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Law of large numbers for the simple exclusion process

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Abstract

We consider simple exclusion processes on \mathbb{Z} for which the underlying random walk has a finite first moment and whose initial distributions are product measures with different densities to the left and to the right of the origin. We prove a strong law of large numbers for the number of particles present at time t in an interval growing linearly with t .

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1. Introduction

The simple exclusion process is a well studied interacting particle system. An excellent introduction to the subject is [6, Chapter 8]. We study the one-dimensional case with state space $\mathcal{X} := \{0, 1\}^{\mathbb{Z}}$ and associated transition probability matrix $p(x, y)$ satisfying

- (1) $p(x, y) = p(0, y - x) \forall x, y \in \mathbb{Z}$,
- (2) $M := \sum_{x \in \mathbb{Z}} |x| p(0, x) < \infty$.

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The mean jump $\alpha := \sum_{x \in \mathbb{Z}} x p(0, x)$ is called the *drift*. For an arbitrary initial distribution μ , we denote by \mathbb{P}_μ the probability measure on the space of trajectories of the process associated to μ . The initial distribution will be a product measure $\nu^{\lambda, \rho}$ with marginals $\nu^{\lambda, \rho}(\{\eta: \eta(x) = 1\}) = \rho$ if $x > 0$ and $\nu^{\lambda, \rho}(\{\eta: \eta(x) = 1\}) = \lambda$ if $x \leq 0$. If $\lambda = \rho$, this measure will also be denoted by ν^ρ . We recall that the measures ν^ρ ($0 \leq \rho \leq 1$) are invariant for the process. Let $\eta_t(x)$ indicate the presence/absence of a particle at site $x \in \mathbb{Z}$ at time $t \geq 0$. Our main result is:

Theorem 1. *Let $0 \leq \rho \leq \lambda \leq 1$ and let $u < v$ be real numbers. Then,*

$$\mathbb{P}_{\nu^{\lambda, \rho}} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \eta_t(x) = \int_u^v f(s) ds \right) = 1,$$

where

$$f(u) := \begin{cases} \lambda & \text{if } u < \alpha(1 - 2\lambda), \\ \frac{1}{2} \left(1 - \frac{u}{\alpha} \right) & \text{if } \alpha(1 - 2\lambda) \leq u \leq \alpha(1 - 2\rho), \\ \rho & \text{if } \alpha(1 - 2\rho) < u, \end{cases} \tag{1}$$

if $\alpha > 0$ and

$$f(u) := \begin{cases} \lambda & \text{if } u < \alpha(1 - \lambda - \rho), \\ \rho & \text{if } \alpha(1 - \lambda - \rho) \leq u, \end{cases} \tag{2}$$

if $\alpha \leq 0$.

Theorem 1 means that the number of particles in the interval $[ut, vt]$ at time t satisfies a strong law of large numbers as t goes to infinity. The function f is the entropic solution at time 1 of the Burgers equation associated to the exclusion process. The limit in the Theorem is usually called “hydrodynamic limit”. Most of the hydrodynamic results in the literature perform limits in probability or in distribution. In particular Rezakhanlou [7] proved the convergence in probability of $\frac{1}{t} \sum_{ut \leq x \leq vt} \eta_t(x)$ for finite range matrices $p(x, y)$ and general initial profiles in general myanthrope processes (the simple exclusion is a particular case of myanthrope). A constructive proof of this convergence has been recently extended for a larger class of attractive processes by Bahadoran, Guiol, Ravishankar and Saada [3]. In the nearest neighbors totally asymmetric case ($p(x, x + 1) = 1$ and $p(x, y) = 0$ otherwise) the almost sure convergence was proven by Rost [8] using the subadditive ergodic theorem. This approach was then taken by Benassi and Fouque [4] who stated the almost sure convergence under the nearest-neighbors assumption $p(x, y) = 0$ if $|x - y| > 1$. Unfortunately their proof contains a mistake, as will be explained later. Nevertheless the main ideas of their proof, the use of a subadditive ergodic theorem and the introduction of various classes of particles, are valuable and essential in this paper. Seppäläinen [9] proved almost sure convergence for general initial profiles in the nearest neighbor totally asymmetric case using a variational approach. Extension of this approach to more general jump rates are not available.

Our result is restricted to initial Riemann profiles, i.e. the initial measure is a product measure with density ρ to the right of the origin and density λ to its left. Bahadoran, Guiol, Ravishankar and Saada [3] extend hydrodynamic limits in probability from Riemann to general initial profiles, under finite range jumps. In the case their technique could be further extended to a.s. convergence and unbounded jumps, our result together with this extension would give a.s. hydrodynamic limits for general initial profiles and unbounded jumps. We leave this question open for future research.

The outline of the paper is as follows. In Section 2 we perform a graphical construction of the process, define couplings and different classes of particles; these are key tools for the proofs. In Section 3 we use the subadditive ergodic theorem to show the hydrodynamic limit for a particular initial measure with densities ρ and λ ; this measure satisfies a stationary condition required by the subadditive ergodic theorem but is not a product measure. In Section 4 we extend the result to initial product measures.

2. Graphical construction and coupling

2.1. Graphical construction

It is convenient to perform a Harris graphical construction of the process (see Harris [5]). Let $\mathcal{N} = ((N_t(x, y), t \geq 0) : x, y \in \mathbb{Z})$ be a family of independent Poisson processes such that the rate of the process indexed by (x, y) is $p(x, y)$. These processes are constructed on a probability space denoted $(\Omega, \mathcal{A}, \mathbb{P})$. We denote \mathbb{E} the expectation with respect to \mathbb{P} . The rate of $\sum_{y \in \mathbb{Z}} N_t(x, y)$ is 1 for all x and the rates of $\sum_{y < x} \sum_{z \geq x} N_t(y, z)$ and $\sum_{z < x} \sum_{y \geq x} N_t(y, z)$ are bounded by M for all x .

