ for all } t \in [0,T], \quad (2.5)$$

$$\left\{\begin{array}{l} f: [0,\infty) \to [0,\infty) \text{ is continuous and nondecreasing}, \\ g: (0,\infty) \to [0,\infty) \text{ is continuous and nonincreasing}, \\ \text{and } f + g: (0,\infty) \to (0,\infty) \end{array}\right\}, \quad (2.6)$$

$$\text{there exists } \alpha > 0 \text{ such that } 1 < \frac{\alpha}{|h|_0 + \kappa_1[f(\alpha) + g(M\alpha)]}, \quad (2.7)$$

and

$$\text{there exists } \beta > 0,\ \beta \neq \alpha, \text{ such that } M > \frac{\beta}{|h|_0 + \kappa_1[f(M\beta) + g(\beta)]} \tag{2.8}$$

hold. Then (2.1) *has at least one positive solution* $y \in C[0,T]$ *and either*

$$\text{(A)}\quad 0 < \alpha < |y|_0 < \beta \qquad \text{and} \qquad y(t) \geq M\alpha,\ t \in [0,T] \text{ if } \alpha < \beta,$$

or

$$\text{(B)}\quad 0 < \beta < |y|_0 < \alpha \qquad \text{and} \qquad y(t) \geq M\beta,\ t \in [0,T] \text{ if } \beta < \alpha,$$

holds.

Proof. Define the operator K_L: $C[0,T] \to C[0,T]$ by

$$K_L y(t) := h(t) + \int_0^T k(t,s)y(s)\,ds, \qquad t \in [0,T].$$

We claim that K_L: $C[0,T] \to C[0,T]$ is completely continuous; that is, for any bounded subset $\Omega \subset C[0,T]$, $\overline{K_L\Omega}$ is compact in $C[0,T]$. Let $\Omega \subset C[0,T]$ be such that there exists $r > 0$ such that $|y|_0 < r$ for all $y \in \Omega$. Then for any $y \in \Omega$,

$$|Ky(t)| \leq |h|_0 + \sup_{t\in[0,T]} \int_0^T k_t(s)|y(s)|\,ds \leq |h|_0 + \sup_{t\in[0,T]} \left(\int_0^T k_t(s)\,ds\right) r$$

and hence $K_L\Omega$ is uniformly bounded. In addition for $t, t' \in [0,T]$ and $y \in \Omega$, we see from (2.4) that

$$|Ky(t) - Ky(t')| \leq |h(t) - h(t')| + \left(\int_0^T |k_t(s) - k_{t'}(s)|\,ds\right) r \to 0 \quad \text{as } t \to t'$$

—thus $K_L\Omega$ is equicontinuous. Consequently the Arzéla–Ascoli theorem assures the relative compactness of $K_L\Omega$ in $C[0,T]$ and hence K_L: $C[0,T] \to C[0,T]$ is completely continuous. This fact also implies the continuity of K_L: $C[0,T] \to C[0,T]$. In summary then we have that

$$K_L\colon C[0,T] \to C[0,T] \text{ is a continuous and completely continuous operator} \tag{2.9}$$

holds.

Krasnoselskii's theorem requires that we find an appropriate cone in which our desired solution should lie. Therefore before examining the nonlinear part of (2.1) we define the cones $\tilde{C}$ and C_M, respectively, by

$$\tilde{C} := \{y \in C[0,T] : y(t) \geq 0 \text{ for all } t \in [0,T]\}$$

and

$$C_M := \{y \in C[0,T] : y(t) \geq M|y|_0 \text{ for all } t \in [0,T]\}.$$

Note that $K_L \colon \tilde{C} \to C_M$, since for $y \in \tilde{C}$ we have by (2.2) and (2.5) that

$$K_L y(t) \geq M|h|_0 + M\int_0^T \kappa(s)y(s)\,ds, \qquad t \in [0,T], \tag{2.10}$$

while by (2.3) we obtain

$$|K_L y|_0 \leq |h|_0 + \int_0^T \kappa(s)y(s)\,ds, \qquad t \in [0,T]. \tag{2.11}$$

Combining (2.10) and (2.11) gives

$$K_L y(t) \geq M|K_L y|_0, \qquad t \in [0,T],$$

and consequently we now have that

$$K_L \colon \tilde{C} \to C_M \text{ is a continuous and completely continuous operator} \tag{2.12}$$

is true.

