Material characterization by dual sharp indenter

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In a recent study, Le [Le, M.-Q., 2008. A computational study on the instrumented sharp indentations with dual indenter. International Journal of Solids and Structures, 45 (10), 2818–2835.] demonstrated that the yield strength Y can be replaced by the loading curvature C and hence the reduced elastic modulus-loading curvature ratio $E^*/C$ and strain hardening exponent $n$ can be used to govern characteristic parameters of indentation load–depth curves. Extending Le’s approach and regarding dimensional analysis, it is found that $C/Y$ and $E^*/Y$ can be used to investigate fundamental issues in instrumented sharp indentation. Based on extensive finite element analysis, a set of dimensionless functions are constructed for cone indenters of half included angles of 60° and 70.3°. Dimensionless relationships with respect to dual indenters are further explored. Several features of hardness are also considered. An inverse analysis procedure is suggested to estimate material properties, giving good inverse results for experimental data from the literature and representative materials. Sensitivity of inverse solution is studied and discussed. The results show that the proposed dual indenter method is quite robust and can be applied to a wide range of materials.

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1. Introduction

Methods to extract material properties from instrumented indentation response have been investigated in a number of studies. Elastic modulus can be estimated from a well-known relation between the true projected contact area $A_m$, the initial unloading slope $S$, and the reduced elastic modulus $E^*$ (see Fig. 1 for notations), see the recent review by Oliver and Pharr (2004):

$$S = \frac{2}{\sqrt{\pi}} E^* \sqrt{A_m}$$

(1)

where $\beta$ is a correction factor and $E^*$ is determined by the elastic modulus $E$ and $E_i$, and Poisson’s ratios $\nu$ and $\nu_i$ of the indented material and the indenter, respectively, as below:

$$E^* = \left( \frac{1 - \nu^2}{E} + \frac{1 - \nu_i^2}{E_i} \right)^{-1}$$

(2)

Due to the self-similarity (Cheng and Cheng, 2004), there can be multiple combinations of mechanical properties ($E$, $Y$, and $n$), that give rise to almost indistinguishable indentation load–depth curves of a single conical/pyramid indenter (Cheng and Cheng, 1999; Capehart and Cheng, 2003; Alkorta et al., 2005; Tho et al., 2005; Luo et al., 2006). The uniqueness of reverse solutions with a single indenter is also suffered from the interdependence of indentation parameters as follows (Le, 2008):

$$\frac{W_r}{W_e} = \frac{m+1}{m} \frac{h_m}{h_e}$$

(3a)

$$\frac{S}{C h_m} = m \frac{h_m}{h_e}$$

(3b)

$$\frac{S}{C h_m} = \frac{3m_1}{m+1} \frac{W_r}{W_e}$$

(3c)

where $m$ and $m_1$ are constants.

Eqs. (3) shows that only two of four indentation parameters of a single indentation load–depth curve, $S$, $C$, $W_r/W_e$, and $h_m/h_e$, are independent, leading to two independent equations, which contain the information on the material properties. Despite this, three independent equations are required in the reverse analysis to estimate three mechanical properties $E^*, Y, n$. Therefore, dual or multiple indenters, which are expected to give more than two independent equations, have appeared as potential methods (Futakawa et al., 2001; Bucaille et al., 2003; Chollacoop et al., 2003; DiCarlo et al., 2004; Cao et al., 2005; Ogawara et al., 2005, 2006; Swaddiwudhipong et al., 2005; Lan and Venkatesh, 2007; Luo and Lin, 2007; Yan et al., 2007; Le, 2008). Most of previous works used the concept of representative strain, which was originally introduced to interpret cone hardness by Atkins and Tabor (1965), to construct dimensionless relationships in instrumented sharp indentation.

It can be seen from the literature that formulations in instrumented sharp indentation appear generally complex when all material parameters are involved. Therefore, removal of a mechanical property, which is considered as unknown in a reverse procedure,
from dimensionless functions may reduce their complicity and accordingly improve reverse results. Further, due to the self-similarity of cone/pyramid indentations (Johnson, 1985; Cheng and Cheng, 2004), several indentation parameters are independent of indentation depth and contain only information on the material properties as well as on the indenter geometry. Thus, the use of an indentation parameter instead of a mechanical property may allow formulating simpler and clearer dimensionless functions.