The process η_t is now constructed as follows: a particle at $x \in \mathbb{Z}$ waits until $\sum_y N_t(x, y)$ jumps. If this jump is due to a jump of $N_t(x, z)$ then the particle either remains at x or jumps to z according to whether or not z is occupied by another particle. We will now show that on a set of probability 1, the trajectories of the process η_t are well defined for all initial configurations. We ignore the set of probability 0 on which two Poisson processes have a simultaneous jump. Fix $z \in \mathbb{Z}$ and $k \in \mathbb{N}$. The number of crossings in the time interval $[0, k]$ of an arbitrary site i ,

$$\sum_{x < i} \sum_{y \geq i} (N_k(x, y) + N_k(y, x)), \tag{3}$$

is a Poisson random variable with mean

$$k \sum_{x < i} \sum_{y \geq i} (p(x, y) + p(y, x)) = k \sum_x |x| p(0, x) = kM < \infty, \tag{4}$$

where the inequality follows from our hypothesis on $p(x, y)$. In particular there is a positive probability of no crossings: $\mathbb{P}(\sum_{x < i} \sum_{y \geq i} (N_k(x, y) + N_k(y, x)) = 0) > 0$. Then, by the Ergodic Theorem, with probability 1 there exist random integers i and j such that $i < z < j$ and

$$\sum_{x < i} \sum_{y \geq i} (N_k(x, y) + N_k(y, x)) = \sum_{x < j} \sum_{y \geq j} (N_k(y, z) + N_k(z, y)) = 0.$$

This means that, on a set $\Omega_{z,k}$ of probability 1, up to time k no particles have crossed (or attempted to cross) the boundaries of the interval $[i, j - 1]$. Hence the movement of the particles which at time 0 were in that interval depend up to time k only on a finite number of Poisson processes and are therefore properly determined. Repeating the argument for each $z \in \mathbb{Z}$ and each $k \in \mathbb{N}$ we see that on the set $\bigcap_{z,k} \Omega_{z,k}$, $\eta_t(z)$ is well defined, as a function of \mathcal{N} for all $t \geq 0$, $z \in \mathbb{Z}$ and $\eta \in \mathcal{X}$. When the dependence on \mathcal{N} needs to be emphasized we will write $\eta_t(\mathcal{N})$. It is convenient to start the process at different times, with different initial configurations; so we also use the notation $\eta_t^{\eta,s}$ to denote the configuration at time t for a process that at time s started with the initial configuration η ; when $s = 0$ we just denote η_t^η . In particular we have

$$\eta_t^{\eta,s}(\mathcal{N}) = \eta_{t-s}^{\eta,0}(\tau_s \mathcal{N}), \tag{5}$$

where $\tau_s \mathcal{N} = ((N_t(x, y) - N_s(x, y), t \geq s): x, y \in \mathbb{Z})$. Denote

$$\mathbb{P}_\mu(\eta_t \in A) := \int \mu(d\eta) \mathbb{P}(\eta_t^\eta \in A); \quad \mathbb{E}_\mu F(\eta_t) := \int \mu(d\eta) \mathbb{E}(F(\eta_t^\eta)) \tag{6}$$

for measurable sets $A \subset \mathcal{X}$ and continuous functions $F: \mathcal{X} \rightarrow \mathbb{R}$.

2.2. Two types of particles (coupling)

The graphical construction allows the simultaneous construction (with the same Poisson processes) of various versions of the process starting with different initial configurations. Let σ and θ two initial configurations and consider the *coupled* process

$$(\sigma_t, \theta_t) := (\eta_t^\sigma(\mathcal{N}), \eta_t^\theta(\mathcal{N})). \tag{7}$$

This is a Markov process in \mathcal{X}^2 . We can continue using \mathbb{P} and \mathbb{E} as the probability and expectation related to the coupled process because it is defined as a function of \mathcal{N} .

Let $(U_x, x \in \mathbb{Z})$ be a sequence of iid random variables uniformly distributed in $[0, 1]$ and define $\sigma(x) = \mathbf{1}\{U_x \leq \rho\}$ and $\theta(x) = \mathbf{1}\{U_x \leq \lambda\}$. Call $\bar{\nu}^{\lambda,\rho}$ the resulting distribution of (σ, θ) . Then, $\bar{\nu}^{\lambda,\rho}$ is a product measure on \mathcal{X}^2 with marginals

$$\bar{\nu}^{\lambda,\rho}(\{(\sigma, \theta): \sigma(x) = 1, x \in A\}) = \rho^{|A|}$$

$$\bar{\nu}^{\lambda,\rho}(\{(\sigma, \theta): \theta(x) = 1, x \in A\}) = \lambda^{|A|}$$

for any finite $A \subset \mathbb{Z}$. Furthermore, if $0 \leq \rho < \lambda \leq 1$, then

$$\bar{\nu}^{\lambda,\rho}(\{(\sigma, \theta): \sigma \leq \theta \text{ coordinatewise}\}) = 1. \tag{8}$$

Lemma 2. *The process (σ_t, θ_t) defined in (7) has an invariant measure $\bar{\mu}^{\lambda,\rho}$ with marginals*

$$\bar{\mu}^{\lambda,\rho}(\{(\sigma, \theta): \sigma(x) = 1, x \in A\}) = \rho^{|A|},$$

$$\bar{\mu}^{\lambda,\rho}(\{(\sigma, \theta): \theta(x) = 1, x \in A\}) = \lambda^{|A|}$$

for any finite $A \subset \mathbb{Z}$. For this measure, if $0 \leq \rho < \lambda \leq 1$, then

$$\bar{\mu}^{\lambda, \rho}(\{(\sigma, \theta): \sigma \leq \theta \text{ coordinatewise}\}) = 1. \tag{9}$$

Furthermore, it is possible to construct a measure $\bar{\nu}$ on $\mathcal{X}^2 \times \mathcal{X}^2$ with marginals $\bar{\mu}^{\lambda, \rho}$ and $\bar{\nu}^{\lambda, \rho}$ such that

$$\bar{\nu}\{((\sigma, \theta), (\tilde{\sigma}, \tilde{\theta})): \sigma = \tilde{\sigma}\} = 1. \tag{10}$$

Proof. For the first part we follow Liggett [6]. Start the process (σ_t, θ_t) with the product measure $\bar{\nu}^{\lambda, \rho}$. Since the marginals are invariant measures for the marginal process, the first and second marginal law of (σ_t, θ_t) are ν^ρ and ν^λ , respectively for all $t \geq 0$. Therefore, we can obtain $\bar{\mu}^{\lambda, \rho}$ as any weak limit as t goes to infinity of

$$\frac{1}{t} \int_0^t \bar{\nu}^{\lambda, \rho} \bar{S}(s) ds.$$

where $\bar{S}(t)$ is the semi group describing the evolution of (σ_t, θ_t) which is defined by $\bar{\nu} \bar{S}(t) f := \int \bar{\nu}(d(\sigma, \theta)) f(\sigma_t, \theta_t)$. The invariance of the limit is proven in Liggett [6]. The domination (9) follows because it is satisfied by the initial distribution and any transition keeps it (this property is usually referred to as *attractiveness*).

