Self-Orthogonal Mendelsohn Triple Systems*

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A Mendelsohn triple system of order \( v \), briefly MTS(\( v \)), is a pair \((X, B)\) where \( X \) is a \( v \)-set (of points) and \( B \) is a collection of cyclic triples on \( X \) such that every ordered pair of distinct points from \( X \) appears in exactly one cyclic triple of \( B \). The cyclic triple \((a, b, c)\) contains the ordered pairs \((a, b), (b, c), \) and \((c, a)\). An MTS(\( v \)) corresponds to an idempotent semisymmetric Latin square (quasigroup) of order \( v \). An MTS(\( v \)) is called self-orthogonal, denoted briefly by SOMTS(\( v \)), if its associated semisymmetric Latin square is self-orthogonal. It is well known \([7]\) that an MTS(\( v \)) exists if and only if \( v \equiv 0 \) or \( 1 \) (mod 3) except \( v \neq 6 \). It is also known that a SOMTS(\( v \)) exists for all \( v \equiv 1 \) (mod 3) except \( v = 10 \) and that a SOMTS(\( v \)) does not exist for \( v = 3, 6, 9, \) and \( 12 \). In this paper it is shown that a SOMTS(\( v \)) exists for all \( v \geq 15 \), where \( v \equiv 0 \) (mod 3), except possibly for \( v = 18 \).

1. INTRODUCTION

A Mendelsohn triple system of order \( v \), briefly MTS(\( v \)), is a pair \((X, B)\) where \( X \) is a \( v \)-set (of points) and \( B \) is a collection of cyclic triples (or blocks) on \( X \) such that every ordered pair of distinct points from \( X \) appears in exactly one cyclic triple of \( B \). The cyclic triple \((a, b, c)\) contains the ordered pairs \((a, b), (b, c), \) and \((c, a)\). It is well known \([7]\) that an MTS(\( v \)) exists if and only if \( v \equiv 0 \) or \( 1 \) (mod 3) and \( v \neq 6 \).

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A quasigroup is an ordered pair \((Q, \cdot)\), where \(Q\) is a set and \((\cdot)\) is a binary operation on \(Q\) such that the equations
\[
a \cdot x = b \quad \text{and} \quad y \cdot a = b
\]
are uniquely solvable for every pair of elements \(a, b\) in \(Q\). It is well known (e.g., see [4]) that the multiplication table of a quasigroup defines a Latin square; that is, a Latin square can be viewed as the multiplication table of a quasigroup with the headline and sideline removed. For a finite set \(Q\), the order of the quasigroup \((Q, \cdot)\) is \(|Q|\). A quasigroup \((Q, \cdot)\) is called idempotent if the identity
\[
x^2 = x
\]
holds for all \(x\) in \(Q\). A quasigroup (Latin square) satisfying the identity
\[
x(yx) = y
\]
is called semisymmetric.

A quasigroup \((Q, \cdot)\) is self-orthogonal if it is orthogonal to its transpose, that is, if
\[
xy = zt \quad \text{and} \quad yz = tz \quad \text{together imply} \quad x = z \quad \text{and} \quad y = t.
\]

An idempotent semisymmetric quasigroup \((Q, \cdot)\) of order \(v\) corresponds to an MTS(\(v\))(\(Q, B\)), where \((x, y, z)\) is a cyclic triple of \(B\) if and only if \(x \cdot y = z\) and \(x^2 = x\) for all \(x \in Q\).

Two MTS(\(v\)) are called orthogonal if their corresponding semisymmetric Latin squares are orthogonal. Equivalently, \((X, B_1)\) and \((X, B_2)\) are orthogonal MTS(\(v\)), denoted by OMTS(\(v\)), if they satisfy the following properties:

1. \(B_1 \cap B_2 = \emptyset\),
2. if \((a, b, w)\) and \((x, y, w)\) \(\in B_1\), and if \((a, b, s)\) and \((x, y, t)\) \(\in B_2\), then \(s \neq t\).