The above-mentioned ideas led to a new approach to consider instrumented sharp indentation problems (Le, 2008). Regarding dimensional analysis, the author showed that the loading curvature C can be used instead of the yield strength Y to govern characteristics of P–h curves. Accordingly, three dimensionless parameters \( W_e/W_p \), \( S/(Ch_m) \) and \( h_m/h_r \) were formulated as linear functions of the reduced elastic modulus-loading curvature ratio \( E'/C \), in which only the strain hardening exponent \( n \) is involved as below:

\[
\frac{S}{Ch_m} = K_{s1}(n) \frac{E'}{C} + K_{s2}(n) \quad (4a)
\]

\[
\frac{W_e}{W_p} = K_{w1}(n) \frac{E'}{C} + K_{w2}(n) \quad (4b)
\]

\[
\frac{h_m}{h_r} = K_{h1}(n) \frac{E'}{C} + K_{h2}(n) \quad (4c)
\]

Moreover, the duality between corresponding parameters has been systematically investigated for common dual indenters of half included angles of 60° and 70.3°:

\[
\frac{E'}{C_{60}} = D_{e1}(n) \frac{E'}{C_{70.3}} + D_{e2}(n) \quad (5a)
\]

\[
\left( \frac{S}{Ch_m} \right)_{60} = D_{s1}(n) \left( \frac{S}{Ch_m} \right)_{70.3} + D_{s2}(n) \quad (5b)
\]

\[
\left( \frac{W_e}{W_p} \right)_{60} = D_{w1}(n) \left( \frac{W_e}{W_p} \right)_{70.3} + D_{w2}(n) \quad (5c)
\]

\[
\left( \frac{h_m}{h_r} \right)_{60} = D_{h1}(n) \left( \frac{h_m}{h_r} \right)_{70.3} + D_{h2}(n) \quad (5d)
\]

The coefficients in Eqs. (4) and (5) are given in Le (2008). Relied on Eqs. (4) and (5), \( E' \) and \( n \) can be simply estimated.

In this paper, Le's approach (Le, 2008) is extended to develop relationships between yield strength \( Y \) and indentation parameters. Relationships between corresponding parameters with respect to dual indenters are further here considered. Several features of hardness are also investigated. A new set of dimensionless functions are accordingly constructed under closed-forms within dimensional analysis and with the aid of finite element analysis (FEA). A dual indenter method is hence proposed to estimate material properties. As inverse analysis was previously investigated in Le (2008) for \( E' \) and \( n \), this study particularly focuses on yield strength and hardness.

### 2. Framework for analysis

Elastic–plastic behavior of many engineering solid materials can be modeled by a power law description. A simple elasto-plastic, true stress–true strain behavior is assumed to be:

\[
\sigma = E \cdot e^\nu, \quad (\sigma \leq Y) \\
\sigma = K \cdot e^n, \quad (\sigma \geq Y)
\]

where \( E \) is the Young's modulus, \( K \) a strength coefficient, \( n \) the strain hardening exponent, \( Y \) the initial compressive uniaxial yield stress and \( e \) the corresponding yield strain, such that

\[
\sigma_y = E e_y = K e^n_y
\]

Fig. 1b illustrates the typical indentation load–depth response of an elasto-plastic material to sharp indentation. Considering dimensional analysis and geometrical similarity of a conical/pyramid indenter, Cheng and Cheng (2004) have demonstrated that the indentation force \( P \) during loading is proportional to the square of the indentation depth \( h \):

\[
P = Ch^2
\]

By using dimensional analysis, Le (2008) has demonstrated the following relations related to the loading and unloading curves:

\[
E' = f_1 \left( \frac{E}{Y} \cdot n \right) \\
\frac{S}{Ch_m} = f_{s1} \left( \frac{E}{Y} \cdot n \right) \\
\frac{W_e}{W_p} = f_{w1} \left( \frac{E}{Y} \cdot n \right) \\
\frac{h_m}{h_r} = f_{h1} \left( \frac{E}{Y} \cdot n \right)
\]

