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Whittaker vectors of the Virasoro algebra in terms of Jack symmetric polynomial

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ABSTRACT

We give an explicit formula of Whittaker vector for Virasoro algebra in terms of the Jack symmetric functions. Our fundamental tools are the Feigin–Fuchs bosonization and the split expression of the Calogero–Sutherland model given by Awata, Matsuo, Odake and Shiraishi.

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1. Introduction

In [1] a remarkable proposal, now called the AGT conjecture, was given on the relation between the Liouville theory conformal blocks and the Nekrasov partition function. Among the related investigations, Gaiotto proposed several degenerated versions of the AGT conjecture in [9]. In that paper, he conjectured that the inner product of a certain element in the Verma module of Virasoro algebra coincides with the Nekrasov partition function for the four-dimensional $\mathcal{N} = 2$ pure gauge theory [15]. Actually, the element considered is a kind of Whittaker vector in the Verma module of the Virasoro algebra.

Whittaker vectors and Whittaker modules are important gadgets in the representation theory since its emergence in the study of finite dimensional Lie algebras [11]. Although numerous analogues and generalizations have been proposed for other algebras, such as affine algebras and quantum groups, not so many investigations have been given for the Whittaker vectors of the Virasoro algebra. A general theory on the properties of Whittaker modules for the Virasoro algebra was recently given in [18].

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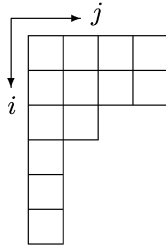


Fig. 1. The Young diagram for (4, 4, 2, 1, 1, 1).

In this paper we give an explicit expression of the Whittaker vector for the Verma module of Virasoro algebra in terms of Jack symmetric functions [13, VI §10]. We use the Feigin–Fuchs bosonization [7] to identify the Verma module and the ring of symmetric function, and then utilize the split expression of the Calogero–Sutherland Hamiltonian [20] to derive a recursion relation on the coefficients of the Whittaker vector in its expansion with respect to Jack symmetric functions.

Our result is related to a conjecture given by Awata and Yamada in [3]. They proposed the five-dimensional AGT conjecture for pure SU(2) gauge theory using the deformed Virasoro algebra, and as a related topic, they also proposed a conjectural formula on the explicit form of the deformed Gaiotto state in terms of Macdonald symmetric functions [3, (3.18)]. Our formula is the non-deformed Virasoro, or four-dimensional, counterpart of their conjectural formula.

The motivation of our study also comes from the work [14], where singular ℓ -vectors of the Virasoro algebra are expressed by Jack polynomials.

Before presenting the detail of the main statement, we need to prepare several notations on Virasoro algebra, symmetric functions and some combinatorics. The main theorem will be given in Section 1.6.

1.1. Partitions

Throughout in this paper, notations of partitions follow [13]. For the positive integer n , a partition λ of n is a (finite) sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \geq \lambda_2 \geq \dots$ and $\lambda_1 + \lambda_2 + \dots = n$. The symbol $\lambda \vdash n$ means that λ is a partition of n . For a general partition we also define $|\lambda| := \sum_i \lambda_i$. The number $\ell(\lambda)$ is defined to be the length of the sequence λ . The conjugate partition of λ is denoted by λ' .

We also consider the empty sequence \emptyset as the unique partition of the number 0.

In addition we denote by \mathcal{P} the set of all the partitions of natural numbers including the empty partition \emptyset . So that we have

$$\mathcal{P} = \{\emptyset, (1), (2), (1, 1), (3), (2, 1), (1, 1, 1), \dots\}.$$

As usual, $p(n) := \#\{\lambda \in \mathcal{P} \mid |\lambda| = n\} = \#\{\lambda \mid \lambda \vdash n\}$ denotes the number of partitions of n .

In the main text, we sometimes use the dominance semi-ordering on the partitions: $\lambda \geq \mu$ if and only if $|\lambda| = |\mu|$ and $\sum_{k=1}^i \lambda_k \geq \sum_{k=1}^i \mu_k$ ($i = 1, 2, \dots$).

We also follow [13] for the convention of the Young diagram. Moreover we will use the coordinate (i, j) on the Young diagram defined as follows: the first coordinate i (the row index) increases as one goes downwards, and the second coordinate j (the column index) increases as one goes rightwards. For example, in Fig. 1 the left-top box has the coordinate $(1, 1)$ and the left-bottom box has the coordinate $(6, 1)$. We will often identify a partition and its associated Young diagram.

Let us also use the notation $(i, j) \in \lambda$, which means that $i, j \in \mathbb{Z}_{\geq 1}$, $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. On the Young diagram of λ the symbol $(i, j) \in \lambda$ corresponds to the box located at the coordinate (i, j) . In Fig. 1, we have $(2, 3) \in \lambda := (4, 4, 2, 1, 1, 1)$ but $(4, 3) \notin \lambda$.

1.2. Virasoro algebra

Let us fix notations on Virasoro algebra and its Verma module. Let $c \in \mathbb{C}$ be a fixed complex number. The Virasoro algebra Vir_c is a Lie algebra over \mathbb{C} with central extension, generated by L_n ($n \in \mathbb{Z}$) with the relation

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \tag{1.1}$$

Vir_c has the triangular decomposition $\text{Vir}_c = \text{Vir}_{c,+} \oplus \text{Vir}_{c,0} \oplus \text{Vir}_{c,-}$ with $\text{Vir}_{c,\pm} := \bigoplus_{\pm n > 0} \mathbb{C}L_n$ and $\text{Vir}_{c,0} := \mathbb{C}L_0 \oplus \mathbb{C}$.

Let h be a complex number. Let \mathbb{C}_h be the one-dimensional representation of the subalgebra $\text{Vir}_{c,\geq 0} := \text{Vir}_{c,0} \oplus \text{Vir}_{c,+}$, where $\text{Vir}_{c,+}$ acts trivially and L_0 acts as the multiplication by h . Then one has the Verma module M_h by

$$M_h := \text{Ind}_{\text{Vir}_{c,\geq 0}}^{\text{Vir}_c} \mathbb{C}_h.$$

Obeying the notation in physics literature, we denote by $|h\rangle$ a fixed basis of \mathbb{C}_h . Then one has $\mathbb{C}_h = \mathbb{C}|h\rangle$ and $M_h = U(\text{Vir}_c)|h\rangle$.

M_h has the L_0 -weight space decomposition:

$$M_h = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M_{h,n}, \quad \text{with } M_{h,n} := \{v \in M_h \mid L_0 v = (h + n)v\}. \tag{1.2}$$

A basis of $M_{h,n}$ can be described simply by partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n we define the abbreviation

$$L_{-\lambda} := L_{-\lambda_k} L_{-\lambda_{k-1}} \cdots L_{-\lambda_1} \tag{1.3}$$

of the element of $U(\text{Vir}_{c,-})$, the enveloping algebra of the subalgebra $\text{Vir}_{c,-}$. Then the set

$$\{L_{-\lambda}|h\rangle \mid \lambda \vdash n\}$$

is a basis of $M_{h,n}$.

1.3. Bosonization

Next we recall the bosonization of the Virasoro algebra [7]. Consider the Heisenberg algebra \mathcal{H} generated by a_n ($n \in \mathbb{Z}$) with the relation

$$[a_m, a_n] = m\delta_{m+n,0}.$$

Consider the correspondence

$$L_n \mapsto \mathcal{L}_n := \frac{1}{2} \sum_{m \in \mathbb{Z}} \overset{\circ}{a}_m \overset{\circ}{a}_{n-m} - (n + 1)\rho a_n, \tag{1.4}$$

where the symbol $\overset{\circ}{\circ}$ means the normal ordering. This correspondence determines a well-defined morphism

$$\varphi : U(\text{Vir}_c) \rightarrow \widehat{U}(\mathcal{H}). \tag{1.5}$$

Here $\widehat{U}(\mathcal{H})$ is the completion of the universal enveloping algebra $U(\mathcal{H})$ in the following sense [8]. For $n \in \mathbb{Z}_{\geq 0}$, let I_n be the left ideal of the enveloping algebra $U(\mathcal{H})$ generated by all polynomials in a_m ($m \in \mathbb{Z}_{\geq 1}$) of degrees greater than or equal to n (where we defined the degree by $\deg a_m := m$). Then we define

$$\widehat{U}(\mathcal{H}) := \varprojlim_n \widehat{U}(\mathcal{H})/I_n.$$

Next we recall the functorial correspondence of the representations. First let us define the Fock representation \mathcal{F}_α of \mathcal{H} . \mathcal{H} has the triangular decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$ with $\mathcal{H}_\pm := \bigoplus_{\pm n \in \mathbb{Z}_+} \mathbb{C}a_n$ and $\mathcal{H}_0 := \mathbb{C}a_0$. Let $\mathbb{C}_\alpha = \mathbb{C}|\alpha\rangle_{\mathcal{F}}$ be the one-dimensional representation of $\mathcal{H}_0 \oplus \mathcal{H}_+$ with the action $a_0|\alpha\rangle_{\mathcal{F}} = \alpha|\alpha\rangle_{\mathcal{F}}$ and $a_n|\alpha\rangle_{\mathcal{F}} = 0$ ($n \in \mathbb{Z}_{\geq 1}$). Then the Fock space \mathcal{F}_α is defined to be

$$\mathcal{F}_\alpha := \text{Ind}_{\mathcal{H}_0 \oplus \mathcal{H}_-}^{\mathcal{H}} \mathbb{C}_\alpha.$$