An MTS(\(v\)) is called self-orthogonal, denoted by SOMTS(\(v\)), if its corresponding semisymmetric quasigroup is self-orthogonal. Denote by SOSQ(\(v\)) a self-orthogonal semisymmetric quasigroup of order \(v\). It is known [1, 5] that a SOSQ(\(v\)) (SOMTS(\(v\))) exists for all \(v \equiv 1 \pmod{3}\) except \(v = 10\) and that a SOMTS(\(v\)) does not exist for \(v = 3, 6, 9,\) and \(12\). In this paper it is shown that a SOMTS(\(v\)) exists for all \(v \geq 15\) where \(v \equiv 0 \pmod{3}\) except possibly for \(v = 18\).
Self-orthogonal Latin squares of various types are of fundamental importance in the construction of other structures. For example, Zhu [9] found our SOSQ(v) results useful in his constructions of a type of orthogonal group divisible design.

2. PRELIMINARIES AND DIRECT CONSTRUCTIONS

For the most part, the known results relating to the existence of SOSQs (SOMTTS) were obtained using finite fields and pairwise balanced designs (PBDs). The nonexistence of SOSQ(12) was established by an exhaustive computer search. These results can be summarized in the following theorem (see [1, 5]).

**Theorem 2.1.** A SOSQ(v) (SOMTS(v)) exists for all v \equiv 1 \pmod{3} except v = 10 and a SOSQ(v) (SOMTS(v)) does not exist for v = 3, 6, 9, and 12.

In order to further enlarge the spectrum of SOMTS (SOSQs), we need some additional direct and recursive constructions. For this purpose, we need the following generalizations of SOSQs.

Let \( H = \{S_1, S_2, \ldots, S_k\} \) be a set of subsets of \( Q \). A holey SOSQ having hole set \( H \) is a triple \( (Q, H, \cdot) \), which satisfies the following properties:

1. \((\cdot)\) is a binary operation defined on \( Q \); however, when both elements \( a \) and \( b \) belong to the same set \( S_i \), \( 1 \leq i \leq k \), there is no definition for \( a \cdot b \);

2. the equations (1) hold when \( a \) and \( b \) are not contained in the same set \( S_i \), \( 1 \leq i \leq k \);

3. the identity (2) holds for any \( x \notin \bigcup_{1 \leq i \leq k} S_i \);

4. the identity (3) holds when \( x \) and \( y \) are not contained in the same set \( S_i \), \( 1 \leq i \leq k \);

5. the constraint (4) holds when neither the pairset \( \{x, y\} \) nor \( \{z, t\} \) is contained in any set \( S_i \), \( 1 \leq i \leq k \).

We denote the holey SOSQ by HSOSQ(n; s_1, s_2, \ldots, s_k), where \( n = |Q| \) is the order and \( s_i = |S_i|, 1 \leq i \leq k \). Each \( S_i \) is called a hole. If \( H = \emptyset \), we obtain a SOSQ(n). If \( H = \{S_1\} \), we call the HSOSQ(n; s_1) an incomplete SOSQ, and denote it by ISOSQ(n, s_1).

If \( H = \{S_1, S_2, \ldots, S_k\} \) is a partition of \( Q \), then a holey SOSQ is called a frame SOSQ. The type of the frame SOSQ is defined to be the multiset \( \{|S_i| : 1 \leq i \leq k\} \). We shall use an “exponential” notation to describe types:
so type $s_1^n s_2^n \cdots s_t^n$ denotes $n_i$ occurrences of $s_i$, $1 \leq i \leq t$, in the multiset. We briefly denote a frame SOSQ of type $s_1^n s_2^n \cdots s_t^n$ by FSOSQ($s_1^n s_2^n \cdots s_t^n$).

We observe that the existence of a SOSQ($n$) is equivalent to the existence of an FSOSQ($1^n$), and the existence of an ISOSQ($n, h$) is equivalent to the existence of an FSOSQ($1^{n-h}h^1$).

If $H = \{S_1, S_2, \ldots, S_t, T\}$, where $\{S_1, S_2, \ldots, S_t\}$ is a partition of $Q$, then a holey SOSQ is called an incomplete frame SOSQ or an $I$-frame SOSQ. The type of the $I$-frame SOSQ is defined to be the multiset $\{(|S_i|, |S_i \cap T|): 1 \leq i \leq k\}$. We may also use an “exponential” notation to describe types of $I$-frame SOSQ.