To construct a measure with the property (10) choose (σ, θ) with $\bar{\mu}^{\lambda, \rho}$ and define

$$\tilde{\sigma}(x) := \sigma(x),$$

$$\tilde{\theta}(x) := \sigma(x) + \mathbf{1}\{\sigma(x) = 0, U_x \in [\rho, \lambda]\},$$

where (U_x) is the sequence of iid uniform random variables in $[0, 1]$. Since the ρ marginal of $\bar{\mu}^{\lambda, \rho}$ is the product measure with density ρ , the law of $((\sigma, \theta), (\tilde{\sigma}, \tilde{\theta}))$ satisfies the requirements. \square

3. The subadditive ergodic theorem

Let $T: \mathcal{X}^2 \rightarrow \mathcal{X}$ be defined by

$$T(\sigma, \theta)(x) = \begin{cases} 1 & \text{if } \sigma(x) = 1, \theta(x) = 1, \\ 1 & \text{if } \sigma(x) = 0, \theta(x) = 1 \text{ and } x \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This operator erases the θ particles to the right of the origin, keeps the θ particles to the left of the origin and all the σ particles and identifies the labels σ and θ . The operator T induces a map from probability measures on \mathcal{X}^2 to probability measures on \mathcal{X} which will be called T too. Note that

$$\nu^{\lambda, \rho} = T \bar{\nu}^{\lambda, \rho}. \tag{11}$$

Proposition 3. For all $u < v, \rho \leq \lambda$, there exists a random variable $G(u, v, \lambda, \rho)$ such that:

$$\mathbb{P}_{T\bar{\mu}^{\lambda, \rho}} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \eta_t(x) = G(u, v, \lambda, \rho) \right) = 1 \tag{12}$$

The proof of this proposition follows Benassi and Fouque [4]. However, in that paper the result is stated for the initial measure $T\bar{\nu}^{\lambda, \rho}$ instead of $T\bar{\mu}^{\lambda, \rho}$. Because the measure $\bar{\nu}^{\lambda, \rho}$ is not invariant for the process (σ_t, θ_t) , their random variables fail to satisfy condition (b) of the Subadditive Ergodic Theorem (see below). We therefore apply that theorem to the initial measure $T\bar{\mu}^{\lambda, \rho}$ and then have to show that a strong law of large numbers for this measure implies that a similar result holds for the more natural initial product measure.

We state now the subadditive ergodic theorem (taken from Liggett [6]), introduce the notion of processes with different classes of particles and remark some hole-particle symmetries inherent to the exclusion process. Then we prove the proposition.

3.1. The subadditive ergodic theorem

Let $\{X_{m,n}\}$ a family of random variables satisfying:

- (a) $X_{0,0} = 0, X_{0,n} \leq X_{0,m} + X_{m,n}$ for $0 \leq m \leq n$.
- (b) $\{X_{(n-1)k, nk}, n \geq 1\}$ is a stationary sequence for each $k \geq 1$.
- (c) $\{X_{m, m+k}, k \geq 0\} = \{X_{m+1, m+k+1}, k \geq 0\}$ in distribution for each m .
- (d) $\mathbb{E}(X_{0,1}) < \infty$.

Then

$$\lim_{n \rightarrow \infty} \frac{X_{0,n}}{n} = X_\infty \text{ exists a.s.}$$

3.2. Particle-hole symmetry

Holes behave as particles but with reflected rates. More precisely, given a particle configuration η define the *reflected hole configuration* $\check{\eta}$ by $\check{\eta}(x) := 1 - \eta(-x)$. The reflected hole process $\check{\eta}_t$ is an exclusion process with (the same) rates $p(x, y)$. If the particle configuration η is distributed with $\nu^{\lambda, \rho}$, then the reflected hole configuration $\check{\eta}$ is distributed with $\nu^{1-\rho, 1-\lambda}$. Analogously, if the two-type particle configuration (σ, θ) is distributed with $\mu^{\lambda, \rho}$, then the two-type reflected hole configuration $(\check{\theta}, \check{\sigma})$ is distributed with $\bar{\mu}^{1-\rho, 1-\lambda}$, an invariant measure for the two-type reflected hole process $(\check{\theta}_t, \check{\sigma}_t)$; furthermore $\bar{\mu}^{1-\rho, 1-\lambda}$ has marginals $\nu^{1-\lambda}$ and $\nu^{1-\rho}$. The coordinates are interchanged because $\check{\theta} \leq \check{\sigma}$ and we want to keep the first coordinate smaller than or equal to the second as in (9). In particular, if (σ, θ) has law $\bar{\mu}^{\lambda, \rho}$, then $\check{\eta} = T(\check{\theta}, \check{\sigma})$ has law $T\bar{\mu}^{1-\rho, 1-\lambda}$.

3.3. Processes with different classes of particles

Suppose σ and ξ are elements of \mathcal{X} such that $\sigma + \xi \in \mathcal{X}$, where the sum is taken coordinatewise. Then we can interpret the process $(\eta_t^\sigma(\mathcal{N}), \eta_t^{\sigma+\xi}(\mathcal{N}))$ as follows: a given site $x \in Z$ is at time t occupied by a first class particle, occupied by a second class particle or vacant according to whether $\eta_t^\sigma(x) + \eta_t^{\sigma+\xi}(x)$ equals 2, 1 or 0, respectively. The reader can easily check that first class particles ignore the presence of second class particles and evolve as an exclusion process. In turn, a second class particle jumps to empty sites as a first class particle, but when a first class particle jumps over a second class one, they interchange sites. When σ and ξ are as above we define:

$$\begin{aligned} \sigma_t &= \eta_t^\sigma(\mathcal{N}) \\ \xi_t &= \eta_t^{\sigma+\xi}(\mathcal{N}) - \eta_t^\sigma(\mathcal{N}). \end{aligned} \tag{13}$$

So that σ_t are the first class particles and ξ_t the second class ones.