Eq. (9) is written under another form as follow:

\[
\frac{C}{Y} = \frac{E'}{Y} f_3 \left( \frac{E}{Y} \cdot n \right)
\]

Details of the finite element model used in the present work and its validation are available in Le (2008). A large number of different combinations of elasto-plastic properties with \( n \) ranging from 0 to 0.6 and \( Y/E \) ranging from 5.0E–5 to 6.0E–2 were used in the computations. This wide range of model materials covers most of metals and engineering alloys. Model materials are assumed to obey Von Mises criterion. The material properties used in the computations are given in Table A1.
3. Characteristic parameters of P-h curves

Relationships between indentation parameters and \((E'/C)\) and \(n\) are described according to Eqs. (4). Relationships between yield stress \(Y\) and indentation parameters are here particularly interested. For this purpose, in a set of mechanical properties \((E', Y, \text{and } n)\), \(Y\) must be kept and then \(E'\) or \(n\) should be removed from formulated dimensionless functions. According to Le (2008), much less sensitivity and error were found for the elastic modulus than those for the strain hardening exponent \(n\) in reverse analysis. For this reason, \(n\) should be removed and \(E'\) should be kept together with \(Y\) in formulated functions to assume as accuracy as possible reverse results on \(Y\).

3.1 Dimensionless relationships for \(S/(Chm)\), \(W_t/W_e\), and \(h_m/h_e\) as functions of \(C/Y\) and \(E'/Y\)

It is noted that whenever \(E'/Y\) and \(C/Y\) are known, \(n\) can be determined according to Eq. (11). Therefore, regarding Eqs. (10), \(C/Y\) can be used instead of \(n\) to express the indentation parameters such as:

\[
\frac{S}{Ch_m} = f_{k_0} \left( \frac{E'}{Y} \right) \left( \frac{C}{Ch_m} \right) \tag{12a}
\]

\[
\frac{W_t}{W_e} = f_{k_1} \left( \frac{E'}{Y} \right) \left( \frac{C}{Ch_m} \right) \tag{12b}
\]

\[
\frac{h_m}{h_e} = f_{k_2} \left( \frac{E'}{Y} \right) \left( \frac{C}{Ch_m} \right) \tag{12c}
\]

It is found that in logarithmic scale \(W_t/W_e, S/(Ch_m)\) and \(h_m/h_e\) vary linearly with \(C/Y\) at a given \(E'/Y\) as depicted in Fig. 2. Eqs. (12) can be conveniently expressed under the following forms:

\[
\frac{C}{Y} = k_0 \left( \frac{E'}{Y} \right)^{1.018} \left( \frac{S}{Ch_m} \right) \tag{13a}
\]

\[
\frac{C}{Y} = k_1 \left( \frac{E'}{Y} \right)^{1.011} \left( \frac{W_t}{W_e} \right) \tag{13b}
\]

\[
\frac{C}{Y} = k_2 \left( \frac{E'}{Y} \right)^{1.052} \left( \frac{h_m}{h_e} \right) \tag{13c}
\]

where \(k_0, k_1, \text{and } k_2\) are constant, and \(G_i, G_{e_i}\), and \(G_n(i = 1 \text{ or } 2)\) are functions of \(E'/Y\) and given in Appendix A.