We now turn our attention to the nonlinear operator

$$Fy(t) := f(y(t)) + g(y(t)), \qquad t \in [0,T].$$

Define Ω_α and Ω_β by

$$\Omega_\alpha := \{y \in C[0,T] : |y|_0 < \alpha\}$$

and

$$\Omega_\beta := \{y \in C[0,T] : |y|_0 < \beta\},$$

respectively, and suppose in what follows that $\beta < \alpha$. (A similar argument holds if $\alpha < \beta$.) If $y \in \overline{\Omega}_\alpha \setminus \Omega_\beta$ then $0 < \beta \leq |y|_0 \leq \alpha$. The fact that $g(0)$ may be undefined means that Fy may not be defined for $y \in \overline{\Omega}_\alpha \setminus \Omega_\beta$, $\tilde{C}$, or C_M. However if $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$ we have the property that

$$0 < M\beta \leq M|y|_0 \leq y(t) \leq |y|_0 \leq \alpha, \qquad t \in [0,T],$$

that is,

$$0 < M\beta \le y(t) \le \alpha, \qquad t \in [0,T]. \tag{2.13}$$

This fact proves essential in our proof should $g(y(t))$ be undefined if $y(t) = 0$ for some $t \in [0,T]$. Now for any $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$ we note that

$$Fy(t) = f(y(t)) + g(y(t)) > 0, \qquad t \in [0,T]$$

and consequently $F\colon C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta) \to \tilde{C}$. In addition $F\colon C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta) \to \tilde{C}$ is clearly continuous and indeed bounded, since for $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$,

$$|Fy|_0 \le f(\alpha) + g(M\beta).$$

Therefore we obtain

$$F\colon C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to \tilde{C} \text{ is a bounded, continuous operator.} \tag{2.14}$$

Finally defining

$$Ky(t) := K_L Fy(t) = h(t) + \int_0^T k(t,s)[f(y(s)) + g(y(s))]\,ds, \qquad t \in [0,T],$$

the combination of (2.12) and (2.14) yields

$$K\colon C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to C_M \text{ is a continuous and completely continuous operator.} \tag{2.15}$$

Of course if $\alpha < \beta$ we obtain

$$K\colon C_M \cap \left(\overline{\Omega}_\beta \setminus \Omega_\alpha\right) \to C_M \text{ is a continuous and completely continuous operator.} \tag{2.16}$$

The desired result now follows from Krasnoselskii's fixed point theorem if we show that

$$|Ky|_0 < |y|_0 \qquad \text{for } y \in C_M \cap \partial\Omega_\alpha \tag{2.17}$$

and

$$|Ky|_0 > |y|_0 \qquad \text{for } y \in C_M \cap \partial\Omega_\beta \tag{2.18}$$

hold. Let $y \in C_M \cap \partial\Omega_\alpha$. Then $|y|_0 = \alpha$ and

$$0 < M\alpha \le y(t) \le \alpha, \qquad t \in [0,T].$$

Therefore by (2.7)

$$|Ky|_0 \le |h|_0 + \int_0^T \kappa(s)[f(y(s)) + g(y(s))]\,ds$$
$$\le |h|_0 + \kappa_1[f(\alpha) + g(M\alpha)] < \alpha = |y|_0.$$

Now suppose that $y \in C_M \cap \partial\Omega_\beta$. Then $|y|_0 = \beta$ and

$$0 \le M\beta \le y(t) \le \beta, \qquad t \in [0,T].$$

Therefore by (2.8),

$$Ky(t) \ge M|h|_0 + M\int_0^T \kappa(s)[f(y(s)) + g(y(s))]\,ds$$
$$\ge M|h|_0 + M\kappa_1[f(M\beta) + g(\beta)] > \beta = |y|_0.$$

Hence (2.17) and (2.18) hold and Krasnoselskii's theorem guarantees a positive solution $y \in C[0,T]$ of (2.1), where $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$ if $\beta < \alpha$ or $y \in C_M \cap (\overline{\Omega}_\beta \setminus \Omega_\alpha)$ if $\alpha < \beta$. ∎

In an attempt to throw some light on conditions (2.6)–(2.8) we consider the following examples. For simplicity we let $h \equiv 0$ in each of the three examples.