Let $T_m, V_m: \mathcal{X} \rightarrow \mathcal{X}$ be the truncations defined by

$$T_m \xi(x) = \xi(x) \mathbf{1}\{x \leq m\}, \quad V_m \xi(x) = \xi(x) \mathbf{1}\{x > m\}. \tag{14}$$

Note that if $\sigma + \xi \in \mathcal{X}$, then $T(\sigma, \sigma + \xi) = \sigma + T_0 \xi$.

Proof of Proposition 3. Let (σ, θ) be distributed according to $\tilde{\mu}^{\lambda, \rho}$,

$$\xi := T_0(\theta - \sigma) \tag{15}$$

and let (σ_t, ξ_t) be the process defined by (13) with initial condition (σ, ξ) . Note that

$$\eta_t := \sigma_t + \xi_t, \tag{16}$$

is an exclusion process with initial distribution $T \tilde{\mu}^{\lambda, \rho}$ and that σ_t is an exclusion process with initial distribution ν^ρ . Since this initial distribution is invariant, by the law of large numbers for triangular arrays we have that for $n \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{un \leq x \leq vn} \sigma_n(x) = \rho(v - u) \quad \mathbb{P}_{\tilde{\mu}^{\lambda, \rho}} \text{ a.s.}, \tag{17}$$

Now taking $n = [t]$ as the integer part of t ,

$$\begin{aligned} & \sum_{u[t] \leq x \leq v[t]} \sigma_{[t]}(x) - \sum_{ut \leq x \leq vt} \sigma_t(x) \\ &= \left(\sum_{u[t] \leq x \leq v[t]} - \sum_{ut \leq x \leq vt} \right) \sigma_{[t]}(x) + \sum_{ut \leq x \leq vt} (\sigma_{[t]}(x) - \sigma_t(x)). \end{aligned} \tag{18}$$

The absolute value of first term is bounded by $|ut - [ut]| + |vt - [vt]|$. The absolute value of the second term is bounded by the number of Poisson jumps crossing either ut or vt in the interval $t - [t]$. Both terms converge to zero almost surely when divided by t . This implies

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \sigma_t(x) = \rho(v - u) \quad \mathbb{P}_{\tilde{\mu}^{\lambda, \rho}} \text{ a.s.}, \tag{19}$$

where the limit is now taken for $t \in \mathbb{R}$.

In Proposition 4, below we show that for $u \geq 0$ there exists a random variable $X(u, \lambda, \rho)$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{x \geq ut} \xi_t(x) = X(u, \lambda, \rho) \quad \mathbb{P}_{\bar{\mu}^{\lambda, \rho}} \text{ a.s.} \tag{20}$$

Then for $u \geq 0$ the proposition follows from (16), (19) and (20); the random variable G is given by

$$G(u, v, \lambda, \rho) := \rho(v - u) + X(u, \lambda, \rho) - X(v, \lambda, \rho). \tag{21}$$

For $u < 0$ we use the particle-hole symmetry. By additivity of the limits, we can assume $v = 0$. For convenience we consider $u > 0$ and compute the limits for $-u$.

$$\sum_{-u \leq x < 0} \eta_t(x) = [ut] - \sum_{0 < x \leq ut} \check{\eta}_t(x), \tag{22}$$

where $\check{\eta}_t(x)$ is the reflected hole process. By the remarks made in the particle-hole symmetry paragraph above, $\check{\eta}_0$ has law $T\bar{\mu}^{1-\rho, 1-\lambda}$, so that we can apply the result for positive u to get that (22) divided by t converges $\mathbb{P}_{T\bar{\mu}^{\lambda, \rho}}$ a.s. to $G(-u, 0, \lambda, \rho) = u - (1 - \lambda)u - X(0, 1 - \rho, 1 - \lambda) + X(u, 1 - \rho, 1 - \lambda)$. This shows the proposition for all $u < v \in \mathbb{R}$. \square

Proposition 4. *Let $u \geq 0$. Let (σ, θ) be $\bar{\mu}^{\lambda, \rho}$ -distributed and let (σ_t, ξ_t) be the two-classes particle process (13) with initial configuration $(\sigma, T_0(\theta - \sigma))$. Then there exists a random variable $X(u, \lambda, \rho)$ such that (20) holds.*

Proof. We prove the proposition for $u > 0$. Then, the case $u = 0$ follows by letting u decrease to 0 in the following inequalities:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{x \geq ut} \xi_t(x) \leq \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{x \geq 0} \xi_t(x) \leq u + \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{x \geq ut} \xi_t(x).$$

We need to introduce more classes of particles. If σ, ξ, ζ and γ are such that $\sigma + \xi + \gamma + \zeta \in \mathcal{X}$, then

$$(\eta_t^\sigma(\mathcal{N}), \eta_t^{\sigma+\xi}(\mathcal{N}), \eta_t^{\sigma+\xi+\gamma}(\mathcal{N}), \eta_t^{\sigma+\xi+\gamma+\zeta}(\mathcal{N}))$$

induces a process with first, second, third and fourth class particles by defining, for $t \geq 0$,

$$\begin{aligned} \sigma_t &:= \eta_t^\sigma(\mathcal{N}), \\ \xi_t &:= \eta_t^{\sigma+\xi}(\mathcal{N}) - \eta_t^\sigma(\mathcal{N}), \\ \gamma_t &:= \eta_t^{\sigma+\xi+\gamma}(\mathcal{N}) - \eta_t^{\sigma+\xi}(\mathcal{N}), \\ \zeta_t &:= \eta_t^{\sigma+\xi+\gamma+\zeta}(\mathcal{N}) - \eta_t^{\sigma+\xi+\gamma}(\mathcal{N}). \end{aligned} \tag{23}$$