For our direct construction, we use the method of difference sets. Instead of listing all of the blocks of a SOMTS($v$), we need only list a set of base blocks, which are developed by a cyclic group of order $v$. An ISOSQ($v, n$) corresponds to a SOMTS($v$) missing a sub-SOMTS($n$), which we shall denote by ISOMTS($v, n$). Let $X = \{x_1, x_2, \ldots, x_n\}$ and $G = Z_v \cup \{x\}$, where we consider $x$ as infinite elements for the group $G$. We may list a set of base blocks and develop them under group $G$ to get the block set of an ISOMTS($v, n$). The difference conditions for the base blocks of an ISOMTS($v, n$) are:

1. the finite differences $b - a$, $c - b$, and $a - c$ from all base blocks $(a, b, c)$ are precisely $G \backslash \{0\}$, where each $x_i$ appears in exactly one base block; and

2. in the corresponding ISOSQ($v, n$) $(X \cup G, \cdot), \{0 \cdot a - a \cdot 0 : a \in G$ and $0 \cdot a, a \cdot 0 \notin X\} \cup \{ \pm (0 \cdot x_i - x_i \cdot 0) : 1 \leq i \leq n\} = G$. 

We give the following example.

**Example 2.2.** An ISOMTS(10, 3). Take $G = Z_7$ and $X = \{x, y, z\}$. Take $(0, 1, 3)$, $(x, 0, 6)$, $(y, 0, 3)$, and $(z, 0, 5)$ as base blocks. We then obtain an associated ISOSQ(15, 4). Using Lemma 2.3, we fill in the size 4 hole with a SOSQ(4) based on $X$ to obtain a SOSQ(15), shown in Fig. 2.2.

**Lemma 2.3.** If there exist ISOSQ($v, n$) and SOSQ($n$), then there exists a SOSQ($v$).

**Example 2.4.** A SOMTS(15). Take $G = Z_11$ and $X = \{x, y, z, w\}$. Take $(0, 2, 8)$, $(0, 5, 4)$, $(0, 1, x)$, $(0, 4, y)$, $(0, 8, z)$, and $(0, 9, w)$ as base blocks. We then obtain an associated ISOSQ(15, 4). Using Lemma 2.3, we fill in the size 4 hole with a SOSQ(4) based on $X$ to obtain a SOSQ(15), shown in Fig. 2.2.
LEMMA 2.5. There exists an ISOMTS\((v, n)\) for \((v, n) = (16, 3), (21, 4), (24, 7), (27, 4), (30, 7), (33, 4), (33, 10), (36, 7), (39, 4), (42, 13), (45, 7), (54, 13), (63, 16), (66, 19), (69, 22), and (78, 25).\

Proof. Take \(G = \mathbb{Z}_{v-n}\) and \(X = \{\infty_1, \infty_2, \ldots, \infty_n\}\). For \((v, n) = (16, 3)\) and \((21, 4)\), the constructions are provided below, where we list the base blocks. For the remaining cases, the constructions can be found in the Appendix. It is readily checked that the difference conditions are satisfied.

\(v = 16, n = 3\). \((0, 1, 8), (0, 2, 5), (0, 6, 2), (\infty_1, 0, 12), (\infty_2, 0, 10), (\infty_3, 0, 4)\).

\(v = 21, n = 4\). \((0, 1, 6), (0, 2, 9), (0, 3, 7), (0, 6, 4), (\infty_1, 0, 9), (\infty_2, 0, 12), (\infty_3, 0, 14), (\infty_4, 0, 16)\). \(\blacksquare\)
3. RECURSIVE CONSTRUCTIONS

In this section, we shall describe several recursive constructions. The first one is simple but useful.

**CONSTRUCTION 3.1 (Filling in Holes).** (1) Suppose there exists an FSOSQ of type \( \{s_i: 1 \leq i \leq n\} \). Let \( a \geq 0 \) be an integer. For \( 1 \leq i \leq n-1 \) if there exists ISOSQ\((a + s_i, a)\), then there is an ISOSQ\((a + s, a + s_i)\), where \( s = \sum_{1 \leq i \leq n} s_i \).