It has the a_0 -weight decomposition

$$\mathcal{F}_\alpha = \bigoplus_{n \geq 0} \mathcal{F}_{\alpha, n}, \quad \mathcal{F}_{\alpha, n} := \{w \in \mathcal{F}_\alpha \mid a_0 w = (n + \alpha)w\}. \tag{1.6}$$

Each weight space $\mathcal{F}_{\alpha, n}$ has a basis

$$\{a_{-\lambda}|\alpha\rangle_{\mathcal{F}} \mid \lambda \vdash n\} \tag{1.7}$$

with $a_{-\lambda} := a_{-\lambda_k} \cdots a_{-\lambda_1}$ for a partition $\lambda = (\lambda_1, \dots, \lambda_k)$. Note also that the action of $\widehat{U}(\mathcal{H})$ on \mathcal{F}_α is well defined.

Similarly the dual Fock space \mathcal{F}_α^* is defined to be $\text{Ind}_{\mathcal{H}_0 \oplus \mathcal{H}_-}^{\mathcal{H}} \mathbb{C}_\alpha^*$, where $\mathbb{C}_\alpha^* = \mathbb{C} \cdot \langle \alpha |_{\mathcal{F}}$ is the one-dimensional right representation of $\mathcal{H}_0 \oplus \mathcal{H}_-$ with the action $\langle \alpha | a_0 = \alpha \cdot \langle \alpha |_{\mathcal{F}}$ and $\langle \alpha | a_{-n} = 0$ ($n \in \mathbb{Z}_{\geq 1}$).

Then one has the bilinear form

$$\cdot : \mathcal{F}_\alpha^* \times \mathcal{F}_\alpha \rightarrow \mathbb{C}$$

defined by

$$\begin{aligned} \mathcal{F}(\alpha | \cdot | \alpha)_{\mathcal{F}} &= 1, & 0 \cdot | \alpha)_{\mathcal{F}} &= \mathcal{F}(\alpha | \cdot 0)_{\mathcal{F}} = 0, \\ \mathcal{F}(\alpha | uu' \cdot | \alpha)_{\mathcal{F}} &= \mathcal{F}(\alpha | u \cdot u' | \alpha)_{\mathcal{F}} = \mathcal{F}(\alpha | \cdot uu' | \alpha)_{\mathcal{F}} \quad (u, u' \in \mathcal{H}). \end{aligned}$$

As in the physics literature, we often omit the symbol \cdot and simply write $\mathcal{F}(\alpha | \alpha)_{\mathcal{F}}$, $\mathcal{F}(\alpha | u | \alpha)_{\mathcal{F}}$ and so on.

Now we can state the bosonization of representation: (1.4) is compatible with the map

$$\psi : M_h \rightarrow \mathcal{F}_\alpha, \quad L_{-\lambda} | h) \mapsto \mathcal{L}_{-\lambda} | \alpha)_{\mathcal{F}} \tag{1.8}$$

with $\mathcal{L}_{-\lambda} := \mathcal{L}_{-\lambda_1} \cdots \mathcal{L}_{-\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}$ and

$$c = 1 - 12\rho^2, \quad h = \frac{1}{2}\alpha(\alpha - 2\rho). \tag{1.9}$$

In other words, we have

$$\psi(xv) = \varphi(x)\psi(v) \quad (x \in \text{Vir}_c, v \in M_h)$$

under the parametrization (1.9) of highest weights.

1.4. Fock space and symmetric functions

The Fock space \mathcal{F}_α is naturally identified with the space of symmetric functions. In this paper the term “symmetric function” means the infinite-variable symmetric “polynomial”. To treat such an object rigorously, we follow the argument of [13, §1.2].

Let us denote by Λ_N the ring of N -variable symmetric polynomials over \mathbb{Z} , and by Λ_N^d the space of homogeneous symmetric polynomials of degree d . The ring of symmetric functions Λ is defined as the inverse limit of the Λ_N in the category of graded rings (with respect to the gradation defined by the degree d). We denote by $\Lambda_K = \Lambda \otimes_{\mathbb{Z}} K$ the coefficient extension to a ring K . Among several bases of Λ , the family of the power sum symmetric functions

$$p_n = p_n(x) := \sum_{i \in \mathbb{Z}_{\geq 1}} x_i^n, \quad p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k},$$

plays an important role. It is known that $\{p_\lambda \mid \lambda \vdash d\}$ is a basis of $\Lambda_{\mathbb{Q}}^d$, the subspace of homogeneous symmetric functions of degree d .

Now following [2], we define the isomorphism between the Fock space and the space of symmetric functions. Let β be a non-zero complex number and consider the next map between \mathcal{F}_α and $\Lambda_{\mathbb{C}}$:

$$\begin{aligned} \iota_\beta : \mathcal{F}_\alpha &\xrightarrow{\quad \quad \quad} \Lambda_{\mathbb{C}} \\ \psi &\quad \quad \quad \psi \\ v &\mapsto \mathcal{F}(\alpha \mid \exp\left(\sqrt{\frac{\beta}{2}} \sum_{n=1}^{\infty} \frac{1}{n} p_n a_n\right) v). \end{aligned} \tag{1.10}$$

Under this morphism, an element $a_{-\lambda} \mid \alpha \rangle_{\mathcal{F}}$ of the base (1.7) is mapped to

$$\iota_\beta(a_{-\lambda} \mid \alpha \rangle_{\mathcal{F}}) = (\sqrt{\beta/2})^{\ell(\lambda)} p_\lambda(x).$$

Since $\{a_{-\lambda} \mid \alpha \rangle_{\mathcal{F}}\}$ is a basis of \mathcal{F}_α and $\{p_\lambda\}$ is a basis of $\Lambda_{\mathbb{Q}}$, ι_β is an isomorphism.

1.5. Jack symmetric function

Now we recall the definition of Jack symmetric function [13, §VI.10]. Let b be an indeterminate¹ and define an inner product on $\Lambda_{\mathbb{Q}(b)}$ by

$$\langle p_\lambda, p_\mu \rangle_b := \delta_{\lambda, \mu} z_\lambda b^{\ell(\lambda)}. \tag{1.11}$$

Here the function z_λ is given by

$$z_\lambda := \prod_{i \in \mathbb{Z}_{\geq 1}} i^{m_i(\lambda)} m_i(\lambda)! \quad \text{with } m_i(\lambda) := \#\{1 \leq i \leq \ell(\lambda) \mid \lambda_j = i\}.$$

Then the (monic) Jack symmetric function $P_\lambda^{(b)}$ is determined uniquely by the two following conditions:

¹ Our parameter b is usually denoted by α in the literature, for example, in [13]. We avoid using α since it is already defined to be the highest weight of the Heisenberg–Fock space \mathcal{F}_α .

(i) It has an expansion via monomial symmetric function m_ν in the form

$$P_\lambda^{(b)} = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu}(b)m_\mu.$$

Here $c_{\lambda,\mu}(b) \in \mathbb{Q}(b)$ and the ordering $<$ among the partitions is the dominance semi-ordering.

(ii) The family of Jack symmetric functions is an orthogonal basis of $\Lambda_{\mathbb{Q}(b)}$ with respect to $\langle \cdot, \cdot \rangle_b$:

$$\langle P_\lambda^{(b)}, P_\mu^{(b)} \rangle_b = 0 \quad \text{if } \lambda \neq \mu.$$

1.6. Main theorem

Finally we can state our main statement.

Consider the Verma module M_h of the Virasoro algebra Vir_c with generic complex numbers c and h . Let a be an arbitrary complex number, and let v_G be an element of the Verma module M_h such that

$$L_1 v_G = a v_G, \quad L_n v_G = 0 \quad (n \geq 2).$$

Then v_G exists uniquely up to scalar multiplication (see Fact 3.2).

Introduce the complex numbers ρ, α and β by the relations

$$c = 1 - 12\rho^2, \quad h = \frac{1}{12}\alpha(\alpha - 2\rho), \quad \rho = \frac{\beta^{1/2} - \beta^{-1/2}}{\sqrt{2}}.$$

Then by the Feigin–Fuchs bosonization $\psi : M_h \rightarrow \mathcal{F}_\alpha$ (1.8) and the isomorphism $\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_{\mathbb{C}}$ (1.10), one has an element $\iota_\beta \circ \psi(v_G) \in \Lambda_{\mathbb{C}}$.