EXAMPLE 2.1. Suppose that $f(y) = y^n$, $n > 1$, and $g \equiv 0$. Then certainly (2.6) is satisfied. For (2.7) and (2.8) to hold we need to find

$$\alpha > 0 \text{ such that } \kappa_1 < \alpha^{1-n} \text{ and } \beta > 0 \text{ such that } M^{n+1}\kappa_1 > \beta^{1-n}.$$

Thus (2.7) and (2.8) are satisfied with $\alpha < \beta$ such that

$$0 < \alpha^{n-1} < \frac{1}{\kappa_1} < \frac{1}{M^{n+1}\kappa_1} < \beta^{n-1}.$$

EXAMPLE 2.2. Suppose that $f(y) = y^m$, $0 \le m < 1$, and $g \equiv 0$. Then (2.6) holds and (2.7) and (2.8) are satisfied with $\beta < \alpha$ such that

$$0 < \beta^{1-m} < M^{m+1}\kappa_1 < \kappa_1 < \alpha^{1-m}.$$

EXAMPLE 2.3. Suppose that $f \equiv 0$ and $g = y^{-n}$, $n > 1$. Then (2.6) holds and (2.7) and (2.8) are satisfied with $\beta < \alpha$ such that

$$0 < \beta^{1+n} < M\kappa_1 < M^{-n}\kappa_1 < \alpha^{1+n}.$$

Similarly if $f \equiv 0$ and $g(y) = y^{-m}$, $0 \le m < 1$, (2.7) and (2.8) hold with $\beta < \alpha$ satisfying

$$0 < \beta^{1+m} < M\kappa_1 < M^{-m}\kappa_1 < \alpha^{1+m}.$$

Remark 2.1. Observe that (2.7) and (2.8) could in fact be replaced with the slightly more precise

$$\text{there exists } \alpha > 0 \text{ such that } 1 < \frac{\alpha}{|h|_0 + K_1[f(\alpha) + g(M\alpha)]} \tag{2.19}$$

and

$$\text{there exists } \beta > 0,\ \beta \neq \alpha \text{ such that } M > \frac{\beta}{|h|_0 + K_2[f(M\beta) + g(\beta)]}, \tag{2.20}$$

where

$$K_1 := \sup_{t\in[0,T]} \int_0^T k(t,s)\,ds \qquad \text{and} \qquad K_2 := \inf_{t\in[0,T]} \int_0^T k(t,s)\,ds. \tag{2.21}$$

Remark 2.2. With additional conditions on the nonlinearity $f+g$, repeated application of Theorem 2.1 will yield additional positive solutions of (2.1). We describe these conditions in the following theorem.

THEOREM 2.2. *Suppose that* (2.2)–(2.6),

$$\left\{\begin{array}{l}\textit{there exists constants } \alpha_i > 0,\ i = 1,\dots,n,\ \textit{some } n \in \mathbf{N},\\ \textit{such that for each } i \in \{1,\dots,n\},\ (2.7) \textit{ is satisfied with } \alpha = \alpha_i\end{array}\right\}, \tag{2.22}$$

and

$$\left\{\begin{array}{l}\textit{there exists constants } \beta_j > 0,\ j = 1,\dots,m,\ \textit{some } m \in \mathbf{N},\\ \textit{such that for each } i \in \{1,\dots,m\},\ (2.8) \textit{ is satisfied with } \beta = \beta_i\end{array}\right\} \tag{2.23}$$

hold.