3.2 Relationships between corresponding parameters with respect to dual indenters

By regarding Eq. (11), it is found that instead of \(n\), \(C_{70.3}/Y\) can be used to govern the evolution of \(C_{60}/Y\). Fig. 3 shows linear relationships between \(C_{60}/Y\) and \(C_{70.3}/Y\) in logarithmic scale for different values of \(Y/E\). Functional relation between \(C_{60}/Y\) and \(C_{70.3}/Y\) can be written as below:

\[
\ln \left[ \frac{C_{60}}{Y} \right] = D_{61} \left( \frac{E'}{Y} \right) \ln \left[ \frac{C_{70.3}}{Y} \right] + D_{62} \tag{14}
\]

where \(D_{61}\) and \(D_{62}\) are functions of \(E'/Y\) and given in Appendix A. From Eqs. (13) and (14), further relationships between corresponding dimensionless parameters can be derived for dual indenters as follows:

\[
\ln \left[ \frac{S}{Ch_m}_{60} \right] = D_{61} \left( \frac{E'}{Y} \right) \ln \left[ \frac{S}{Ch_m}_{70.3} \right] + D_{62} \tag{15a}
\]

\[
\ln \left[ \frac{W_t}{W_e}_{60} \right] = D_{61} \left( \frac{W_t}{W_e}_{70.3} \right) \ln \left[ \frac{W_t}{W_e}_{70.3} \right] + D_{62} \tag{15b}
\]

\[
\ln \left[ \frac{h_m}{h_e}_{60} \right] = D_{61} \left( \frac{h_m}{h_e}_{70.3} \right) \ln \left[ \frac{h_m}{h_e}_{70.3} \right] + D_{62} \tag{15c}
\]

where \(D_{61}, D_{62}, \text{and } D_{62}(i = 1 \text{ or } 2)\) are functions of \(E'/Y\) and given in Appendix A. Fig. 4 shows variations between corresponding indentation parameters at given yield strength-elastic modulus ratios \(Y/E\) for dual indenters. It should be emphasized that we have the following relation in logarithmic scale:
Therefore, it could be concluded that linear features related to characteristics of \( P - h \) curves in instrumented sharp indentation (between \( S / (Ch_m) \), \( W_t / W_e \), and \( h_m / h_e \), and \( E / C \) for a single indenter; and between corresponding parameters for dual indenters: \( E / C_{60} \) and \( E / C_{70.3} \), \( S / (Ch_m)_{60} \) and \( S / (Ch_m)_{70.3} \), \( W_t / W_e \) and \( (W_t / W_e)_{70.3} \), \( (h_m / h_e)_{60} \) and \( (h_m / h_e)_{70.3} \)) at a given strain hardening exponent is also found in logarithmic scale at a given yield strength-elastic modulus ratio, \( Y/E \).

3.3. Further relationships with respect to dual indenters

It is demonstrated that \( C \) can be used instead of the reduced elastic modulus \( E' \) or yield strength \( Y \) to study characteristics parameters of \( P - h \) curves of a single indenter. Accordingly, one material parameter \( (Y \text{ or } n) \) is removed from established functions. However, two unknowns are always involved in such functions. It should be emphasized that \( E' / Y \) is mathematically considered here as one unknown. Therefore, Eqs. 5b, 5c, 5d and (15), which correlate the duality between corresponding parameters for dual indenters, contain only one unknown: \( n \) or \( E' / Y \). Consider now another possibility of deriving equations with one unknown. Le (2008) has established dimensionless relationships for \( W_t / W_e \) as functions of \( E' / C \) and \( n \) as follows:

\[
\frac{W_t}{W_e} = f_5 \left( \frac{E'}{C}, n \right)
\]

(17)

It can be seen that if \( C_{70.3} \) is used instead of \( E' \) in the set of variables \( (E' / C_{60} \text{ and } n) \) in Eq. (17), hence, \( C_{70.3} / C_{60} \) and \( n \) appear as variables in resulting dimensionless function \( (W_t / W_e)_{60} \) as below:

\[
\left( \frac{W_t}{W_e} \right)_{60} = f_{60} \left( \frac{C_{70.3}}{C_{60}}, n \right)
\]

(18a)

Conducting similar analysis yields:

\[
\left( \frac{W_t}{W_e} \right)_{70.3} = f_{70.3} \left( \frac{C_{60}}{C_{70.3}}, n \right)
\]

(18b)