Let (σ, θ) be distributed according to $\bar{\mu}^{\lambda, \rho}$ and

$$\xi := T_0(\theta - \sigma), \quad \gamma := 0, \quad \zeta := \theta - \sigma - \xi \tag{24}$$

and let $(\sigma_t, \xi_t, \gamma_t, \zeta_t)$ be the four-classes process defined by (23) with this initial condition. For each integer $m \geq 0$, we define a process on \mathcal{X}^4 in the time interval $[m/u, \infty)$ as follows: the initial (random) configurations at time m/u are

$$\begin{aligned} \sigma^m &:= \sigma_{m/u}, & \xi^m &:= T_m(\xi_{m/u} + \zeta_{m/u}), \\ \gamma^m &:= V_m(\xi_{m/u}), & \zeta^m &:= V_m(\zeta_{m/u}) \end{aligned} \tag{25}$$

and for $t \geq m/u$,

$$\begin{aligned} \sigma_t^m &:= \eta_t^{\sigma^m, m/u}(\mathcal{N}), \\ \xi_t^m &:= \eta_t^{\sigma^m + \xi^m, m/u}(\mathcal{N}) - \eta_t^{\sigma^m, m/u}(\mathcal{N}), \\ \gamma_t^m &:= \eta_t^{\sigma^m + \xi^m + \gamma^m, m/u}(\mathcal{N}) - \eta_t^{\sigma^m + \xi^m, m/u}(\mathcal{N}), \\ \zeta_t^m &:= \eta_t^{\sigma^m + \xi^m + \gamma^m + \zeta^m, m/u}(\mathcal{N}) - \eta_t^{\sigma^m + \xi^m + \gamma^m, m/u}(\mathcal{N}). \end{aligned} \tag{26}$$

Note that for $m = 0$ this is the same process as $(\sigma_t, \xi_t, \gamma_t, \zeta_t)$, and that for any m the process $(\sigma_t^m, \sigma_t^m + \xi_t^m + \gamma_t^m + \zeta_t^m)$ defined for $t \geq m/u$ is a version of the coupled process (7) under the invariant measure $\bar{\mu}^{\lambda, \rho}$.

Let $u > 0$ and for $0 \leq m \leq n$ define an array $X_{m,n}$ of random variables:

$$X_{m,n} := \begin{cases} 0 & \text{if } m = n, \\ \sum_{y > n} \xi_{n/u}^m(y) & \text{if } 0 \leq m < n. \end{cases} \tag{27}$$

This array satisfies the hypothesis of the subadditive ergodic theorem:

(a) Assume $0 < m < n$ since the other cases are trivial. Define

$$\bar{\xi}^m := T_m \xi_{m/u}. \tag{28}$$

It follows from Definitions (25) and (28) that for $t \geq m/u$,

$$\begin{aligned} \xi_t &= \eta_t^{\sigma^m + \bar{\xi}^m + \gamma^m, m/u}(\mathcal{N}) - \eta_t^{\sigma^m, m/u}(\mathcal{N}) \\ &\leq \eta_t^{\sigma^m + \xi^m + \gamma^m, m/u}(\mathcal{N}) - \eta_t^{\sigma^m, m/u}(\mathcal{N}). \end{aligned}$$

Hence $\xi_{n/u}$ is bounded above by

$$\begin{aligned} \eta_{n/u}^{\sigma^m + \xi^m + \gamma^m, m/u}(\mathcal{N}) - \eta_{n/u}^{\sigma^m + \xi^m, m/u}(\mathcal{N}) + \eta_{n/u}^{\sigma^m + \xi^m, m/u}(\mathcal{N}) - \eta_{n/u}^{\sigma^m, m/u}(\mathcal{N}) \\ = \gamma_{n/u}^m + \xi_{n/u}^m. \end{aligned}$$

Therefore

$$X_{0,n} = \sum_{y > n} \xi_{n/u}(y) \leq \sum_{y \in \mathbb{Z}} \gamma_{n/u}^m(y) + \sum_{y > n} \xi_{n/u}^m(y)$$

and, since all the γ particles are created at time m/u , the first term of the right-hand side is equal to $\sum_{y \in \mathbb{Z}} \gamma^m(y)$, which by (25) is the same as $\sum_{y > m} \xi_{m/u}(y)$. Hence,

$$X_{0,n} \leq \sum_{y > m} \xi_{m/u}(y) + \sum_{y > n} \xi_{n/u}^m(y) = X_{0,m} + X_{m,n}.$$

(b) and (c) These stationary conditions follow from the space and time translation invariance of the Poisson processes and the stationarity of the initial measure $\bar{\mu}^{\lambda,\rho}$. See the comment after (26).

(d) Since ξ particles can only jump when a Poisson jump is present, $X_{0,1}$, the number of ξ particles to the right of the origin at time 1, is bounded by $\sum_{x \leq 0} \sum_{y > 0} (N_1(x, y) + N_1(y, x))$, the number of Poisson jumps across the origin in the interval $[0, 1]$. As in (3) and (4) (with $k = 1$) this is a Poisson variable with mean M which is finite by hypothesis.

Then, by the subadditive ergodic theorem,

$$\lim_{n \rightarrow \infty} \frac{X_{0,n}}{n} = X_\infty \quad \text{exists } \mathbb{P}_{\bar{\mu}^{\lambda,\rho}} \text{ a.s.}$$

By the definition of $X_{0,n}$, calling $n = [ut]$ (integer part)

$$\sum_{x \geq ut} \xi_t(x) = X_{0,n} + \text{rest}, \tag{29}$$

where the absolute value of the rest in (29) is

$$\begin{aligned} \left| \sum_{x \geq ut} \xi_t(x) - \sum_{x > n} \xi_{n/u}(x) \right| &\leq \sum_{x > n} |\xi_t(x) - \xi_{n/u}(x)| + 1 \\ &\leq \left(\sum_{x \leq n} \sum_{y > n} + \sum_{y \leq n} \sum_{x > n} \right) (N_t(x, y) - N_{[ut]/u}(x, y)) + 1 \end{aligned} \tag{30}$$

(The difference of the number of ξ particles to the right of n at two different times is dominated by the number of Poisson crossings of n between those times; the addition of 1 is to cover the case $ut = n$.) Reasoning as in (4) (with $k = ut - [ut]$), the sum in the second line of (30) is a Poisson random variable with mean

$$\sum_x |x| p(0, x) (ut - [ut]) / u \leq \frac{M}{u}.$$

Hence, the rest in (29) divided by t goes to zero almost surely as $t \rightarrow \infty$ and we have proved that for $u > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{x \geq ut} \xi_t(x) = u X_\infty := X(u, \lambda, \rho). \quad \square \tag{31}$$

4. Proof of Theorem 1

In the previous section we proved a law of large numbers when the initial measure is $T\bar{\mu}^{\lambda,\rho}$. In this section we show that the same is true for $\nu^{\lambda,\rho}$ and identify the limit G as being the function f .