Theorem. *We have*

$$\iota_\beta \circ \psi(v_G) = \sum_{\lambda \in \mathcal{P}} a^{|\lambda|} c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})}, \tag{1.12}$$

where λ runs over all the partitions and the coefficient $c_\lambda(\alpha, \beta)$ is given by

$$c_\lambda(\alpha, \beta) = \prod_{(i,j) \in \lambda} \frac{1}{\lambda_i - j + 1 + \beta(\lambda'_j - i)} \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} \frac{\beta}{(j+1) + \sqrt{2}\beta^{1/2}\alpha - (i+1)\beta}. \tag{1.13}$$

(See Section 1.1 for the symbol “ $(i, j) \in \lambda$.”)

The proof of this theorem will be given in Section 3.1.

In the main theorem above, the element v_G is the Whittaker vector associated to the degenerate Lie algebra homomorphism $\eta : \text{Vir}_{c,+} \rightarrow \mathbb{C}$, that is, $\eta(L_2) = 0$. We shall call this element “Gaiotto state”, following [3]. A general theory of Whittaker vectors usually assumes the non-degeneracy of the homomorphism η , i.e., $\eta(L_1) \neq 0$ and $\eta(L_2) \neq 0$. This non-degenerate case will be treated in Proposition 3.11, although there seem to be no factored expressions for the coefficients as (1.13).

The content of this paper is as follows. In Section 2 we recall the split expression of the Calogero–Sutherland Hamiltonian, which is a key point in our proof. In Section 3 we investigate the Whittaker vectors in terms of symmetric functions. The main theorem will be proved Section 3.1, using some combinatorial identities shown in Section 4. The Whittaker vector with respect to the non-degenerate homomorphism will be treated in Section 3.3. In the final Section 5 we give some remarks on possible

generalizations and the related works. We also added Appendix A concerning the AGT relation and its connection to our argument.

2. Preliminaries on Jack symmetric functions and bosonized Calogero–Sutherland Hamiltonian

This section is a preliminary for the proof of the main theorem. We need the following Definition 2.1 and Proposition 2.2:

Definition 2.1. (1) Let λ be a partition and b, β be generic complex numbers. Define $f_\lambda^{(b,\beta)} \in \mathcal{F}_\alpha$ to be the element such that

$$\iota_\beta(f_\lambda^{(b,\beta)}) = P_\lambda^{(b)}, \tag{2.1}$$

where ι_β is the isomorphism given in (1.10).

(2) For a complex number β , define an element of $\widehat{U}(\mathcal{H})$ by

$$\widehat{E}_\beta = \sqrt{2\beta} \sum_{n>0} a_{-n} \mathcal{L}_n + \sum_{n>0} a_{-n} a_n (\beta - 1 - \sqrt{2\beta} a_0). \tag{2.2}$$

Here $\mathcal{L}_n \in \widehat{U}(\mathcal{H})$ is the bosonized Virasoro generator (1.4), and we have put the assumption

$$\rho = (\beta^{1/2} - \beta^{-1/2})/\sqrt{2}. \tag{2.3}$$

Proposition 2.2. For a generic complex number β we have

$$\widehat{E}_\beta f_\lambda^{(\beta^{-1},\beta)} = \epsilon_\lambda(\beta) f_\lambda^{(\beta^{-1},\beta)}, \tag{2.4}$$

$$\epsilon_\lambda(\beta) := \sum_i (\lambda_i^2 + \beta(1 - 2i)\lambda_i), \tag{2.5}$$

for any partition λ .

The proof of this proposition is rather complicated, since we should utilize Jack symmetric polynomials with finite variables.

2.1. Jack symmetric polynomials

Recall that in Section 1.4 we denoted by Λ_N the space of symmetric polynomials of N variables, and by Λ_N^d its degree d homogeneous subspace. In order to denote N -variable symmetric polynomials, we put the superscript “ (N) ” on the symbols for the infinite-variable symmetric functions. For example, we denote by $p_\lambda^{(N)}(x) := p_{\lambda_1}^{(N)}(x)p_{\lambda_2}^{(N)}(x)\cdots$ the product of the power sum polynomials $p_k^{(N)}(x) := \sum_{i=1}^N x_i^k$, and by $m_\lambda^{(N)}(x)$ the monomial symmetric polynomial.

Let us fix $N \in \mathbb{Z}_{\geq 1}$ and an indeterminate² t . For a partition λ with $\ell(\lambda) \leq N$, the N -variable Jack symmetric polynomial $P_\lambda^{(N)}(x; t)$ is uniquely specified by the two following properties:

(i)
$$P_\lambda^{(N)}(x; t) = m_\lambda^{(N)}(x) + \sum_{\mu < \lambda} \tilde{c}_{\lambda,\mu}(t) m_\mu^{(N)}(x), \quad \tilde{c}_{\lambda,\mu}(t) \in \mathbb{Q}(t).$$

² In the literature this indeterminate is usually denoted by $\beta = \alpha^{-1}$, and we will also identify it with our β given in (2.3) later. But at this moment we don't use it to avoid confusion.

(ii)

$$H_t^{(N)} P_\lambda^{(N)}(x; t) = \epsilon_\lambda^{(N)}(t) P_\lambda^{(N)}(x; t), \tag{2.6}$$

$$H_t^{(N)} := \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i} \right)^2 + t \sum_{1 \leq i < j \leq N} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right), \tag{2.7}$$

$$\epsilon_\lambda^{(N)}(t) := \sum_i (\lambda_i^2 + t(N + 1 - 2i)\lambda_i). \tag{2.8}$$

The differential operator (2.7) is known to be equivalent to the Calogero–Sutherland Hamiltonian (see [2, §2] for the detailed explanation). In (i) we have used the dominance partial semi-ordering on the partitions. If $N \geq d$, then $\{P_\lambda^{(N)}(x; t)\}_{\lambda \vdash d}$ is a basis of $\Lambda_{N, \mathbb{Q}(t)}^d$.

Definition 2.3. For $M \geq N$, we denote the restriction map from Λ_M to Λ_N by

$$\begin{aligned} \rho_{M,N} : \quad \Lambda_M &\rightarrow \Lambda_N \\ \downarrow &\quad \downarrow \\ f(x_1, \dots, x_M) &\mapsto f(x_1, \dots, x_N, \mathbf{0}, \dots, \mathbf{0}), \end{aligned}$$

and the induced restriction map from Λ to Λ_N by

$$\begin{aligned} \rho_N : \quad \Lambda &\rightarrow \Lambda_N \\ \downarrow &\quad \downarrow \\ f(x_1, x_2, \dots) &\mapsto f(x_1, \dots, x_N, \mathbf{0}, \dots). \end{aligned}$$

We denote the maps on the tensored spaces $\Lambda_{M, \mathbb{C}} \rightarrow \Lambda_{N, \mathbb{C}}$ and $\Lambda_{\mathbb{C}} \rightarrow \Lambda_{N, \mathbb{C}}$ by the same symbols $\rho_{M,N}$ and ρ_N .

Fact 2.4. For any $\lambda \in \mathcal{P}$, every $N \in \mathbb{Z}_{\geq 1}$ with $N \geq \ell(\lambda)$, and any generic $t \in \mathbb{C}$ we have

$$\rho_N(P_\lambda^{(t^{-1})}) = P_\lambda^{(N)}(x; t).$$

2.2. Split form of the Calogero–Sutherland Hamiltonian

We recall the collective field method in the Calogero–Sutherland model following [2, §3]. Recall that the Calogero–Sutherland Hamiltonian $H_t^{(N)}$ (2.7) acts on the space of symmetric polynomials $\Lambda_{N, \mathbb{Q}(t)}$.

Fact 2.5. (1) Let t, t' be non-zero complex numbers. Define an element of $\widehat{U}(\mathcal{H})$ by

$$\widehat{H}_{t,t'}^{(N)} := \sum_{n,m>0} \left(t' a_{-m-n} a_m a_n + \frac{t}{t'} a_{-m} a_{-n} a_{m+n} \right) + \sum_{n>0} (n(1-t) + Nt) a_{-n} a_n.$$

Then for any $v \in \mathcal{F}_\alpha$ and every $N \in \mathbb{Z}_{\geq 1}$ we have

$$\rho_N \circ \iota_t(\widehat{H}_{t,t'}^{(N)} v) = H_t^{(N)}(\rho_N \circ \iota_t(v)). \tag{2.9}$$

(2) Under the relation

$$\rho = (t^{1/2} - t^{-1/2})/\sqrt{2}$$

we have

$$\widehat{H}_{t,\sqrt{t}/2}^{(N)} = \sqrt{2t} \sum_{n>0} a_{-n} \mathcal{L}_n + \sum_{n>0} a_{-n} a_n (Nt + t - 1 - \sqrt{2t} a_0). \tag{2.10}$$

Here $\mathcal{L}_n \in \widehat{U}(\mathcal{H})$ is the bosonized Virasoro generator (1.4).

Proof. These are well-known results (for example, see [19, Prop. 4.47], [2] and the references therein). We only show the sketch of the proof.