(2) Suppose there exists an I-frame SOSQ of type \( \{(s_i, h_i): 1 \leq i \leq n\} \), and let \( a \geq 0 \) be an integer. For \( 1 \leq i \leq n \), suppose there exists an ISOSQ\((a + s_i, a + h_i)\). Then there is an ISOSQ\((a + s, a + h)\), where \( s = \sum_{1 \leq i \leq n} s_i \) and \( h = \sum_{1 \leq i \leq n} h_i \).

The next recursive construction for FSOSQ uses group divisible designs. A group divisible design (or GDD), is a triple \((X, G, B)\) which satisfies the following properties:

1. \( G \) is a partition of \( X \) into subsets called *groups*,
2. \( B \) is a set of subsets of \( X \) (called *blocks*) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in a unique block.

The group type of the GDD is the multiset \( \{|G|: G \in G\} \). A TD\((k, n)\) is a GDD of group type \( n^k \) and block size \( k \). It is well known that the existence of a TD\((k, n)\) is equivalent to the existence of \( k - 2\text{MOLS}(n) \). We wish to remark that a special GDD with all groups size one is essentially a PBD, denoted by \( (X, B) \).

The following PBD construction is essentially \([1, \text{Lemma 2.3}]\).

**CONSTRUCTION 3.2.** Suppose there exists a PBD\((X, B)\) and for each block \( B \in B \) there exists a SOSQ\(|B|\). Then there exists a SOSQ\(|X|\).

More generally, we can apply the Wilson's fundamental construction for GDDs \([8]\) to obtain the similar construction for FSOSQ.

**CONSTRUCTION 3.3 (Weighting).** Suppose \((X, G, B)\) is a GDD and let \( w: X \to \mathbb{Z}^+ \cup \{0\} \). Suppose there exist FSOSQ of type \( \{w(x): x \in B\} \) for every \( B \in B \). Then there exists an FSOSQ of type \( \{\sum_{x \in G} w(x): G \in G\} \).

If we start with an incomplete TD, a TD\((k, n) - TD(k, h)\), we have the following modification.
CONSTRUCTION 3.4. Suppose there is a TD\((k, n) - TD(k, h)\). If there exists an FSOSQ\((m^k)\), then there exists an \(I\)-frame SOSQ of type \((mn, mh)^k\).

To apply the above constructions the following known results are useful.

**Theorem 3.5** [4].

1. For any prime power \(p\), there exists a TD\((k, p)\), where \(3 \leq k \leq p + 1\).
2. There exists a TD\((4, m)\) for any positive integer \(m\), \(m \geq 2, 6\).

**Theorem 3.6** [6]. There exists a TD\((4, n) - TD(4, h)\) for all positive integers \(n\) and \(h\) satisfying \(n \geq 3h\) except that a TD\((4, 6) - TD(4, 1)\) does not exist.

**Theorem 3.7** [3]. There exists a TD\((8, n)\) for any positive integer \(n \geq 77\).

We will use the following lemmas in the next section.

**Lemma 3.8.** Suppose there exists an ISOSQ\((n + k, k)\), where \(k \equiv 1 \pmod{3}\) and \(n \geq 77\). Then there exists an SOSQ\((v)\), where \(v = 7n + k + 3t\) and \(13 \leq k + 3t \leq k + 3n\).

**Proof.** Start with a TD\((8, n)\) which exists from Theorem 3.7. Each point in the last group receives weight zero or three and other points of the TD get weight one each. From Theorem 2.1 and Example 2.2, we have an FSOSQ\((17^1)\) and an FSOSQ\((17^31)\). Applying Construction 3.3 with the two FSOSQs as input designs, we obtain an FSOSQ\((n^7(3t)^1)\), where we assume \(t\) points receive weight three. Further apply Construction 3.1 (1) and use ISOSQ\((n + k, k)\) to fill in the size \(n\) holes. We obtain an ISOSQ\((7n + k + 3t, k + 3t)\), where \(0 \leq t \leq n\). Since \(k + 3t \equiv 1 \pmod{3}\) and \(k + 3t \geq 13\), there exists a SOSQ\((k + 3t)\) by Theorem 2.1. We apply Lemma 2.3 and fill in the size \(k + 3t\) hole to get a SOSQ\((7n + k + 3t)\). This completes the proof.