(I) *If* $m = n+1$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_n < \alpha_n < \beta_{n+1}$, *then* (2.1) *has at least* $2n$ *nonnegative solutions* $y_1,\dots,y_{2n} \in C[0,T]$ *such that*

$$0 < \beta_1 < |y_1|_0 < \alpha_1 < \cdots < \alpha_n < |y_{2n}|_0 < \beta_{n+1}.$$

(II) *If* $m = n$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_n < \alpha_n$, *then* (2.1) *has at least* $2n-1$ *nonnegative solutions* $y_1,\dots,y_{2n-1} \in C[0,T]$ *such that*

$$0 < \beta_1 < |y_1|_0 < \alpha_1 < \cdots < \beta_n < |y_{2n-1}|_0 < \alpha_n.$$

(III) *If* $m = n$ *and* $0 < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < \alpha_{n+1}$, *then* (2.1) *has at least* $2n$ *nonnegative solutions* $y_1,\dots,y_{2n} \in C[0,T]$ *such that*

$$0 < \alpha_1 < |y_1|_0 < \beta_1 < \cdots < \beta_n < |y_{2n}|_0 < \alpha_{n+1}.$$

(IV) *If* $m = n$ *and* $0 < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n$, *then* (2.1) *has at least* $2n - 1$ *nonnegative solutions* $y_1, \dots, y_{2n-1} \in C[0, T]$ *such that*

$$0 < \alpha_1 < |y_1|_0 < \beta_1 < \cdots < \alpha_n < |y_{2n-1}|_0 < \beta_n.$$

Proof. The proof follows by repeated use of Theorem 2.1. We omit the detail. ∎

EXAMPLE 2.4. Suppose that $f = 1 + y^m + y^n$, where $0 \leq m < 1 < n$ and $g \equiv 0$. Since

$$\frac{y}{|h|_0 + \kappa_1\left[1 + (My)^m + (My)^n\right]} \to 0 \qquad \text{as } y \to 0^+ \text{ and } y \to \infty,$$

there exists $0 < \tilde{\beta}_1 < \tilde{\beta}_2$ such that f satisfies (2.23) with both $\beta = \beta_1$ and $\beta = \beta_2$, where $\beta_1 \in [0, \tilde{\beta}_1)$ and $\beta_2 \in (\tilde{\beta}_2, \infty)$. In addition if

$$\sup_{y \in [0,\infty)} \frac{y}{|h|_0 + \kappa_1[1 + y^m + y^n]} > 1$$

then there exists $\alpha > 0$ for which f satisfies (2.7) and (2.22)). Here $0 < \beta_1 < \alpha < \beta_2$.

EXAMPLE 2.5. Indeed if $f + g$ is such that

$$\frac{y}{|h|_0 + \kappa_1[f(y) + g(My)]} \to \infty \qquad \text{as } y \to 0^+ \text{ and/or } y \to \infty,$$

then one can find α_1 "small enough" and/or α_2 "large enough" such that (2.7) (and (2.22)) is satisfied. Similarly if $f + g$ is such that

$$\frac{y}{|h|_0 + \kappa_1[f(My) + g(y)]} \to 0 \qquad \text{as } y \to 0^+ \text{ and/or } y \to \infty,$$

then one can find β_1 "small enough" and/or β_2 "large enough" such that (2.8) (and (2.23)) is satisfied. (This was illustrated in Example 2.4.) The ideas expressed in this example are reminiscent of those detailed in [2, 3].

Reexamining Theorem 2.1, we see that if the possibly singular function g is identically zero, then we may have more scope in choosing an appropriate cone for the application of Krasnoselskii's fixed point theorem. It was imperative in this theorem that any $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$ (assuming $\beta < \alpha$) had to be such that $y(t) > 0$ for all $t \in [0, T]$. Therefore if $g \equiv 0$ perhaps we can choose a slightly larger cone than C_M and in doing so, reduce some of the conditions on the kernel k. Such problems have already been considered in the literature, for example [2–5]. Therefore we just state one such result, the proof of which can be found in [5].

THEOREM 2.3. *Suppose that* (2.3), (2.4), (2.6) *with* $g \equiv 0$,

$$0 \le k_t = k(t,s) \in L^1[0,T] \qquad \text{for each } t \in [0,T], \tag{2.24}$$

$$\left\{\begin{array}{l} \text{there exists } 0 < M < 1,\ \kappa \in L^1[0,T], \text{ and an interval } [a,b] \subseteq [0,T], \\ a < b, \text{ such that } k(t,s) \ge M\kappa(s) \ge 0,\ t \in [a,b],\ a.e.\ s \in [0,T] \end{array}\right\}, \tag{2.25}$$