Proposition 5. *There exists a measure $\bar{\pi}^{\lambda,\rho}$ on \mathcal{X}^2 with marginals $T\bar{\mu}^{\lambda,\rho}$ and $T\bar{\nu}^{\lambda,\rho}$ such that the coupled process $(\eta_t, \tilde{\eta}_t)$ defined as in (7) satisfies that for all $K > M$,*

$$\mathbb{P}_{\bar{\pi}^{\lambda,\rho}} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{-Kt \leq x \leq Kt} (\eta_t(x) - \tilde{\eta}_t(x)) = 0 \right) = 1. \tag{32}$$

Proof. Choose $((\sigma, \theta), (\sigma, \tilde{\theta}))$ distributed according to $\bar{\nu}$ of Lemma 2 and define $\xi = \theta - \sigma$ and $\tilde{\xi} = \tilde{\theta} - \sigma$. Let $\bar{\pi}^{\lambda,\rho}$ be the law of

$$(\eta, \tilde{\eta}) := (\sigma + T_0\xi, \sigma + T_0\tilde{\xi}). \tag{33}$$

It is clear that $\bar{\pi}^{\lambda,\rho}$ has marginals $T\bar{\mu}^{\lambda,\rho}$ and $T\bar{\nu}^{\lambda,\rho}$. The coupling with initial distribution $\bar{\pi}^{\lambda,\rho}$ is defined by

$$(\eta_t, \tilde{\eta}_t) := (\eta_t^\eta(\mathcal{N}), \eta_t^{\tilde{\eta}}(\mathcal{N})). \tag{34}$$

Define the flux $J_t^{\eta,r}$ as the number of η particles that at time zero were to the left of r and at time t are strictly to the right of r minus the number of η particles that at time zero were strictly to the right of r and at time t are to the left of r . Then for arbitrary positive K we can write

$$\begin{aligned} \sum_{-Kt \leq x \leq Kt} (\eta_t(x) - \tilde{\eta}_t(x)) &= \sum_{-Kt \leq x \leq Kt} (\eta_0(x) - \tilde{\eta}_0(x)) \\ &\quad + J_t^{\eta,-Kt} - J_t^{\eta,Kt} - J_t^{\tilde{\eta},-Kt} + J_t^{\tilde{\eta},Kt}. \end{aligned} \tag{35}$$

But

$$\begin{aligned} \sum_{-Kt \leq x \leq Kt} (\eta_0(x) - \tilde{\eta}_0(x)) &= \sum_{-Kt \leq x \leq 0} (\theta_0(x) - \tilde{\theta}_0(x)) \\ &\quad + \sum_{0 < x \leq Kt} (\sigma_0(x) - \sigma_0(x)). \end{aligned}$$

By the law of large numbers for the marginals of both $T\bar{\mu}^{\lambda,\rho}$ and $T\bar{\nu}^{\lambda,\rho}$ the first term divided by t goes to zero. Indeed both θ_0 and $\tilde{\theta}_0$ have law ν^ρ . The second term is zero.

We prove now that $\lim_{t \rightarrow \infty} (1/t)(J_t^{\eta,Kt} - J_t^{\tilde{\eta},Kt}) = 0$ almost surely. We couple three exclusion processes with initial configurations σ , $\sigma + T_0\xi$ and $\sigma + T_0\tilde{\xi}$ and define:

$$\sigma_t := \eta_t^\sigma(\mathcal{N}),$$

$$\xi_t := \eta_t^{\sigma+T_0\tilde{\xi}}(\mathcal{N}) - \eta_t^\sigma(\mathcal{N}), \tag{36}$$

$$\tilde{\xi}_t := \eta_t^{\sigma+T_0\tilde{\tilde{\xi}}}(\mathcal{N}) - \eta_t^\sigma(\mathcal{N}).$$

In this way, $(\eta_t, \tilde{\eta}_t) = (\sigma_t + \xi_t, \sigma_t + \tilde{\xi}_t)$ and

$$J_t^{\eta, Kt} - J_t^{\tilde{\eta}, Kt} = J_t^{\xi, Kt} - J_t^{\tilde{\xi}, Kt}. \tag{37}$$

The second-class particle configurations ξ and $\tilde{\xi}$ are dominated by $T_0(1 - \sigma)$ and dominate the null configuration. Since the dynamics is attractive, this domination is valid at all positive times as well. This implies that the absolute value of (37) assumes its maximal value when $\xi = T_0(1 - \sigma)$ and $\tilde{\xi} \equiv 0$, which we assume for the sequel. We label the ξ particles at time zero and follow the positions of the labeled particles. Call R_t^x the position of the ξ particle starting at $x \leq 0$. Then

$$J_t^{\xi, Kt} - J_t^{\tilde{\xi}, Kt} \leq \sum_{x \leq 0} (1 - \sigma(x)) \mathbf{1}\{R_t^x > Kt\}. \tag{38}$$

To dominate this, consider independent Poisson random variables $N_x, x < 0$, with mean $1 + \varepsilon$. Let $Y_t^{x, \ell}, \ell = 1, \dots, N_x$ be independent random walks that jump from y to $y + n$ at rate $p(0, n) + p(0, -n)$ for all $y \in \mathbb{Z}$. Order the Y particles at time zero and call the ordered particles Z_0^i so that $Z_0^i \geq Z_0^{i+1}$ for all i (here the superlabel does not coincide necessarily with the initial position as for the R and Y particles). Since the mean number of Y particles in each x is bigger than one, using independence and the Poisson law of N_x , we have $Z_0^i \geq R_0^i$ for all but a finite (random) number W of i 's. The law of W decays exponentially.