**Lemma 3.9.** Suppose there exists an ISOSQ\((n + k, k)\), where \(k \equiv 1 \pmod{3}\) and \(n \geq \min\{6, 2k\}\). Then there exists a SOSQ\((v)\), where \(v = 4n + k + 3t, 0 \leq t \leq k\), and \(k + 3t \neq 10\).

**Proof.** Start with a TD\((4, n + t) - TD(4, t)\) which exists from Theorem 3.6. Apply Construction 3.4 and give weight one to each point of the incomplete TD. We use an FSOSQ\((1^4)\) as input designs, which exists from Theorem 2.1. We then obtain an \(I\)-frame SOSQ of type \((n + t, t)^4\). Further apply Construction 3.1 (2) and use ISOSQ\((n + k, k)\) to fill in the size \(n + t\) holes. We obtain an ISOSQ\((4n + k + 3t, k + 3t)\), where \(0 \leq t \leq k\). Since
$k + 3t \equiv 1 \pmod{3}$ and $k + 3t \not\equiv 13$, there exists a $\text{SOSQ}(k + 3t)$ by Theorem 2.1. We apply Lemma 2.3 and fill in the size $k + 3t$ hole to get a $\text{SOSQ}(4n + k + 3t)$. This completes the proof.

4. Existence of $\text{SOSQ}(v)$, $v \equiv 0 \pmod{3}$

In this section, we shall investigate the existence of $\text{SOSQ}(v)$ for $v \equiv 0 \pmod{3}$.

Lemma 4.1. There exists a $\text{SOSQ}(v)$ for $v = 15, 21, 24, 27, 30, 33, 36, 39, 42, 45, 54, 63, 66, 69, and 78.$

Proof. A $\text{SOSQ}(15)$ is shown in Fig. 2.2. For each of the remaining values of $v$, there exists by Lemma 2.5 an $\text{ISOSQ}(v, n)$ for $n = 4, 7, 13, 16, 19, 22, or 25$. Since there exists a $\text{SOSQ}(n)$ for each of these $n$ from Theorem 2.1, we may apply Lemma 2.3 to obtain a $\text{SOSQ}(v)$.

In what follows, we shall mainly use Lemmas 3.8 and 3.9, where the required $\text{TD}(8, n)$ and $\text{TD}(4, n + t) - \text{TD}(4, t)$ come from Theorems 3.5–3.7.

Lemma 4.2. There exists a $\text{SOSQ}(v)$ for $v = 48, 51, 57, 60, 72,$ and $75.$

Proof. Apply Lemma 3.9 with $n = 11$ and $k = 4$, where an $\text{ISOSQ}(15, 4)$ exists from Example 2.4. We obtain a $\text{SOSQ}(v)$ for $v \equiv 0 \pmod{3}$, $48 \leq v \leq 60$, and $v \neq 54$. For $v = 72$ and $75$, we again apply Lemma 3.9 with $n = 17$ and $k = 4$, where an $\text{ISOSQ}(21, 4)$ exists from Lemma 2.5. We then obtain a $\text{SOSQ}(72)$ and a $\text{SOSQ}(75)$.

Lemma 4.3. If $v \equiv 0 \pmod{3}$ and $81 \leq v \leq 96$, then there exists a $\text{SOSQ}(v)$.

Proof. Apply Lemma 3.9 with $n = 17$ and $k = 7$, where an $\text{ISOSQ}(24, 7)$ exists from Lemma 2.5. We obtain a $\text{SOSQ}(v)$ for $v \equiv 0 \pmod{3}$, $81 \leq v \leq 96$.

Lemma 4.4. If $v \equiv 0 \pmod{3}$ and $90 \leq v \leq 114$, then there exists a $\text{SOSQ}(v)$.