Combining Eqs. (18a) and (18b), and noting the relation of \( Y/E \) with the loading curvature \( C \) and maximum indentation depth \( h_m \), a new dimensionless function is derived as below:

\[
\frac{C_{70.3}}{C_{60}} = \frac{C_{70.3}}{C_{60}} f_{60} \left( \frac{C_{70.3}}{C_{60}}, n \right) = f_7 \left( \frac{C_{70.3}}{C_{60}}, n \right)
\]

(19)

where \( C_a = W_t / (h_m)^2 \). Linear variations of \( C_{70.3} / C_{60} \) versus \( C_{70.3} / C_{60} \) are plotted in Fig. 5 for different values of \( n \). By using a least square fitting procedure, Eq. (19) is written as below:

\[
\frac{C_{70.3}}{C_{60}} = (a_1 n + a_0) \frac{C_{70.3}}{C_{60}} + (b_1 n + b_0)
\]

(20)

The coefficients in Eq. (20) are given in Appendix A. From Eq. (20), \( n \) is simply derived under an explicit functional form as below:

\[
n = \frac{C_{70.3}}{C_{60}} - b_0 \frac{C_{70.3}}{C_{60}} + b_1
\]

(21a)

or

\[
n = \frac{W_t}{W_e} - a_0 \frac{W_t}{W_e} - b_0 \left( \frac{h_m}{h_e} \right)^{\frac{3}{2}}
\]

(21b)

Eqs. (21) shows that \( n \) is fundamentally related to the ratio between total indentation works of dual indenters as well as this between elastic indentation works of dual indenters.
4. Hardness

4.1. Independent relationships

Elastic recovery around cone indentations in metals was studied experimentally by Stilwell and Tabor (1961), who observed that while there was little change in the diameter of an indentation during unloading, there was significant decrease in its depth. Thus, for metals there should be negligible difference between the residual projected contact areas, \( A \), and the true projected contact area under maximum indentation load, \( A_m \). Hence, the average contact pressure, \( p_{\text{ave}} \), is commonly identified with the material hardness, \( H \):

\[
H = p_{\text{ave}} = \frac{P}{A_m}
\]

(22)

Relationships between hardness, elastic modulus and characteristics of \( P-h \) curves have been previously investigated by several authors (Cheng et al., 2002; Cheng and Cheng, 2004; Oliver and Pharr, 2004; Alkorta et al., 2006). \( H/E \) and \( H/E^2 \) are commonly used to consider such relationships. Cheng et al. (2002) have previously established relationships between \( H/E \) and characteristic parameters of \( P-h \) curves for a wide range of the half included angles of indenters (60° ≤ \( \theta \) ≤ 80°) as follows:

\[
\frac{H}{E} = k \frac{W_e}{W_t}
\]

(23)

While Cheng et al. (2002) proposed that \( k \) is independent of material properties, \( k = 0.787/[1.50 \tan(\theta) + 0.327] \), Alkorta et al. (2006) showed that \( k \) depends significantly on strain hardening exponent \( n \).

Relation between \( H/(E)^2 \) and \( P/S^2 \), which is established by combining Eqs. (1) and (22), was also explored (Joslin and Oliver, 1990; Oliver and Pharr, 2004):

\[
\frac{P}{S^2} = \frac{\pi}{(2\beta)^2 (E)^2} \frac{H}{(E)^2}
\]

(24)

In the other hand, it exists one-to-one correspondent linear relationships between one and other among three dimensionless indentation parameters \( S/(Ch_m) \), \( W_t/W_e \), and \( h_m/h_e \) as indicated in Eqs. (3). Previous results (Joslin and Oliver, 1990; Cheng et al., 2002; Oliver and Pharr, 2004; Le, 2008) suggest to find independent relationships between hardness, reduced elastic modulus and three dimensionless indentation parameters \( S/(Ch_m) \), \( W_t/W_e \), and \( h_m/h_e \). Combining Eqs. (8) and (24) and regarding Eqs. (3b) and (3c) yield:

\[
\frac{S}{Ch_m} = \beta \frac{2}{\sqrt{\pi}} \frac{E}{\sqrt{CH}}
\]

(25a)

\[
\frac{W_t}{W_e} = \alpha \frac{2}{\sqrt{\pi}} \frac{E}{\sqrt{CH}}
\]