$$h \in C[0,T] \text{ with } h(t) \ge 0,\ t \in [0,T], \text{ and } \min_{t\in[a,b]} h(t) \ge M|h|_0, \tag{2.26}$$

$$\begin{gathered} \text{there exists } \alpha > 0 \text{ such that } 1 < \frac{\alpha}{|h|_0 + K_1 f(\alpha)}, \\ \text{where } K_1 = \sup_{t\in[0,T]} \int_0^T k(t,s)\,ds > 0, \end{gathered} \tag{2.27}$$

and

$$\begin{gathered} \text{there exists } \beta > 0,\ \beta \ne \alpha, \text{ such that } 1 > \frac{\beta}{K_3 f(M\beta)}, \\ \text{where } K_3 := \sup_{t\in[0,T]} \int_a^b k(t,s)\,ds > 0 \end{gathered} \tag{2.28}$$

hold. Then

$$y(t) = h(t) + \int_0^T k(t,s) f(y(s))\,ds, \qquad t \in [0,T] \tag{2.29}$$

has at least one positive solution $y \in C[0,T]$ *and either*

(A) $0 < \alpha < |y|_0 < \beta$ *and* $y(t) \ge M\alpha,\ t \in [a,b]$ *if* $\alpha < \beta$,

or

(B) $0 < \beta < |y|_0 < \alpha$ *and* $y(t) \ge M\beta,\ t \in [a,b]$ *if* $\beta < \alpha$,

holds.

Proof. The complete proof can be found in [5]. Therefore we just mention a few of the details below. The chosen cone is

$$C := \left\{ y \in C[0,T] : y(t) \ge 0,\ t \in [0,T] \text{ and } \min_{t\in[a,b]} y(t) \ge M|y|_0 \right\}$$

and we define

$$K_L y(t) := h(t) + \int_0^T k(t,s) y(s)\,ds \text{ and } Fy(t) := f(y(t)), \qquad t \in [0,T].$$

It can be shown that

$$K_L : \tilde{C} \to C \qquad \text{and} \qquad F : \tilde{C} \to \tilde{C},$$

and in fact we have that

$$K := K_L F : \tilde{C} \to C \text{ is a continuous and completely continuous operator.}$$

Note that since it is assumed that $g \equiv 0$ we do not have to restrict K to $C \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$ since for the proof it is not necessary that $y(t) > 0$ for all $t \in [0, T]$. ∎

While slightly off the point, it is perhaps worth mentioning another result taken from [5]. The authors show, using a nonlinear alternative of Leray–Schauder type, that on extracting some of the hypotheses from Theorem 2.3, an existence principle may be represented which guarantees the existence of at least one nonnegative solution of (2.29). In this result it is possible that $y \equiv 0$ may be the solution, whereas in Theorem 2.3 we had that the guaranteed solution was positive on some interval $[a, b] \subseteq [0, T]$. For completeness we just state the following two results, the proofs of which can be found in [5].

THEOREM 2.4. *Suppose that* (2.4), (2.6) *with* $g \equiv 0$, (2.24), (2.27), *and*

$$h \in C[0, T] \text{ with } h(t) \geq 0 \text{ for all } t \in [0, T] \tag{2.30}$$

hold. Then (2.29) *has at least one nonnegative solution* $y \in C[0, T]$ *such that* $0 \leq y(t) < \alpha$, $t \in [0, T]$.

THEOREM 2.5. *Suppose that* (2.3), (2.4), (2.6) *with* $g \equiv 0$, *and* (2.24)–(2.28) *hold with* $0 < \alpha < \beta$. *Then* (2.29) *has at least two nonnegative solutions* $y_1, y_2 \in C[0, T]$ *such that* $0 \leq |y_1|_0 < \alpha < |y|_2 < \beta$ *and* $y_2(t) \geq M\alpha$ *for* $t \in [a, b]$.

Remark 2.3. With extra conditions on the nonlinearity f (of the type described in Theorem 2.2) repeated use of Theorem 2.3 will yield additional positive solutions of (2.29). For a precise statement of this result we refer the reader to [5].