As for (1), note that $\{a_{-\lambda}|\alpha\}_{\mathcal{F}} \mid \lambda \in \mathcal{P}\}$ is a basis of \mathcal{F} . So it is enough to show (2.9) for each λ . One can calculate the left hand side using the commutation relation of \mathcal{H} only. On the right hand side, one may use $\iota_t(a_{-\lambda}|\alpha)_{\mathcal{F}} \propto p_\lambda$, and calculate it using the expression (2.7).

(2) is proved by a direct calculation. \square

Remark 2.6. The form (2.10) is called the split expression in [20, §1].

2.3. Proof of Proposition 2.2

By Fact 2.5 we have the left commuting diagram in (2.11). Note that we set the parameters t and t' in $\widehat{H}_{t,t'}^{(N)}$ to be β and $\sqrt{\beta/2}$, so that we may use Fact 2.5(2). In the right diagram of (2.11) we show how the element $f_\lambda^{(b,\beta)} \in \mathcal{F}_\alpha$ given in (2.1) behaves under the maps indicated in the left diagram. Here we set the parameter b to be β^{-1} so that $\iota_\beta f_\lambda^{(\beta^{-1},\beta)} = P_\lambda^{(\beta^{-1})} \in \Lambda_{\mathbb{C}}$ and $\rho_N \circ \iota_\beta f_\lambda^{(\beta^{-1},\beta)} = P_\lambda^{(N)}(x; \beta) \in \Lambda_{N,\mathbb{C}}$. At the bottom line we have used the eigen-equation of Jack symmetric polynomial (2.6).

$$\begin{array}{ccc}
 \mathcal{F}_\alpha \xrightarrow{\widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)}} \mathcal{F}_\alpha & & f_\lambda^{(\beta^{-1},\beta)} \longmapsto \widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)}(f_\lambda^{(\beta^{-1},\beta)}) \\
 \downarrow \iota_\beta \quad \wr & & \downarrow \\
 \Lambda_{\mathbb{C}} \quad \circ & & P_\lambda^{(\beta^{-1})} \quad \quad \quad \iota_\beta \circ \widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)}(f_\lambda^{(\beta^{-1},\beta)}) \\
 \downarrow \rho_N & & \downarrow \\
 \Lambda_{N,\mathbb{C}} \xrightarrow{H_\beta^{(N)}} \Lambda_{N,\mathbb{C}} & & P_\lambda^{(N)}(x; \beta) \longmapsto P_\lambda^{(N)}(x; \beta) \cdot \epsilon_\lambda^{(N)}(\beta)
 \end{array} \tag{2.11}$$

Since these diagrams hold for every N with $N \geq \ell(\lambda)$, we have

$$\widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)} f_\lambda^{(\beta^{-1},\beta)} = \epsilon_\lambda^{(N)}(\beta) f_\lambda^{(\beta^{-1},\beta)}.$$

Therefore we have

$$\begin{aligned}
 & \left[\sqrt{2t} \sum_{n>0} a_{-n} \mathcal{L}_n + \sum_{n>0} a_{-n} a_n (Nt + t - 1 - \sqrt{2t} a_0) \right] f_\lambda^{(\beta^{-1},\beta)} \\
 &= f_\lambda^{(\beta^{-1},\beta)} \cdot \sum_i (\lambda_i^2 + t(N + 1 - 2i)\lambda_i).
 \end{aligned}$$

We can subtract N -dependent terms from both sides. The result is nothing but the desired statement of Proposition 2.2.

3. Whittaker vectors

Recall the notion of the Whittaker vector for a finite dimensional Lie algebra \mathfrak{g} given in [11]. Let \mathfrak{n} be a maximal nilpotent Lie subalgebra of \mathfrak{g} and $\eta : \mathfrak{n} \rightarrow \mathbb{C}$ be a homomorphism. Let V be any $U(\mathfrak{g})$ -module. Then a vector $w \in V$ is called a Whittaker vector with respect to η if $xw = \eta(x)w$ for all $x \in \mathfrak{n}$.

We shall discuss an analogue of this Whittaker vector in the Virasoro algebra Vir_c . In the triangular decomposition $\text{Vir}_c = \text{Vir}_{c,+} \oplus \text{Vir}_{c,0} \oplus \text{Vir}_{c,-}$, the elements $L_1, L_2 \in \text{Vir}_{c,+}$ generate $\text{Vir}_{c,+}$. Thus if we take $\text{Vir}_{c,+}$ as the η in the above definition, what we should consider is a homomorphism $\eta : \text{Vir}_{c,+} \rightarrow \mathbb{C}$, which is determined by $\eta(L_1)$ and $\eta(L_2)$.

In [18], a characterization of Whittaker vectors in general $U(\text{Vir})$ -modules is given under the assumption that η is non-degenerate, i.e. $\eta(L_1) \neq 0$ and $\eta(L_2) \neq 0$.

In this section we shall express Whittaker vectors in the Verma module M_h using Jack symmetric functions. Before starting the general treatment, we first investigate a degenerate version of the Whittaker vector, i.e. we assume $\eta(L_2) = 0$. We will call this vector Gaiotto state of Virasoro algebra, although the paper [9] treated both degenerate and non-degenerate Whittaker vectors.

3.1. Gaiotto state via Jack polynomials

Definition 3.1. Fix a non-zero complex number a . Let v_G be a non-zero element of the Verma module M_h satisfying

$$L_1 v_G = a v_G, \quad L_n v_G = 0 \quad (n \in \mathbb{Z}_{\geq 2}).$$

We call such an element v_G a Gaiotto state of M_h .

Fact 3.2. Assume that c and h are generic. Then v_G exists uniquely up to constant multiplication.

Proof. This statement is shown in [18]. \square

Lemma 3.3. Decompose a Gaiotto state v_G in the way (1.2) as

$$v_G = \sum_{n \in \mathbb{Z}_{\geq 0}} a^n v_{G,n}, \quad v_{G,n} \in M_{h,n}.$$

Then we have

$$v_{G,n} = L_1 v_{G,n+1} \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3.1}$$

Proof. This follows from the commutation relation $[L_0, L_1] = -L_1$. \square

Now consider the bosonized Gaiotto state

$$w_{G,n} := \psi(v_{G,n}) \in \mathcal{F}_{\alpha,n}$$

where $\psi : M_h \rightarrow \mathcal{F}_\alpha$ is the Feigin–Fuchs bosonization (1.8) and $\mathcal{F}_{\alpha,n}$ is the a_0 -weight space (1.6). At this moment the Heisenberg parameters ρ, α are related to the Virasoro parameters c, h by the relations

$$c = 1 - 12\rho^2, \quad h = \frac{1}{12}\alpha(\alpha - 2\rho).$$

From the condition (3.1) we have

$$\mathcal{L}_1 w_{G,n+1} \in \mathcal{F}_{h,n}, \quad w_{G,n} = \mathcal{L}_1 w_{G,n+1}. \tag{3.2}$$

Next we map this bosonized state $w_{G,n}$ into a symmetric function by the isomorphism $\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_{\mathbb{C}}$ (1.10):

$$\iota_\beta(w_{G,n}) = \iota_\beta \circ \psi(v_{G,n}) \in \Lambda_{\mathbb{C}}^n.$$

Here $\Lambda_{\mathbb{C}}^n$ is the space of degree n symmetric functions. We take the parameter β so that the Heisenberg parameter ρ is expressed by

$$\rho = (\beta^{1/2} - \beta^{-1/2})/\sqrt{2}.$$

Recall also that the family of Jack symmetric functions $\{P_\lambda^{(\beta^{-1})} \mid \lambda \vdash n\}$ is a basis of $\Lambda_{\mathbb{C}}^n$ for a generic $\beta \in \mathbb{C}$. Thus we can expand $\iota_\beta(w_{G,n}) \in \Lambda_{\mathbb{C}}^n$ by $P_\lambda^{(\beta^{-1})}$'s. Let us express this expansion as:

$$\iota_\beta(w_{G,n}) = \iota_\beta \circ \psi(v_{G,n}) = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})}, \quad c_\lambda(\alpha, \beta) \in \mathbb{C}. \tag{3.3}$$

Note that this expansion is equivalent to

$$w_{G,n} = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) f_\lambda^{(\beta^{-1}, \beta)} \in \mathcal{F}_\alpha \tag{3.4}$$

by (2.1). Now the correspondence of the parameters becomes:

$$c = 1 - 12\rho^2, \quad h = \frac{1}{12}\alpha(\alpha - 2\rho_0), \quad \rho = \frac{\beta^{1/2} - \beta^{-1/2}}{\sqrt{2}}. \tag{3.5}$$

The main result of this paper is

Theorem 3.4. Assume that c and h are generic. (Then v_G exists uniquely up to constant multiplication by Fact 3.2.) If $c_\emptyset(\alpha, \beta)$ is set to be 1 in the expansion (3.3), then the other coefficients are given by

$$c_\lambda(\alpha, \beta) = \prod_{(i,j) \in \lambda} \frac{1}{\lambda_i - j + 1 + \beta(\lambda'_j - i)} \prod_{(i,j) \in \lambda \setminus \{(1,1)\}} \frac{1}{(j+1)\beta + \sqrt{2}\beta^{1/2}\alpha - (i+1)}. \tag{3.6}$$

Here we have used the notation $(i, j) \in \lambda$ as explained in Section 1.1.