(25b)

\[
\frac{h_m}{h_e} = \gamma \frac{2}{\sqrt{\pi}} \frac{E}{\sqrt{CH}}
\]

(25c)

For the case of small deformation of an elastic material indented by a rigid axisymmetric punch of smooth profile, \( \beta \) is exactly 1. When this condition is violated due to non-axisymmetric indenters, or relatively tight cone angles, or elasto-plastic behavior of materials, \( \beta \) can deviate significantly from unity, see the recent review by Oliver and Pharr (2004). \( \beta \) is fairly dependent on the half included angle of conical indenters and Poisson’s ratio (Hay et al., 1999). Oliver and Pharr (2004) reported that for a Berkovich indenter \( \beta \) falls in the range 1.0225 ≤ \( \beta \) ≤ 1.05, and hence, \( \beta = 1.05 \) is highly recommended.

Fig. 6 shows linear variations of \( S/(Ch_m) \), \( W_t/W_e \), and \( h_m/h_e \) versus \( E^2/(CH)^{0.5} \). The correction factors \( \alpha, \beta, \gamma \) in Eqs. (25) are given in Appendix A. Their deviations fall within 5% for \( \theta = 60^\circ \) and 70.3°.

4.2. Dependent relationships

Since the dimensionless indentation parameter \( E^2/(CH)^{0.5} \) correlates linearly to each of three characteristics of \( P-h \) curves \( S/(Ch_m), W_t/W_e, \) and \( h_m/h_e \) as shown just above, all features, which are indicated for these three dimensionless parameters \( S/(Ch_m), W_t/W_e, \) and \( h_m/h_e \) in Section 3 and in Le (2008), are therefore found for \( E^2/(CH)^{0.5} \). Fig. 7 shows relationships between

![Fig. 6. Variations of characteristic parameters of P-h curves: S/(Chm), Wt/We, and hm/he versus E^2/(CH)^{0.5}.](image)

![Fig. 7. Relationships between E^2/(CH)^{0.5} and the loading curvature C with respect to: (a) the reduced elastic modulus E' at given strain hardening exponents n, and (b) the yield strength Y at given yield strength-elastic modulus ratios, Y/E.](image)
Table 1
Instrumented indentation data for aluminum alloys (Dao et al., 2001; Chollacoop et al., 2003).

<table>
<thead>
<tr>
<th>Aluminum alloys</th>
<th>C_{0.3} (GPa)</th>
<th>(S/Chm)_{70.3}</th>
<th>C_{60} (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al 6061-T6511</td>
<td>27.4</td>
<td>16.4455</td>
<td>11.27</td>
</tr>
<tr>
<td>Al 7075-T651</td>
<td>41.2</td>
<td>10.2256</td>
<td>17.60</td>
</tr>
</tbody>
</table>

\( E^*/(CH)^{0.5} \) and the loading curvature \( C \) with respect to the reduced elastic modulus \( E^* \) at given strain hardening exponents \( n \), and to the yield strength \( Y \) at given yield strength-elastic modulus ratios, \( Y/E \). Detail functional forms read:

\[
\frac{E^*}{\sqrt{CH}} = \frac{K_{H1}(n)}{C} + \frac{K_{H2}(n)}{C} \quad (26a)
\]

\[
\ln \left( \frac{E^*}{\sqrt{CH}} \right) = G_{H1}\left( \frac{E}{Y} \right) \ln \left( \frac{C}{Y} \right) + G_{H2}\left( \frac{E}{Y} \right) \quad (26b)
\]

The coefficients in Eqs. (26a) are given in Appendix A for \( \theta = 70.3^\circ \). Eqs. (26a) and (26b) give two hardness formulations, which are related to \( E^* \) and \( n \) and to \( E^* \) and \( Y \), respectively:

\[
H = \frac{E^*}{\sqrt{K_{H1}(n)E + K_{H2}(n)C}} \quad (27a)
\]

\[
H = \left( \frac{C}{Y} \right)^{F_{H1}} \exp\left[F_{H2} \right] \quad (27b)
\]

where \( F_{H1} = -2(G_{H1} + 1) \) and \( F_{H2} = 2(1 - G_{H2}) \).