Since the R particles jump from site x to site $x + n$ at most at rate $p(0, n) + p(0, -n)$, obvious coupling shows

$$Z_0^i \geq R_0^i \text{ implies } Z_t^i \geq R_t^i \text{ for all } t \geq 0. \tag{39}$$

Then the number of R particles to the right of Kt is dominated by the number of Z particles to the right of Kt plus W :

$$\sum_{x \leq 0} (1 - \sigma(x)) \mathbf{1}\{R_t^x > Kt\} \leq Z_t + W, \tag{40}$$

where

$$Z_t := \sum_{i \leq 0} \mathbf{1}\{Z_t^i > Kt\}. \tag{41}$$

The variable Z_t is Poisson with mean

$$\mathbb{E}Z_t = (1 + \varepsilon) \sum_{x \leq 0} \mathbb{P}(Y_t^x > Kt) = (1 + \varepsilon) \sum_{j \geq Kt} \mathbb{P}(Y_t^0 > j), \tag{42}$$

where Y_t^x is a random walk starting at x that jumps from x to $x + n$ with rate $p(0, n) + p(0, -n)$. Hence

$$\frac{1}{t} \mathbb{E}Z_t = (1 + \varepsilon) \frac{1}{t} \sum_{j \geq 0} \mathbb{P}\left(\frac{Y_t^0}{t} > K + \frac{j}{t}\right) \tag{43}$$

$$\leq (1 + \varepsilon) \frac{[t] + 1}{t} \sum_{j \geq 0} \mathbb{P} \left(\frac{Y_t^0}{t} > K + j \right) \tag{44}$$

$$\leq (1 + \varepsilon) \frac{[t] + 1}{t} \mathbb{E} \left(\frac{Y_t^0}{t} - K \right)^+ \tag{45}$$

which goes to zero as $t \rightarrow \infty$ by the law of large numbers for the random walk Y_t^0 if $K > M$, because

$$M = \sum_n n(p(0, n) + p(0, -n)) = \mathbb{E}Y_t^0/t. \tag{46}$$

On the other hand, using the subadditive ergodic theorem we can show that $(1/t)Z_t$ converges almost surely. The Poisson distribution of the Y particles is used here to show the stationary conditions (b) and (c) of the subadditive ergodic theorem. Indeed, the product measures with Poisson marginals with constant mean are invariant for the process of independent particles. Since $(1/t)Z_t$ is positive and its expectation converges to zero, $(1/t)Z_t$ converges almost surely to zero.

This, (37), (38) and (40), imply

$$\forall K > M \quad \lim_{t \rightarrow \infty} \frac{1}{t} (J_t^{\eta, Kt} - J_t^{\tilde{\eta}, Kt}) = 0 \quad \mathbb{P}_{\tilde{\pi}^{\lambda, \rho}} \text{ a.s.} \tag{47}$$

The flux of particles is equal to minus the flux of holes. Indeed, each time a particle jumps from x to y there is a hole jumping from y to x . Hence, by the particle-hole symmetry, the flux $J_t^{\eta, -Kt} - J_t^{\tilde{\eta}, -Kt}$ for $(\eta, \tilde{\eta})$ chosen with $\tilde{\pi}^{\lambda, \rho}$ has the same law as $J_t^{\eta, Kt} - J_t^{\tilde{\eta}, Kt}$ for $(\eta, \tilde{\eta})$ chosen with $\tilde{\pi}^{1-\rho, 1-\lambda}$. This implies that

$$\forall K > M \quad \lim_{t \rightarrow \infty} \frac{1}{t} (J_t^{\eta, -Kt} - J_t^{\tilde{\eta}, -Kt}) = 0 \quad \mathbb{P}_{\tilde{\pi}^{\lambda, \rho}} \text{ a.s.} \quad \square \tag{48}$$

The following is a corollary to Propositions 3 and 5.

Corollary 6. For all $K > M$

$$\mathbb{P}_{T\tilde{\nu}^{\lambda, \rho}} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{-Kt \leq x \leq Kt} \eta_t(x) = G(-K, K, \rho, \lambda) \right) = 1.$$

Proposition 7. For all $u < v$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \eta_t(x) = \int_u^v f(s) ds \quad \text{in } \mathbb{P}_{T\tilde{\nu}^{\lambda, \rho}} \text{ probability,}$$

where f is defined in (1).

Remark. If $p(x, y)$ vanishes when $|x - y|$ is bigger than a constant, then the above proposition is contained in Rezakhanlou [7]. To include random walks with infinite range, we will derive it from Theorems (2.4) and (2.10) in Andjel and Vares [1], Andjel and Vares [2]. Although their proofs are written for the zero range process,

they also apply to the exclusion process (see Remark (5.3) in that reference). For this process their results can be stated as follows:

$$\lim_{t \rightarrow \infty} (T\bar{v}^{\lambda, \rho})S(t)\tau_{[ut]} = v^{f(u)} \quad \forall u \text{ if } \alpha > 0 \text{ and } \forall u \neq 1 - \lambda - \rho \text{ if } \alpha \leq 0, \quad (49)$$

where f is given in (1) (if $\alpha > 0$) and in (2) (if $\alpha \leq 0$), and τ_x is the translation operator $\tau_x\eta(y) = \eta(y - x)$. Note that Theorem (2.4) of Andjel and Vares [1] is stated for random walks with nonzero drift but their proof applies to random walks with no drift too. The limit (49) is usually referred to as *local equilibrium*. It says that at the *macroscopic* point u at macroscopic time 1 the law of the process is the invariant distribution (equilibrium) with parameter determined by the value of the function f in site u .

Proof. In the sequel we write \mathbb{P} instead of $\mathbb{P}_{T\bar{v}^{\lambda, \rho}}$ to simplify notation and $\sum_{x=a}^b$ instead of $\sum_{a \leq x \leq b}$ (for $a, b \in \mathbb{R}$). To prove the proposition we start showing that for all $u < v$ and $\epsilon > 0$ we have:

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \geq (v - u)(f(u) + \epsilon) \right] = 0. \quad (50)$$

Since $\frac{1}{t} \sum_{x=ut}^{(u+r)t-1} \eta_t(x) \leq r$, we may assume that $u \neq 1 - \lambda - \rho$ if $\alpha < 0$ and apply (49). Let $\delta > 0$. Then, it follows from (49) that there exist n and t_0 such that:

$$\mathbb{P} \left[\frac{1}{n} \sum_{x=ut}^{ut+n-1} \eta_t(x) \geq f(u) + \delta \right] \leq \delta^2 \quad \forall t \geq t_0.$$

Since $(\eta_t(x), x \in \mathbb{Z})$ is stochastically larger than $(\eta_t(x + 1), x \in \mathbb{Z})$, we have

$$\mathbb{P} \left[\frac{1}{n} \sum_{x=ut+kn}^{ut+(k+1)n-1} \eta_t(x) \geq f(u) + \delta \right] \leq \delta^2 \quad \forall t \geq t_0, k \in \mathbb{N}. \quad (51)$$