One observation from Theorem 2.3, and indeed to a greater extent from Theorem 2.1 is that the conditions on the kernel k are quite restrictive. On examination we see that one reason for this is that we are considering a rather large class of nonlinear functions $f + g$ and, as a compromise, we transfer to the shoulders of the kernel the onus of ensuring that the image of the appropriate space under $K_L F$ lies in the appropriate cone. Therefore it seems reasonable to ask how we might strengthen the conditions on the nonlinearity F, allowing it to better utilise the properties of the domain on which it operates and thus (hopefully) reducing the restrictions on the kernel. (To some extent we saw this happen in Theorem 2.3, when the possibly singular function g was assumed to be zero, and the conditions on the kernel from Theorem 2.1 were lessened.)

To be more concrete, recall that in Theorem 2.1 we saw that

$$K_L : \tilde{C} \to C_M \qquad \text{and} \qquad F: C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to \tilde{C} \tag{2.31}$$

were true, giving the desired result

$$K = K_L F: C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to C_M.$$

(Here we are assuming without loss of generality that $\beta < \alpha$.) Looking at (2.31) it is reasonable to assume that if we can find an operator F such that $F(C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta))$ is contained is a smaller cone than $\tilde{C}$, then it is likely that the restrictions on k can be reduced since K_L will not have to bring the larger cone $\tilde{C}$ into C_M anymore.

Suppose then that we have a kernel k: $[0,T] \times [0,T] \to [0,T]$ that satisfies (2.4) and (2.24), and suppose that K_1 and K_2 are such that

$$K_1 := \sup_{t \in [0,T]} \int_0^T k(t,s)\, ds, \qquad K_2 := \inf_{t \in [0,T]} \int_0^T k(t,s)\, ds > 0, \qquad \text{and} \qquad K_1 \neq K_2 \tag{2.32}$$

hold. In addition assume that $h \equiv 0$ and let K_L: $C[0,T] \to C[0,T]$ be defined by

$$K_L y(t) := \int_0^T k(t,s) y(s)\, ds, \qquad t \in [0,T].$$

Out of interest, let us restrict the linear operator K_L to the cone

$$C_M := \{ y \in C[0,T] : y(t) \geq M|y|_0 \text{ for all } t \in [0,T]\}.$$

(Here $0 < M < 1$ is a predetermined constant.) Then for $y \in C_M$ we see that

$$|K_L y|_0 \leq \sup_{t \in [0,T]} \int_0^T k(t,s) y(s)\, ds \leq K_1 |y|_0,$$

which in addition to (2.4) and (2.24) implies

$$K_L y(t) \geq \inf_{t \in [0,T]} \int_0^T k(t,s) y(s)\, ds \geq K_2 M |y|_0 \geq M \frac{K_2}{K_1} |K_L y|_0. \tag{2.33}$$

By (2.32) we have that $K_2/K_1 < 1$ and hence $1 > M > MK_2/K_1$. Now (2.33) implies that

$$K_L\colon C_M \to C_{M(K_2/K_1)}, \tag{2.34}$$

that is, K_L maps C_M to the larger cone C_{MK_2/K_1}. This suggests that we are looking for an operator F that takes C_M into a smaller cone.

Suppose then that $F\colon C[0,T] \to C[0,T]$ is given by

$$Fy(t) := y^m(t), \qquad t \in [0,T] \text{ and } 0 \le m < 1.$$

For $y \in C_M$ we see that $y(t) \ge M|y|_0 \ge 0$ for all $t \in [0,T]$, thus implying that

$$y^m(t) \ge M^m|y|_0^m = M^m|y^m|_0,$$

that is

$$y^m \in C_{M^m} := \{y \in C[0,T] : y(t) \ge M^m|y|_0 \text{ for all } t \in [0,T]\}.$$