3.2. Proof of Theorem 3.4

Before starting the proof, we need to prepare the following Proposition 3.7. Recall the Pieri formula of Jack symmetric function. We only need the case of “adding one box”, that is, the case of multiplying the degree one power sum function $p_1 = x_1 + x_2 + \dots$.

Definition 3.5. For partitions μ and λ , we denote $\mu \prec_k \lambda$ if $|\mu| = |\lambda| - k$ and $\mu \subset \lambda$.

Fact 3.6. (See [13, p. 340, VI (6.24), p. 379, VI (10.10)].) We have

$$p_1 P_\mu^{(\beta^{-1})} = \sum_{\lambda >_1 \mu} \psi'_{\lambda/\mu}(\beta) P_\lambda^{(\beta^{-1})}, \tag{3.7}$$

$$\psi'_{\lambda/\mu}(\beta) := \prod_{i=1}^{l-1} \frac{\lambda_i - \lambda_l + \beta(l - i + 1)}{\lambda_i - \lambda_l + 1 + \beta(l - i)} \frac{\lambda_i - \lambda_l + 1 + \beta(l - i - 1)}{\lambda_i - \lambda_l + \beta(l - i)}. \tag{3.8}$$

In the expression in (3.8) the partitions λ and μ are related by $\lambda_l = \mu_l + 1$ and $\lambda_i = \mu_i$ for $i \neq l$.

Proposition 3.7. $c_\lambda(\alpha, \beta)$ satisfies the next recursion relation:

$$(\epsilon_\lambda(\beta) + |\lambda|(1 + \sqrt{2\beta\alpha} - \beta))c_\lambda(\alpha, \beta) = \beta \sum_{\mu <_1 \lambda} \psi'_{\lambda/\mu}(\beta)c_\mu(\alpha, \beta). \tag{3.9}$$

Here the function $\epsilon_\lambda(\beta)$ is given in (2.5).

Proof. We will calculate $\widehat{E}_\beta w_{G,n} \in \mathcal{F}_\alpha$ in two ways. By comparing both expression we obtain the recursion relation.

First, by the definition of \widehat{E}_β given in (2.2) and by the condition (3.2) of $v_{G,n}$ we have

$$\begin{aligned} \widehat{E}_\beta w_{G,n} &= \left[\sqrt{2\beta} \sum_{m \geq 1} a_{-m} \mathcal{L}_m + \sum_{m \geq 1} a_{-m} a_m (\beta - 1 - \sqrt{2\beta} a_0) \right] w_{G,n} \\ &= \left[\sqrt{2\beta} a_{-1} \mathcal{L}_1 + \sum_{m \geq 1} a_{-m} a_m (\beta - 1 - \sqrt{2\beta} a_0) \right] w_{G,n} \\ &= \sqrt{2\beta} a_{-1} w_{G,n-1} + n(\beta - 1 - \sqrt{2\beta\alpha}) w_{G,n} \in \mathcal{F}_\alpha. \end{aligned}$$

Now applying the isomorphism $\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_{\mathbb{C}}$ on both sides and substituting $w_{G,n}$ and $w_{G,n-1}$ by their expansions (3.3), we have

$$\iota_\beta(\widehat{E}_\beta w_{G,n}) = \beta p_1 \sum_{\mu \vdash n-1} c_\mu(\alpha, \beta) P_\mu^{(\beta^{-1})} + n(\beta - 1 - \sqrt{2\beta\alpha}) \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})} \in \Lambda_{\mathbb{C}}.$$

Using the Pieri formula (3.7) in the first term, we have

$$\begin{aligned} \iota_\beta(\widehat{E}_\beta w_{G,n}) &= \beta \sum_{\mu \vdash n-1} c_\mu(\alpha, \beta) \sum_{\lambda >_1 \mu} \psi'_{\lambda/\mu}(\beta) P_\lambda^{(\beta^{-1})} \\ &\quad + n(\beta - 1 - \sqrt{2\beta\alpha}) \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})}. \end{aligned} \tag{3.10}$$

Next, by (3.4) and by (2.4) we have

$$\widehat{E}_\beta w_{G,n} = \widehat{E}_\beta \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) f_\lambda^{(\beta^{-1}, \beta)} = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) \epsilon_\lambda(\beta) f_\lambda^{(\beta^{-1}, \beta)} \in \mathcal{F}_\alpha.$$

Therefore we have

$$t_\beta(\widehat{E}_\beta w_{G,n}) = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) \epsilon_\lambda(\beta) P_\lambda^{(\beta^{-1})} \in \Lambda_{\mathbb{C}}. \tag{3.11}$$

Then comparing (3.10) and (3.11) we have

$$\sum_{\lambda \vdash n} (\epsilon_\lambda(\beta) + n(1 + \sqrt{2\beta}\alpha - \beta)) c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})} = \beta \sum_{\mu \vdash n-1} c_\mu(\alpha, \beta) \sum_{\lambda >_1 \mu} \psi'_{\lambda/\mu}(\beta) P_\lambda^{(\beta^{-1})} \in \Lambda_{\mathbb{C}}^n.$$

Since $\{P_\lambda^{(\beta)} \mid \lambda \vdash n\}$ is a basis of $\Lambda_{\mathbb{C}}^n$, we have

$$(\epsilon_\lambda(\beta) + n(1 + \sqrt{2\beta}\alpha - \beta)) c_\lambda(\alpha, \beta) = \beta \sum_{\mu <_1 \lambda} c_\mu(\alpha, \beta) \psi'_{\lambda/\mu}(\beta). \quad \square$$

Proof of Theorem 3.4. The recursion relation (3.9) of Propositions 3.7 determines $c_\lambda(\alpha, \beta)$ uniquely if we set the value of $c_\beta(\alpha, \beta)$. Since the existence and uniqueness of v_G is known by Fact 3.2, we only have to show that the ansatz (3.6) satisfies (3.9).

For partitions λ and μ which are related by $\lambda_I = \mu_I + 1$ and $\lambda_i = \mu_i$ for $i \neq I$, we have the two following formulas:

$$\begin{aligned} & \left[\prod_{(i,k) \in \mu} \frac{1}{\lambda_i - k + 1 + \beta(\lambda'_k - i)} \right] / \left[\prod_{(i,k) \in \lambda} \frac{1}{\lambda_i - k + 1 + \beta(\lambda'_k - i)} \right] \\ &= \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + 1 + \beta(I - i)}{\lambda_i - \lambda_I + 1 + \beta(I - 1 - i)} \times \prod_{i=1}^{\lambda_I - 1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)}, \\ & \left[\prod_{\substack{(i,k) \in \mu \\ \mu \neq (1,1)}} \frac{\beta}{(k+1) + \sqrt{2\beta}\alpha - (i+1)\beta} \right] / \left[\prod_{\substack{(i,k) \in \lambda \\ \mu \neq (1,1)}} \frac{\beta}{(k+1) + \sqrt{2\beta}\alpha - (i+1)\beta} \right] \\ &= \frac{(\lambda_I + 1) + \sqrt{2\beta}\alpha - (I + 1)\beta}{\beta}. \end{aligned}$$

Substituting the $c_\mu(\alpha, \beta)$ in the right hand side of (3.9) by the ansatz (3.6) and using the above two equations, we have

$$\begin{aligned} \text{RHS of (3.9)} &= \sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + \beta(I - i)} \times \prod_{i=1}^{\lambda_I - 1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} \\ &\quad \times ((\lambda_I + 1) + \sqrt{2\beta}\alpha - (I + 1)\beta) c_\lambda(\alpha, \beta), \end{aligned} \tag{3.12}$$

where $C(\lambda)$ is the set of boxes \square in the Young diagram of λ such that $\lambda \setminus \{\square\}$ is also a partition. In particular, if $\square = (I, \lambda_I) \in C(\lambda)$, then $\mu := \lambda \setminus \{\square\}$ is the partition satisfying $\mu_I = \lambda_I - 1$ and $\mu_i = \lambda_i$ for $i \neq I$, recovering the previous description.