Proof. Apply the construction in Lemma 3.8 with $n = 11$ and $k = 4$, where an $\text{ISOSQ}(15, 4)$ exists from Example 2.4.

Lemma 4.5. If $v \equiv 0 \pmod{3}$ and $105 \leq v \leq 132$, then there exists a $\text{SOSQ}(v)$. 

TABLE 4.1

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>7</td>
<td>132–177</td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>174–231</td>
</tr>
<tr>
<td>29</td>
<td>13</td>
<td>216–303</td>
</tr>
<tr>
<td>41</td>
<td>1</td>
<td>300–411</td>
</tr>
<tr>
<td>56</td>
<td>1</td>
<td>405–561</td>
</tr>
</tbody>
</table>

Proof. Apply Lemma 3.9 with $n=23$ and $k=10$, where an ISOSQ(33, 10) exists from Lemma 2.5.

Lemma 4.6. If $v \equiv 0 \pmod{3}$ and $21 \leq v \leq 561$, then there exists a SOSQ($v$).

Proof. Combining Lemmas 4.1–4.5, we need only to show the existence of a SOSQ($v$) for $v \equiv 0 \pmod{3}$ and $135 \leq v \leq 561$. We shall apply the construction in Lemma 3.8 with parameters $n, k$, and $v$ shown in Table 4.1. The existence of a TD(8, 56) can be found in [3]. All the required ISOSQ($n+k, k$) are known from Lemmas 2.5 and 4.1.

Lemma 4.7. If $v \equiv 0 \pmod{3}$ and $v \geq 552$, then there exists a SOSQ($v$).

Proof. For any integer $v \equiv 0 \pmod{3}$ where $v \geq 552$, we can write $v = 7n + 1 + 3t$ where $n + 1 \equiv 0 \pmod{3}$, $n \geq 77$ and $13 \leq 1 + 3t \leq 1 + 3n$. Apply Lemma 3.8 with the parameter $n$ and $k = 1$. We know that a SOSQ($v$) exists if a SOSQ($n + 1$) exists. Therefore, we may use induction to show the existence of a SOSQ($v$) since a SOSQ($u$) exists for any $u \equiv 0 \pmod{3}$ and $75 \leq u \leq 549$ from Lemma 4.6. This completes the proof.

Combining Lemma 4.1 and Lemmas 4.6–4.7, we obtain the main result of this section.

Theorem 4.8. There exists a SOSQ($v$) (SOMTS($v$)) for $v = 15$ and for all $v \geq 21$ where $v \equiv 0 \pmod{3}$.

5. CONCLUDING REMARKS

Combining Theorem 2.1, Theorem 4.8, and the obvious necessary condition, we may summarize the known results in the following theorem.
TABLE 5.1
OMTS(9) Based on \{0, 1, ..., 8\}

<table>
<thead>
<tr>
<th>First MTS(9)</th>
<th>Second MTS(9)</th>
</tr>
</thead>
</table>

THEOREM 5.1. The necessary condition for the existence of a SOMTS(v) (SOSQ(v)), that is, \( v \equiv 0 \text{ or } 1 \pmod{3} \), is also sufficient except for \( v = 3, 6, 9, 10, 12 \), and possibly excepting \( v = 18 \).

Since the transpose of a SOSQ(v) is also a SOSQ(v), Theorem 5.1 also addresses the problem of existence of a pair of OMTS(v) (see [2]). Recently, B. McCune (private communication) has found the pair of OMTS(9) shown in Table 5.1. We have made a computer search for order 10 which ruled out the existence. Thus we have the following corollary.

COROLLARY 5.2. The necessary condition for the existence of a pair of OMTS(v), that is, \( v \equiv 0 \text{ or } 1 \pmod{3} \), is also sufficient except for \( v = 3, 6, 10 \), and possibly excepting \( v \in \{12, 18\} \).