Hence

$$F\colon C_M \to C_{M^m} \subseteq C_M. \tag{2.35}$$

The fact that C_{M^m} is a smaller cone than C_M (since $K_2/K_1 < 1$) is ideal for us, since if we define

$$M := \left(\frac{K_2}{K_1}\right)^{1/(1-m)} \tag{2.36}$$

and then from (2.34) we see that

$$K_L\colon C_{M^m} \to C_{M^m(K_2/K_1)} = C_M. \tag{2.37}$$

Consequently (2.35) and (2.37) imply that $K = K_L F\colon C_M \to C_M$ and, in particular (assuming that $\beta < \alpha$),

$$K\colon C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to C_M.$$

Alternatively suppose that

$$Fy(t) := y^{-m}(t), \qquad t \in [0,T] \text{ and } 0 \le m < 1,$$

and define M as in (2.36). If $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$ then $0 < M\beta \le M|y|_0 \le y(t) \le |y|_0 \le \alpha$. Consequently for $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$,

$$|Ky|_0 \le \sup_{t \in [0,T]} \int_0^T k(t,s) y^{-m}(s)\, ds \le K_1 M^{-m} |y|_0^{-m},$$

while

$$Ky(t) \geq \inf_{t \in [0,T]} \int_0^T k(t,s) y^{-m}(s)\, ds \geq K_2 |y|_0^{-m} \geq \frac{K_2}{K_1} M^m |Ky|_0 = M|Ky|_0, \quad t \in [0,T].$$

Therefore in this case also

$$K\colon C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to C_M.$$

We can summarise the above in the following theorem:

THEOREM 2.6. *Suppose that* (2.4), (2.24), (2.32), *and either*

$$f(y) = y^m, \qquad \text{where } 0 \leq m < 1, \tag{2.38}$$

or

$$f(y) = y^{-m}, \qquad \text{where } 0 \leq m < 1, \tag{2.39}$$

hold. Then

$$y(t) = \int_0^T k(t,s) f(y(s))\, ds, \qquad t \in [0,t], \tag{2.40}$$

has a positive solution $y \in C[0,T]$ *with* $0 < \beta < |y|_0 < \alpha$ *and* $y(t) \geq M\beta$, $t \in [0,T]$. *Here M is as defined in* (2.36), *and* α *and* β *are as described in* (2.41) (*see below*) *if* (2.38) *holds, whereas if* (2.39) *holds,* α *and* β *satisfy* (2.42) (*see below*).

Proof. From the details outlined above and using some of the ideas from the proof of Theorem 2.1, we see that

$$K = K_L F\colon C_M \cap \left(\overline{\Omega}_\alpha \setminus \Omega_\beta\right) \to C_M \text{ is a continuous and completely continuous operator.}$$

In addition if (2.38) holds, then there exists $0 < \beta < \alpha$ such that

$$0 < \beta^{1-m} < K_2 M^m \leq K_2 \leq K_1 < \alpha^{1-m}, \tag{2.41}$$

implying that (2.17) and (2.18) are true. The result now follows from Krasnoselskii's fixed point theorem.

Alternatively, suppose that (2.39) is true. Then there exists $0 < \beta < \alpha$ such that

$$0 < \beta^{1+m} < K_2 \leq K_1 \leq K_2 M^{-m} < \alpha^{1+m}, \tag{2.42}$$

implying that (2.17) and (2.18) are satisfied. Once again the desired result follows directly from Krasnoselskii's fixed point theorem. ▮

Remark 2.4. In (2.32) we choose k such that $K_1 \neq K_2$. If $K_1 = K_2$ then it is immediate that (2.40) has a constant solution.

Our final result is a general form of Theorem 2.6, which, as above, allows us to keep minimal conditions on the kernel k, while enabling us to consider certain nonlinear functions $f + g$ with f nondecreasing and g nonincreasing and possibly singular.

THEOREM 2.7. *Suppose that* (2.4), (2.6), (2.24), (2.32),

$$\left\{\begin{array}{l} \textit{there exists a continuous function } \psi : (0,1) \to (0,\infty) \\ \textit{such that for any } 0 < \tilde{M} < 1 \textit{ and } u > 0, \textit{ we have} \\ f(\tilde{M}u) + g(u) \geq \psi(\tilde{M})\big[f(u) + g(\tilde{M}u)\big] \end{array}\right\}, \tag{2.43}$$

$$\textit{there exists } 0 < M < 1 \textit{ such that } \frac{M}{\psi(M)} \leq \frac{K_2}{K_1}, \tag{2.44}$$

$$\textit{there exists } \alpha > 0 \textit{ such that } 1 < \frac{\alpha}{K_1[f(\alpha) + g(M\alpha)]}, \tag{2.45}$$