As for the left hand side of (3.9), we have by (2.5):

$$\epsilon_\lambda(\beta) + |\lambda|(1 + \sqrt{2\beta}\alpha - \beta) = |\lambda|(1 + \sqrt{2\beta}\alpha) + \sum_i (\lambda_i^2 - 2i\lambda_i\beta). \tag{3.13}$$

Thus by (3.12) and (3.13), Eq. (3.9) under the substitution (3.6) is equivalent to the next one:

$$\begin{aligned}
 & |\lambda|(1 + \sqrt{2\beta\alpha}) + \sum_i (\lambda_i^2 - 2i\lambda_i\beta) \\
 &= \sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + \beta(I - i)} \times \prod_{i=1}^{\lambda_I - 1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} \\
 &\quad \times (1 + \sqrt{2\beta\alpha} + \lambda_I - (I + 1)\beta).
 \end{aligned}$$

This is verified by Propositions 4.1 and 4.2 which will be shown in Section 4. \square

3.3. Non-degenerate Whittaker vector via Jack polynomials

Definition 3.8. Fix non-zero complex numbers a and b . Let v_W be an element of the Verma module M_h satisfying

$$L_1 v_W = a v_W, \quad L_2 v_W = b v_W, \quad L_n v_W = 0 \quad (n \in \mathbb{Z}_{\geq 3}).$$

We call such an element v_W (non-degenerate) Whittaker vector of M_h .

Fact 3.9. Assume that c and h are generic complex numbers. Then v_W exists uniquely up to scalar multiplication.

Proof. This is shown in [18]. \square

Lemma 3.10. Let us decompose v_W as

$$v_W = \sum_{n \in \mathbb{Z}_{\geq 0}} a^n v_{W,n}, \quad v_{W,n} \in M_{h,n}.$$

Then we have

$$L_1 v_{W,n+1} = v_{W,n}, \quad L_2 v_{W,n+2} = a^{-2} b \cdot v_{W,n}.$$

Proof. This follows from the commutation relations $[L_0, L_1] = -L_1$ and $[L_0, L_2] = -2L_2$. \square

Now we expand the bosonized Whittaker vector

$$w_{W,n} := \psi(v_{W,n}) \in \mathcal{F}_{\alpha,n}$$

by $f_{\lambda}^{(\beta^{-1}, \beta)}$,_S (2.1) and express it as

$$w_{W,n} = \sum_{\lambda \vdash n} d_{\lambda}(\alpha, \beta) f_{\lambda}^{(\beta^{-1}, \beta)}, \quad d_{\lambda}(\alpha, \beta) \in \mathbb{C}.$$

Proposition 3.11. Using the notation $\lambda \succ_k \mu$ given in Definition 3.5, we have the next recursion relation for $d_{\lambda}(\alpha, \beta)$:

$$\begin{aligned}
 & (\epsilon_\lambda(\beta) + |\lambda|(1 + \sqrt{2\beta}\alpha - \beta))c_\lambda(\alpha, \beta) \\
 &= \beta \sum_{\nu <_2 \lambda} d_\nu(\alpha, \beta)\psi'_{\lambda/\nu}{}^{(2)}(\beta) + \beta \sum_{\mu <_1 \lambda} d_\mu(\alpha, \beta)\psi'_{\lambda/\mu}(\beta).
 \end{aligned} \tag{3.14}$$

$\psi'_{\lambda/\nu}{}^{(2)}(\beta)$ is the coefficient in the next Pieri formula:

$$p_2 P_\nu^{(\beta^{-1})} = \sum_\lambda \psi'_{\lambda/\nu}{}^{(2)}(\beta) P_\lambda^{(\beta^{-1})}.$$

Proof. Similar as the proof of Proposition 3.7. \square

Remark 3.12. The author doesn't know whether d_λ has a good explicit formula, although c_λ has the factored formula (3.6).

4. Combinatorial identities of rational functions

Proposition 4.1. For a partition λ , let $C(\lambda)$ be the set of boxes \square of λ such that $\lambda \setminus \{\square\}$ is also a partition. Then

$$\sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + \beta(I - i)} \times \prod_{i=1}^{\lambda_I-1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} = |\lambda|. \tag{4.1}$$

Proof. Let λ be the partition such that

$$\lambda = (\overbrace{n_1, \dots, n_1}^{j_1}, \overbrace{n_2, \dots, n_2}^{j_2}, \dots, \overbrace{n_l, \dots, n_l}^{j_l}). \tag{4.2}$$

Then we have

$$C(\lambda) = \{(m_1, n_1), (m_2, n_2), \dots, (m_l, n_l)\} \tag{4.3}$$

with $m_k := j_1 + \dots + j_k$ ($k = 1, \dots, l$), where we have used the coordinate (i, j) of Young diagram associated to λ as explained in Section 1.1.

Let us choose an element $\square = (m_k, n_k)$ of $C(\lambda)$, and calculate the corresponding factor in (4.1). The first product reads

$$\begin{aligned}
 & \prod_{1 \leq i \leq m_1} \frac{(n_1 - n_k) + \beta(m_k - i + 1)}{(n_1 - n_k) + \beta(m_k - i)} \prod_{m_1+1 \leq j \leq m_2} \frac{(n_2 - n_k) + \beta(m_k - i + 1)}{(n_2 - n_k) + \beta(m_k - i)} \\
 & \times \dots \times \prod_{m_{k-1}+1 \leq i \leq m_k-1} \frac{m_k - i + 1}{m_k - i} \\
 &= \prod_{i=1}^{k-1} \frac{(n_i - n_k) + \beta(m_k - m_{i-1})}{(n_i - n_k) + \beta(m_k - m_i)} \times (m_k - m_{k-1}).
 \end{aligned}$$

Here we have used the notation $m_0 := 0$. The second product reads

$$\begin{aligned} & \prod_{1 \leq j \leq n_l} \frac{(n_k - j + 1) + \beta(m_l - n_k)}{(n_k - j) + \beta(m_l - n_k)} \prod_{n_{l+1} \leq j \leq n_{l-1}} \frac{(n_k - j + 1) + \beta(m_{l-1} - n_k)}{(n_k - j) + \beta(m_{l-1} - n_k)} \\ & \quad \times \cdots \times \prod_{n_{k+1} + 1 \leq j \leq n_{k-1}} \frac{(n_k - j + 1)}{(n_k - j)} \\ & = (n_k - n_{k+1}) \times \prod_{j=k+1}^l \frac{(n_k - n_{j+1}) + \beta(m_j - m_k)}{(n_k - n_j) + \beta(m_j - m_k)}. \end{aligned}$$

Here we have used the notation $n_{l+1} := 0$.

Now let us define

$$F_1(\{m_k\}, \{n_k\}, \beta) := \sum_{k=1}^l F_{1,k}(\{m_k\}, \{n_k\}, \beta),$$

$$\begin{aligned} F_{1,k}(\{m_k\}, \{n_k\}, \beta) & := (m_k - m_{k-1})(n_k - n_{k+1}) \\ & \times \prod_{i=1}^{k-1} \frac{(n_i - n_k) + \beta(m_k - m_{i-1})}{(n_i - n_k) + \beta(m_k - m_i)} \prod_{j=k+1}^l \frac{(n_k - n_{j+1}) + \beta(m_j - m_k)}{(n_k - n_j) + \beta(m_j - m_k)}. \end{aligned}$$

Then for the proof of (4.1) it is enough to show that F_1 is equal to $|\lambda|$ if $\{m_k\}$ and $\{n_k\}$ correspond to λ as in (4.2) and (4.3).

Hereafter we consider F_1 as a rational function of the valuables $\{m_k\}$, $\{n_k\}$ and β . As a rational function of β , F_1 has the apparent poles at $\beta_{j,k} := -(n_j - n_k)/(m_k - m_j)$ ($j = 1, 2, \dots, k-1, k+1, \dots, l$). We may assume that these apparent poles are mutually different so that all the poles are at most single. Then the residue at $\beta = \beta_{j,k}$ comes from the factors $F_{1,j}$ and $F_{1,k}$. Now we may assume $j < k$. Then a direct computation yields

$$\text{Res}_{\beta=\beta_{j,k}} F_{1,j} = \frac{(m_j - m_{j-1})(n_j - n_{j+1})(n_k - n_{k+1})}{(m_j - m_k)} \tag{4.4}$$

$$\times \prod_{i=1}^{j-1} \frac{(n_i - n_j)(m_k - m_j) - (n_j - n_k)(m_j - m_{i-1})}{(n_i - n_j)(m_k - m_j) - (n_j - n_k)(m_j - m_i)} \tag{4.5}$$

$$\times \prod_{i=j+1}^{k-1} \frac{(n_j - n_{i+1})(m_k - m_j) - (n_j - n_k)(m_i - m_j)}{(n_j - n_i)(m_k - m_j) - (n_j - n_k)(m_i - m_j)} \tag{4.6}$$

$$\times \prod_{i=k+1}^l \frac{(n_j - n_{i+1})(m_k - m_j) - (n_j - n_k)(m_i - m_j)}{(n_j - n_i)(m_k - m_j) - (n_j - n_k)(m_i - m_j)}, \tag{4.7}$$

and

$$\text{Res}_{\beta=\beta_{j,k}} F_{1,k} = \frac{(m_k - m_{k-1})(n_k - n_{k+1})(n_j - n_k)(m_{j-1} - m_j)}{(m_k - m_j)^2} \tag{4.8}$$

$$\times \prod_{i=1}^{j-1} \frac{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_{i-1})}{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_i)} \tag{4.9}$$

$$\times \prod_{i=j+1}^{k-2} \frac{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_{i-1})}{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_i)} \tag{4.10}$$

$$\times \prod_{i=k+1}^l \frac{(n_k - n_{i+1})(m_k - m_j) - (n_j - n_k)(m_i - m_k)}{(n_k - n_i)(m_k - m_j) - (n_j - n_k)(m_i - m_k)}. \tag{4.11}$$