APPENDIX: ISOMTS(v, n)

\[ v = 24, n = 7. \]

\((0, 3, 8), (0, 4, 15), (0, 6, 7), (\infty_1, 0, 7), (\infty_2, 0, 8), (\infty_3, 0, 12), (\infty_4, 0, 13), (\infty_5, 0, 14), (\infty_6, 0, 15), (\infty_7, 0, 16).\]

\[ v = 27, n = 4. \]

\((0, 1, 10), (0, 2, 20), (0, 4, 11), (0, 5, 13), (0, 11, 7), (0, 14, 6), (\infty_1, 0, 6), (\infty_2, 0, 20), (\infty_3, 0, 21), (\infty_4, 0, 22).\]

\[ v = 30, n = 7. \]

\((0, 1, 9), (0, 2, 17), (0, 3, 10), (0, 4, 13), (0, 11, 6), (\infty_1, 0, 5), (\infty_2, 0, 12), (\infty_3, 0, 16), (\infty_4, 0, 19), (\infty_5, 0, 20), (\infty_6, 0, 21), (\infty_7, 0, 22).\]

\[ v = 33, n = 4. \]

\((0, 1, 22), (0, 2, 12), (0, 3, 11), (0, 4, 15), (0, 5, 20), (0, 12, 5), (0, 13, 9), (0, 19, 6), (\infty_1, 0, 6), (\infty_2, 0, 26), (\infty_3, 0, 27), (\infty_4, 0, 28).\]

\[ v = 33, n = 10. \]

\((0, 1, 5), (0, 2, 10), (0, 3, 12), (0, 7, 1), (\infty_1, 0, 5), (\infty_2, 0, 6), (\infty_3, 0, 10), (\infty_4, 0, 12), (\infty_5, 0, 14), (\infty_6, 0, 15), (\infty_7, 0, 16), (\infty_8, 0, 19), (\infty_9, 0, 20), (\infty_{10}, 0, 21).\]
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$v = 36, n = 7$. 
(0, 1, 12), (0, 2, 24), (0, 3, 13), (0, 4, 17), (0, 7, 21),
(0, 15, 9), (0, 18, 8), (∞₁, 0, 6), (∞₂, 0, 9), (∞₃, 0, 24),
(∞₄, 0, 25), (∞₅, 0, 26), (∞₆, 0, 27), (∞₇, 0, 28).

$v = 39, n = 4$. 
(0, 1, 12), (0, 2, 10), (0, 3, 22), (0, 4, 19), (0, 5, 23),
(0, 6, 15), (0, 10, 7), (0, 14, 9), (0, 17, 11), (0, 21, 8),
(∞₁, 0, 7), (∞₂, 0, 31), (∞₃, 0, 33), (∞₄, 0, 34).

$v = 42, n = 13$. 
(0, 1, 4), (0, 2, 14), (0, 6, 1), (0, 7, 16), (0, 8, 18),
(∞₁, 0, 4), (∞₂, 0, 5), (∞₃, 0, 14), (∞₄, 0, 16),
(∞₅, 0, 17), (∞₆, 0, 18), (∞₇, 0, 19), (∞₈, 0, 20),
(∞₉, 0, 21), (∞₁₀, 0, 22), (∞₁₁, 0, 23), (∞₁₂, 0, 26),
(∞₁₃, 0, 27).

$v = 45, n = 7$. 
(0, 31, 5), (0, 19, 9), (0, 24, 11), (0, 10, 12), (0, 8, 15),
(0, 13, 16), (0, 1, 17), (0, 4, 18), (0, 9, 20), (0, 15, 21),
(0, ∞₁, 1), (0, ∞₂, 2), (0, ∞₃, 3), (0, ∞₄, 4), (0, ∞₅, 6),
(0, ∞₆, 8), (0, ∞₇, 33).

$v = 54, n = 13$. 
(0, 13, 3), (0, 33, 4), (0, 17, 6), (0, 16, 7), (0, 25, 12),
(0, 10, 15), (0, 15, 17), (0, 19, 20), (0, 9, 23), (0, ∞₁, 1),
(0, ∞₂, 2), (0, ∞₃, 5), (0, ∞₄, 14), (0, ∞₅, 18),
(0, ∞₆, 19), (0, ∞₇, 21), (0, ∞₈, 30), (0, ∞₉, 33),
(0, ∞₁₀, 34), (0, ∞₁₁, 35), (0, ∞₁₂, 37), (0, ∞₁₃, 38).