and

$$\textit{there exists } \beta > 0,\ \beta \neq \alpha \textit{ such that } 1 > \frac{\beta}{K_2[f(M\beta) + g(\beta)]} \tag{2.46}$$

hold. Then

$$y(t) = \int_0^T k(t,s)\big[f(y(s)) + g(y(s))\big]\,ds, \qquad t \in [0,T] \tag{2.47}$$

has at least one positive solution $y \in C[0,T]$ *and either*

(A) $0 < \alpha < |y|_0 < \beta$ *and* $y(t) \geq M\alpha,\ t \in [0,T]$ *if* $\alpha < \beta$,

(B) $0 < \beta < |y|_0 < \alpha$ *and* $y(t) \geq M\beta,\ t \in [0,T]$ *if* $\beta < \alpha$,

holds.

Proof. Define the operator

$$Ky(t) := \int_0^T k(t,s)\big[f(y(s)) + g(y(s))\big]\,ds, \qquad t \in [0,T],$$

and the cone

$$C_M := \{y \in C[0,T] : y(t) \geq M|y|_0,\ t \in [0,T]\},$$

where M is as defined in (2.44). Suppose that $\beta < \alpha$ (a similar argument holds if $\alpha < \beta$) and let $y \in C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta)$. We have that $0 < M|y|_0 \le y(t) \le |y|_0$, $t \in [0, T]$, and therefore from (2.6) one obtains

$$|Ky|_0 \le K_1[f(|y|_0) + g(M|y|_0)]. \tag{2.48}$$

In addition, (2.6), (2.43), (2.48), and (2.44) give, for $t \in [0, T]$,

$$\begin{aligned} Ky(t) &\ge K_2[f(M|y|_0) + g(|y|_0)] \\ &\ge K_2\psi(M)[f(|y|_0) + g(M|y|_0)] \\ &\ge \frac{K_2\psi(M)}{K_1}|Ky|_0 \ge M|Ky|_0 \end{aligned}$$

and hence $K\colon C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta) \to C_M$. The continuity and complete continuity of $K\colon C_M \cap (\overline{\Omega}_\alpha \setminus \Omega_\beta) \to C_M$ follow in an identical fashion to that detailed in Theorem 2.1.

It remains to show that (2.17) and (2.18) hold. Once again the technique (using (2.45) and (2.46)) is similar to that illustrated in Theorem 2.1. Therefore we omit the detail. The desired result now follows from Krasnoselskii's fixed point theorem. ∎

Obviously Theorem 2.6 is a special case of the above, with $g \equiv 0$ and f as described in either (2.38) or (2.39). In both of these cases $\psi(M) = M^m$ and $M = (K_2/K_1)^{1/(1-m)}$ satisfies (2.43) and (2.44), respectively. We finish with an additional example.

EXAMPLE 2.6. Suppose that

$$f(y) = y^m \text{ and } g(y) = y^{-n}, \qquad \text{where } 0 \le n \le m < 1.$$

Then for any $M \in (0, 1)$,

$$\begin{aligned} f(Mu) + g(y) &= M^m u^m + u^{-n} = M^m[u^m + M^{-m}u^{-n}] \\ &\ge M^m[u^m + M^{-n}u^{-n}] = M^m[f(u) + g(Mu)]. \end{aligned}$$

Thus f and g satisfy (2.43) with $\psi(M) = M^m$ and (2.44) is satisfied with $0 < M = (K_2/K_1)^{1/(1-m)}$.

Alternatively suppose that

$$f(y) = y^m \text{ and } g(y) = y^{-n}, \qquad \text{where now } 0 \le m \le n < 1.$$

Then for any $M \in (0, 1)$,

$$\begin{aligned} f(Mu) + g(u) &= M^m u^m + u^{-n} \ge M^n u^m + u^{-n} \\ &= M^n[u^m + M^{-n}u^{-n}] = M^n[f(u) + g(Mu)]. \end{aligned}$$

Thus f and g satisfy (2.43) with $\psi(M) = M^n$ and (2.44) is satisfied with $0 < M = (K_2/K_1)^{1/(1-n)}$.

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