Using the identity $(a - b)(x - y) - (c - b)(x - z) = (a - c)(x - y) - (c - b)(y - z)$, one finds that the factors (4.5) and (4.9) are equal. Similarly (4.7) and (4.11) are equal. Shifting the index i in (4.6) and using the above identity, one also finds that

$$\frac{(4.6)}{(4.10)} = - \frac{(n_j - n_k)(m_k - m_{k-1})}{(n_{j+1} - n_j)(m_k - m_j)}.$$

Thus we have

$$\frac{\text{Res}_{\beta=\beta_{j,k}} F_{1,j}}{\text{Res}_{\beta=\beta_{j,k}} F_{1,k}} = - \frac{(n_j - n_k)(m_k - m_{k-1})}{(n_{j+1} - n_j)(m_k - m_j)} \frac{(4.5)}{(4.9)} = -1.$$

Therefore we have

$$\text{Res}_{\beta=\beta_{j,k}} F_1(\{m_k\}, \{n_k\}, \beta) = 0,$$

such that F_1 is a polynomial of β .

Then from the behavior F_1 in the limit $\beta \rightarrow \infty$, we find that F_1 is a constant as a function of β . This constant can be calculated by setting $\beta = 0$, and the result is

$$F_1(\{m_k\}, \{n_k\}, \beta) = \sum_{k=1}^l (m_k - m_{k-1}) n_k.$$

It equals to $|\lambda|$ if λ is given by (4.2) and $m_k = j_1 + \dots + j_k$. This is the desired consequence. \square

Proposition 4.2. *Using the same notation as in Proposition 4.1, we have*

$$\begin{aligned} & \sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + \beta(I - i)} \times \prod_{i=1}^{\lambda_I - 1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} \\ & \times (\lambda_I - (I + 1)\beta) = \sum_i (\lambda_i^2 - 2i\lambda_i\beta). \end{aligned} \tag{4.12}$$

Proof. As in the proof of Proposition 4.1, set $\lambda = (\overbrace{n_1, \dots, n_1}^{j_1}, \overbrace{n_2, \dots, n_2}^{j_2}, \dots, \overbrace{n_l, \dots, n_l}^{j_l})$ and $m_k := j_1 + \dots + j_k$ ($k = 1, \dots, l$). We can write the left hand side of (4.12) as

$$F_2(\{m_k\}, \{n_k\}, \beta) = \sum_{k=1}^l F_{2,k}(\{m_k\}, \{n_k\}, \beta),$$

$$F_{2,k}(\{m_k\}, \{n_k\}, \beta) := (n_k - (m_k + 1)\beta)(m_k - m_{k-1})(n_k - n_{k+1}) \\ \times \prod_{i=1}^{k-1} \frac{(n_i - n_k) + \beta(m_k - m_{i-1})}{(n_i - n_k) + \beta(m_k - m_i)} \prod_{j=k+1}^l \frac{(n_k - n_{j+1}) + \beta(m_j - m_k)}{(n_k - n_j) + \beta(m_j - m_k)}.$$

The residues of F_2 are the same as those of F_1 , and by a similar calculation as in Proposition 4.1, one can find that F_2 is a polynomial of β . The behavior of F_2 in the limit $\beta \rightarrow \infty$ shows that F_2 is a linear function of β .

Using the original expression (4.12), we find that

$$F_2(\{m_k\}, \{n_k\}, 0) = \sum_i \lambda_i^2.$$

In order to determine the coefficient of β in F_2 , we rewrite F_2 as the rational function of β^{-1} , and take the limit $\beta^{-1} \rightarrow \infty$. The result is

$$\lim_{\beta \rightarrow \infty} (\beta^{-1} F_2(\{m_k\}, \{n_k\}, \beta)) = - \sum_{k=1}^l (m_k + 1)(m_k - m_{k-1})(n_k - n_{k+1}) \frac{m_k}{m_k - m_{k-1}} \\ = - \sum_{k=1}^l n_k (m_k - m_{k-1})(m_k + m_{k-1} + 1).$$

A moment thought shows that this becomes $-\sum_i 2i\lambda_i$ if $\{m_k\}$ and $\{n_k\}$ correspond to λ . Thus the proof is complete. \square

5. Conclusion and remarks

We have investigated the expansions of Whittaker vectors for the Virasoro algebra in terms of Jack symmetric functions. As we have mentioned in Section 1, the paper [3, (3.18)] proposed a conjecture on the factored expression for the Gaiotto state of the deformed Virasoro algebra using Macdonald symmetric functions. However, our proof cannot be applied to this deformed case. The main obstruction is that the zero-mode T_0 of the generating field $T(z)$ of the deformed Virasoro algebra behaves badly, such that one cannot analyze its action on Macdonald symmetric functions, and cannot obtain a recursive formula similar to the one in Proposition 3.7.

It is also valuable to consider the $\mathcal{W}(\mathfrak{sl}_n)$ -algebra case. In [21] a degenerate Whittaker vector is expressed in terms of the contravariant form of the $\mathcal{W}(\mathfrak{sl}_3)$ -algebra. At this moment, however, we don't know how to treat Whittaker vectors for $\mathcal{W}(\mathfrak{sl}_n)$ -algebra. It seems to be related to the higher rank analogues of the AGT conjecture (see [23] for examples).

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Appendix A. AGT relation

This appendix is devoted to the explanation of the AGT relation for pure SU(2) gauge theory, and its connection to the formula given in our main theorem. This section is not necessary for the main argument of this paper.

A.1. AGT relation for pure SU(2) gauge theory

The original AGT conjecture [1] states the equivalence between the Liouville conformal blocks and the Nekrasov partition functions [15]. In [9] the degenerated versions of the conjecture were proposed. As the most simplified case, it was conjectured that the norm of the Gaiotto state of Virasoro algebra coincides with the Nekrasov partition function for the four-dimensional pure SU(2) gauge theory.

First we introduce the contravariant form (Shapovalov form) on the Verma module M_h . Let us denote the (restricted) dual Verma module by M_h^* . This is a right Vir_c -representation generated by $\langle h |$ with $\langle h | \text{Vir}_{c,+} = 0$ and $\langle h | L_0 = h \langle h |$. The contravariant form is the bilinear map

$$\cdot : M_h^* \times M_h \rightarrow \mathbb{C}$$

determined by

$$\langle h | \cdot | h \rangle = 1, \quad 0 \cdot | h \rangle = \langle h | \cdot 0 = 0, \quad \langle h | u \cdot | h \rangle = \langle h | \cdot u | h \rangle \quad (u \in \text{Vir}).$$

Fix a complex number Λ .³ In this section we denote by $|G\rangle \in M_h^4$ the Gaiotto state

$$L_1 |G\rangle = \Lambda^2 |G\rangle, \quad L_n |G\rangle = 0 \quad (n > 1).$$

normalized as

$$|G\rangle = |h\rangle + \dots$$

This normalization condition means that the homogeneous component of $|G\rangle$ in $M_{h,0}$ is $|h\rangle$, i.e., the coefficient $c_\emptyset(\alpha, \beta)$ in (1.12) is set to be one.

Let us also define the anti-homomorphism

$$\dagger : U(\text{Vir}_{c,-}) \rightarrow U(\text{Vir}_{c,+}), \quad L_{-n} \mapsto L_n.$$

We will also denote the action of this map as $L_{-n}^\dagger = L_n$. It induces a linear map $M_h \rightarrow M_h^*$, which is also written by \dagger . We define $\langle G | := (|G\rangle)^\dagger$.

Next we recall the Nekrasov partition function (see [15] and [4,5,16,17]). It has a geometric meaning, but here we only give the next combinatorial expression. Let $r \in \mathbb{Z}_{\geq 2}$ and $x, \epsilon_1, \epsilon_2, \vec{a} = (a_1, \dots, a_r)$ be indeterminates. Then the Nekrasov partition function $Z^{\text{rank}=r}(x; \epsilon_1, \epsilon_2, \vec{a})$ for pure SU(r) gauge theory is defined to be:

$$\begin{aligned} Z^{\text{rank}=r}(x; \epsilon_1, \epsilon_2, \vec{a}) &:= \sum_{\vec{Y}} \frac{x^{|\vec{Y}|}}{\prod_{1 \leq \alpha, \beta \leq r} n_{\alpha, \beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a})}, \\ n_{\alpha, \beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}) &:= \prod_{\square \in Y_\alpha} [-\ell_{Y_\beta}(\square)\epsilon_1 + (a_{Y_\alpha}(\square) + 1)\epsilon_2 + a_\beta - a_\alpha] \\ &\quad \times \prod_{\blacksquare \in Y_\beta} [(\ell_{Y_\alpha}(\blacksquare) + 1)\epsilon_1 - a_{Y_\beta}(\blacksquare)\epsilon_2 + a_\beta - a_\alpha]. \end{aligned} \tag{A.1}$$

³ In this subsection we use the notations in the physics literatures. Do not confuse this parameter Λ and the notation Λ of the ring of symmetric functions.