Let $k = k(t) =: \max\{\ell: [ut + \ell n - 1 \leq [vt]]\}$ then:

$$\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) = \left(\frac{1}{t} \sum_{i=0}^{k-1} \sum_{x=ut+in}^{ut+(i+1)n-1} \eta_t(x) \right) + \frac{g(t, \eta)}{t}, \quad (52)$$

where $g(t, \eta) \leq n$ for all t . The first term of the right-hand side of (52) can be written as

$$\frac{nk}{t} \frac{1}{k} \sum_{i=0}^{k-1} A_i(t, \eta),$$

where $\lim_{t \rightarrow \infty} nk/t = v - u$ and $A_i(t, \eta) = 1/n \sum_{x=ut+in}^{ut+(i+1)n-1} \eta_t(x)$. Since $A_i(t, \eta) \leq 1$, we have:

$$A_i(t, \eta) \leq A_i(t, \eta) \mathbf{1}\{A_i(t, \eta) \leq f(u) + \delta\} + B_i(t, \eta),$$

where $0 \leq B_i(t, \eta) = \mathbf{1}\{A_i(t, \eta) > f(u) + \delta\}$. Therefore,

$$\begin{aligned} \mathbb{P} \left(\frac{1}{k} \sum_{i=0}^{k-1} A_i(t, \eta) \geq f(u) + 2\delta \right) &\leq \mathbb{P} \left(\frac{1}{k} \sum_{i=0}^{k-1} B_i(t, \eta) \geq \delta \right) \\ &\leq \frac{1}{k\delta} \mathbb{E} \left(\sum_{i=0}^{k-1} B_i(t, \eta) \right) \leq \delta, \quad \forall t \geq t_0, \end{aligned} \tag{53}$$

where the last inequality follows from (51). Using (52) we get:

$$\mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \geq \frac{nk}{t} (f(u) + 2\delta) + \frac{g(t, \eta)}{t} \right] \leq \delta \quad \forall t \geq t_0.$$

Since $\lim_{t \rightarrow \infty} nk/t = v - u$ and $\lim_{t \rightarrow \infty} g(t, \eta)/t = 0$, for all t large enough we have:

$$\mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \geq (v - u)(f(u) + 3\delta) \right] \leq \delta,$$

which implies (50).

Similarly one shows that

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \leq (v - u)(f(v) - \epsilon) \right] = 0. \tag{54}$$

We now derive the proposition from (50) and (54). Let k be a positive integer and write:

$$\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) = \sum_{i=0}^{k-1} \frac{1}{t} \sum_{x=ut+(v-u)t \frac{i}{k}}^{ut+(v-u)t \frac{i+1}{k}} \eta_t(x).$$

Then apply (50) to each of the terms of the sum on i to obtain:

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \geq \sum_{i=0}^{k-1} \frac{v - u}{k} \left(f \left(u + \frac{i(v - u)}{k} \right) + \epsilon \right) \right] = 0,$$

for all $\epsilon > 0$. Letting k go to infinity and then ϵ go to 0 we see that:

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) > \int_u^v f(s) ds \right] = 0.$$

Similarly, using (54) we get

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) < \int_u^v f(s) ds \right] = 0,$$

and the proposition follows. \square

Corollary 8. For all $K > M$

$$\mathbb{P}_{T\bar{v}^{\lambda,\rho}} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{-Kt \leq x \leq Kt} \eta_t(x) = \int_{-K}^K f(s) ds \right) = 1.$$

Proposition 9. For all $u < v$ and $\rho \leq \lambda$

$$\mathbb{P}_{T\bar{v}^{\lambda,\rho}} \left(\liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \eta_t(x) \geq \int_u^v f(s) ds \right) = 1.$$

Proof. Let (σ_0, θ_0) be distributed according to $\bar{v}^{\lambda,\rho}$. Then, let $\xi_0 = T_0(\theta_0 - \sigma_0)$ and let (σ_t, ξ_t) be the process defined by (13) with this initial condition. Then,

$$\eta_t := \sigma_t + \xi_t \tag{55}$$

is the exclusion process with initial distribution $T\bar{v}^{\lambda,\rho}$, and σ_t is the exclusion process with initial distribution v^ρ . Then, as in the proof of (19),

$$\mathbb{P}_{\bar{v}^{\lambda,\rho}} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \sigma_t(x) = \rho(v - u) \right) = 1.$$

This and (55) imply that

$$\mathbb{P}_{T\bar{v}^{\lambda,\rho}} \left(\liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{ut \leq x \leq vt} \eta_t(x) \geq \rho(v - u) \right) = 1.$$

Since $f(s) = \rho$ if $s \geq \alpha(1 - 2\rho)$, this proves the proposition for $\alpha(1 - 2\rho) \leq u < v$.

For the case $u < v \leq \alpha(1 - 2\rho)$, note that $T\bar{v}^{\lambda,\rho} \geq T\bar{v}^{\lambda,0} = T\bar{\mu}^{\lambda,0}$. Thus, the result for these values of u and v follows from Proposition 7 and the fact that the function f in (1) does not change its values in the interval $(-\infty, \alpha(1 - 2\rho)]$ if we substitute 0 for ρ .

Finally for $u < \alpha(1 - 2\rho) < v$ the result follows from the inequality

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{x=ut}^{\alpha(1-2\rho)t} \eta_t(x) + \liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{x=\alpha(1-2\rho)t}^{vt} \eta_t(x)$$

and the two previous cases. \square

Proof of Theorem 1. Fix $u < v$ and let $K = 1 + \max\{M, |u|, |v|\}$. Then, by Corollary 8 we have $\mathbb{P}_{T\bar{v}^{\lambda,\rho}}$ -a.s.

$$\begin{aligned} \int_{-K}^K f(s) ds &= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{x=-Kt}^{Kt} \eta_t(x) \\ &\geq \liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{x=-Kt}^{ut-1} \eta_t(x) + \limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) + \liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{x=vt+1}^{Kt} \eta_t(x). \end{aligned}$$

Therefore, by Proposition 9

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{x=ut}^{vt} \eta_t(x) \leq \int_{-K}^K f(s) ds - \int_{-K}^u f(s) ds - \int_v^K f(s) ds = \int_u^v f(s) ds,$$

$P_{T_{\vec{v}, \rho}}$ -a.s. The theorem follows from Proposition 9, this last inequality and (11). \square

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