$v = 63, n = 16$. 
(0, 24, 9), (0, 30, 10), (0, 7, 13), (0, 12, 14), (0, 5, 16),
(0, 1, 18), (0, 19, 22), (0, 8, 24), (0, 4, 25), (0, 14, 27),
(0, ∞₁, 1), (0, ∞₂, 2), (0, ∞₃, 3), (0, ∞₄, 4), (0, ∞₅, 5),
(0, ∞₆, 6), (0, ∞₇, 7), (0, ∞₈, 8), (0, ∞₉, 11),
(0, ∞₁₀, 12), (0, ∞₁₁, 19), (0, ∞₁₂, 21), (0, ∞₁₃, 29),
(0, ∞₁₄, 32), (0, ∞₁₅, 37), (0, ∞₁₆, 38).

$v = 66, n = 19$. 
(0, 6, 10), (0, 1, 11), (0, 16, 18), (0, 14, 19), (0, 12, 20),
(0, 20, 23), (0, 18, 25), (0, 17, 26), (0, 15, 28), (0, ∞₁, 1),
(0, ∞₂, 2), (0, ∞₃, 3), (0, ∞₄, 4), (0, ∞₅, 5), (0, ∞₆, 6),
(0, ∞₇, 7), (0, ∞₈, 8), (0, ∞₉, 9), (0, ∞₁₀, 12),
(0, ∞₁₁, 13), (0, ∞₁₂, 14), (0, ∞₁₃, 15), (0, ∞₁₄, 16),
(0, ∞₁₅, 17), (0, ∞₁₆, 21), (0, ∞₁₇, 22), (0, ∞₁₈, 24),
(0, ∞₁₉, 36).

$v = 69, n = 22$. 
(0, 1, 4), (0, 2, 23), (0, 6, 1), (0, 7, 36), (0, 8, 17),
(0, 10, 32), (0, 12, 28), (0, 13, 27), (∞₁, 0, 4), (∞₂, 0, 5),
(∞₃, 0, 17), (∞₄, 0, 18), (∞₅, 0, 23), (∞₆, 0, 25),
(∞₇, 0, 26), (∞₈, 0, 27), (∞₉, 0, 28), (∞₁₀, 0, 31),
(∞₁₁, 0, 32), (∞₁₂, 0, 33), (∞₁₃, 0, 34), (∞₁₄, 0, 35),
(∞₁₅, 0, 36), (∞₁₆, 0, 37), (∞₁₇, 0, 38), (∞₁₈, 0, 39),
(∞₁₉, 0, 40), (∞₂₀, 0, 41), (∞₂₁, 0, 44), (∞₂₂, 0, 45).
\(v = 78, \ n = 25.\) 
\((0, 1, 4), \ (0, 2, 16), \ (0, 6, 1), \ (0, 7, 26), \ (0, 8, 25),\)
\((0, 9, 32), \ (0, 10, 22), \ (0, 11, 24), \ (0, 15, 33), \ (\infty_1, 0, 4),\)
\((\infty_2, 0, 5), \ (\infty_3, 0, 16), \ (\infty_4, 0, 22), \ (\infty_5, 0, 24),\)
\((\infty_6, 0, 25), \ (\infty_7, 0, 26), \ (\infty_8, 0, 30), \ (\infty_9, 0, 32),\)
\((\infty_{10}, 0, 33), \ (\infty_{11}, 0, 34), \ (\infty_{12}, 0, 35), \ (\infty_{13}, 0, 36),\)
\((\infty_{14}, 0, 38), \ (\infty_{15}, 0, 39), \ (\infty_{16}, 0, 40), \ (\infty_{17}, 0, 41),\)
\((\infty_{18}, 0, 42), \ (\infty_{19}, 0, 43), \ (\infty_{20}, 0, 44), \ (\infty_{21}, 0, 45),\)
\((\infty_{22}, 0, 46), \ (\infty_{23}, 0, 47), \ (\infty_{24}, 0, 50), \ (\infty_{25}, 0, 51).\)

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**References**