⁴ Do not confuse this symbol $|G\rangle$ for the Gaiotto state and the symbol $|h\rangle$ for the highest weight vector.

Table 1
Parameter correspondence.

Virasoro	Nekrasov
c	$13 + 6(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1)$
h	$((\epsilon_1 + \epsilon_2)^2 - (a_2 - a_1)^2)/4\epsilon_1\epsilon_2$
Λ	$x^{1/4}/(\epsilon_1\epsilon_2)^{1/2}$

Here $\vec{Y} = (Y_1, \dots, Y_r)$ is an r -tuple of partitions, $|\vec{Y}| := |Y_1| + \dots + |Y_r|$, and $a_Y(\square)$, $\ell_Y(\square)$ are the arm and leg of the box \square with respect to Y as

$$a_\lambda(\square) := \lambda_i - j, \quad \ell_\lambda(\square) := \lambda'_j - i.$$

Note that for the case $i > \ell(\lambda)$ the number λ_i should be taken as $\lambda_i = 0$, and for $j > \lambda_1$ the number λ'_j taken as $\lambda'_j = 0$. Thus $a_\lambda(\square)$ and $\ell_\lambda(\square)$ could be minus in general, although such cases don't occur in the norm of Jack symmetric functions.

Now the statement of the simplest case of the Gaiotto conjectures is

$$\langle G|G \rangle \stackrel{?}{=} Z^{\text{rank}=2}(x; \epsilon_1, \epsilon_2, \vec{a}). \tag{A.2}$$

Here the parameters are related as in Table 1.

Note that this degenerate version of the AGT conjecture is proved by the method of Zamolodchikov-type recursive formula in the papers [6] and [10].

A.2. Comparison of the inner products

Our formula (1.12) describes the Gaiotto state $|G\rangle$ by Jack symmetric functions. In order to calculate the norm $\langle G|G \rangle$, we should compare the contravariant form $\cdot : M_h^* \rightarrow M_h$ and the inner product $\langle \cdot, \cdot \rangle$ on Λ .

Let us recall the isomorphism ι_β :

$$\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_{\mathbb{C}}, \quad a_{-n}|\alpha\rangle \mapsto p_n \cdot \beta^{1/2}/\sqrt{2} \quad (n > 0).$$

In order to give the consistency between the bilinear form $\cdot : \mathcal{F}_\alpha^* \times \mathcal{F}_\alpha \rightarrow \mathbb{C}$ on the Heisenberg–Fock space and the contravariant form $\cdot : M_h^* \times M_h \rightarrow \mathbb{C}$ on the Verma module of the Virasoro algebra, we need to give the anti-homomorphism

$$\omega : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\alpha^*$$

such that

$$\langle h|u_1^\dagger \cdot u_2|h \rangle = \mathcal{F} \langle \alpha | \omega(\varphi(u_1)) \cdot \varphi(u_2) | \alpha \rangle_{\mathcal{F}}$$

holds for any $u_1, u_2 \in U(\text{Vir}_{c,+})$, where $\varphi : U(\text{Vir}_c) \rightarrow \widehat{U}(\mathcal{H})$ is the bosonization map (1.5).

The consistent definition of ω is given as follows [12,22]:

$$\omega(a_n) = a_{-n} - 2\rho\delta_{n,0}, \quad \omega(\rho) = -\rho.$$

It implies for the parametrization $\rho = -(\beta^{1/2} - \beta^{-1/2})/\sqrt{2}$ that $\omega(\beta^{1/2}) = -\beta^{1/2}$. Then we can spell out the inner product on Λ which is consistent with the contravariant form on the Verma module M_h :

$$\langle p_n, p_m \rangle = \mathcal{F} \langle \alpha | \omega(\sqrt{2}\beta^{-1/2}a_{-n}) \cdot \sqrt{2}\beta^{-1/2}a_{-m} | \alpha \rangle_{\mathcal{F}} = -2n/\beta \cdot \delta_{n,m}.$$

This is the inner product $\langle \cdot, \cdot \rangle_{-2/\beta}$ defined in (1.11). But the Jack symmetric function orthogonal with respect to it is $P_\lambda^{(-2/\beta)}$, not $P_\lambda^{(1/\beta)}$ which is used in our expansion.

Thus the AGT relation (A.2) is equivalent to

$$\sum_{\lambda, \mu \in \mathcal{P}} \Lambda^{2|\mu|+2|\lambda|} c_\lambda(\alpha, \beta) c_\mu(\alpha, \beta) \langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta} \stackrel{?}{=} Z^{\text{rank}=2}(\mathbf{x}; \epsilon_1, \epsilon_2, \vec{a}). \tag{A.3}$$

Now one may easily find that

$$\langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta} = 0 \quad \text{unless } |\lambda| = |\mu|.$$

Using this fact and comparing the homogeneous parts (the coefficients of Λ^{4d} and those of x^d) of both sides in (A.3), one finds that (A.2) is equivalent to

$$\begin{aligned} & \sum_{\lambda, \mu \vdash d} c_\lambda(\alpha, \beta) c_\mu(\alpha, \beta) \langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta} \\ & \stackrel{?}{=} (\epsilon_1 \epsilon_2)^{2d} \sum_{\substack{\lambda, \mu \in \mathcal{P} \\ |\lambda|+|\mu|=d}} \frac{1}{\prod_{1 \leq \alpha, \beta \leq 2} n_{\alpha, \beta}^{(\lambda, \mu)}(\epsilon_1, \epsilon_2, \vec{a})} \end{aligned} \tag{A.4}$$

for each $d \in \mathbb{Z}_{\geq 0}$. In the right hand side we changed the notation $\vec{Y} \in \mathcal{P}^2$ to the pair $(\lambda, \mu) \in \mathcal{P}^2$. Note that the ranges of running indexes in the left and right sides are different. Eq. (A.4) seems to contain non-trivial relations among the ‘non-diagonal’ pairings $\langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta}$. According to the computer experiment, these pairings have complicated looks (in particular, no factored expressions) in general, although each summand in the right hand side of (A.4) is factored. A combinatorial proof of (A.4) would be another justification of the AGT relation (A.2), but we have no clue to show it directly at this moment.

We have another combinatorial restatement of (A.2). If $\lambda \vdash d$, then one can expand $P_\lambda^{(1/\beta)} \in \Lambda_{\mathbb{C}}^d$ by the basis $\{P_\nu^{(-2/\beta)} \mid \nu \vdash d\}$ of $\Lambda_{\mathbb{C}}^d$. Let us express it as

$$P_\lambda^{(1/\beta)} = \sum_{\nu \vdash d} \gamma_\lambda^\nu(\beta) P_\nu^{(-2/\beta)}, \quad \gamma_\lambda^\nu(\beta) \in \mathbb{C}. \tag{A.5}$$

Then by an elementary calculation one finds that (A.2) is equivalent to

$$\begin{aligned} & \sum_{\lambda, \mu, \nu \vdash d} c_\lambda(\alpha, \beta) c_\mu(\alpha, \beta) \gamma_\lambda^\nu(\beta) \gamma_\mu^\nu(\beta) N_\nu(-2/\beta) \\ & \stackrel{?}{=} (\epsilon_1 \epsilon_2)^{2d} \sum_{\substack{\lambda, \mu \in \mathcal{P} \\ |\lambda|+|\mu|=d}} \frac{1}{\prod_{1 \leq \alpha, \beta \leq 2} n_{\alpha, \beta}^{(\lambda, \mu)}(\epsilon_1, \epsilon_2, \vec{a})}. \end{aligned} \tag{A.6}$$

Here we have used the norm of Jack symmetric function

$$N_\nu(b) := \langle P_\nu^{(b)}, P_\nu^{(b)} \rangle_b = \prod_{\square \in \nu} \frac{a_\nu(\square) + b \ell_\nu(\square) + 1}{a_\nu(\square) + b \ell_\nu(\square) + b}.$$

According to the computer experiment, the coefficient $\gamma_\lambda^\mu(\beta)$ in the expansion (A.5) doesn’t have a factored expression in general, although it looks a little simpler than the pairing $\langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta}$.

One might find an explicit formula of $\gamma_\lambda^\mu(\beta)$. However a direct proof of (A.6) will require a manipulation on the changes of indexes from $\lambda, \mu, \nu \vdash d$ to $\lambda, \mu \in \mathcal{P}$ with $|\lambda| + |\mu| = d$, which seems to be hard at this moment.

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