Table 2
Reverse analysis results for Al 6061 and Al 7075 aluminum alloys.

<table>
<thead>
<tr>
<th>Material</th>
<th>( E^* ) (GPa)</th>
<th>% Error ( E^* )</th>
<th>( n )</th>
<th>( Y ) (MPa)</th>
<th>% Error ( Y )</th>
<th>( H ) (GPa)</th>
<th>% Error ( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Al 6061-T6511</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experiments</td>
<td>70.2</td>
<td>0.08</td>
<td>284</td>
<td>1.047</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub-problem 1</td>
<td>68.7</td>
<td>-2.14</td>
<td>0.0613</td>
<td>0.9307</td>
<td>-11.11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub-problem 2</td>
<td>69.15</td>
<td>-1.49</td>
<td>314.0</td>
<td>10.56</td>
<td>0.9144</td>
<td>-12.67</td>
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</tr>
<tr>
<td>Chollacoop et al. (2003)</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>70.1</td>
<td>-0.14</td>
<td>255.6</td>
<td>-10.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) Al 7075-T651</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experiments</td>
<td>73.4</td>
<td>0.122</td>
<td>500</td>
<td>1.741</td>
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<td>Sub-problem 1</td>
<td>67.78</td>
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<td>1.5520</td>
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<tr>
<td>Sub-problem 2</td>
<td>70.38</td>
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<td>6.95</td>
<td>1.5957</td>
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</tr>
<tr>
<td>Chollacoop et al. (2003)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>79.3</td>
<td>8.04</td>
<td>419.4</td>
<td>-16.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All deviations were computed as \( \frac{|X_{predicted} - X_{experiment}|}{X_{experiment}} \), where X represents a variable.
Table 4
Indentation parameters of the representative materials.

<table>
<thead>
<tr>
<th>Materials</th>
<th>C_{033} (GPa)</th>
<th>(W/1)_{3/60}</th>
<th>(W/1)_{7/93}</th>
<th>C_{033}/C_{01}</th>
<th>C_{033}/C_{00}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>17.981</td>
<td>34.772</td>
<td>26.274</td>
<td>2.031</td>
<td>6.687</td>
</tr>
<tr>
<td>Aluminum</td>
<td>4.207</td>
<td>125.730</td>
<td>78.000</td>
<td>2.480</td>
<td>6.004</td>
</tr>
<tr>
<td>Gold</td>
<td>9.442</td>
<td>56.819</td>
<td>36.976</td>
<td>2.354</td>
<td>6.164</td>
</tr>
<tr>
<td>Lead</td>
<td>1.251</td>
<td>108.291</td>
<td>63.306</td>
<td>2.659</td>
<td>6.492</td>
</tr>
<tr>
<td>Silver</td>
<td>15.721</td>
<td>33.004</td>
<td>22.728</td>
<td>2.265</td>
<td>5.365</td>
</tr>
<tr>
<td>tungsten</td>
<td>55.355</td>
<td>65.164</td>
<td>37.169</td>
<td>2.710</td>
<td>4.757</td>
</tr>
<tr>
<td>Iron</td>
<td>56.048</td>
<td>19.805</td>
<td>13.386</td>
<td>2.240</td>
<td>3.131</td>
</tr>
<tr>
<td>Titanium 1</td>
<td>30.352</td>
<td>28.840</td>
<td>17.939</td>
<td>2.435</td>
<td>3.916</td>
</tr>
<tr>
<td>Steel 1</td>
<td>60.025</td>
<td>25.826</td>
<td>15.906</td>
<td>2.453</td>
<td>3.988</td>
</tr>
<tr>
<td>Nickel</td>
<td>139.381</td>
<td>7.1984</td>
<td>5.3318</td>
<td>2.0288</td>
<td>2.7389</td>
</tr>
<tr>
<td>Steel 2</td>
<td>127.279</td>
<td>8.8941</td>
<td>6.2877</td>
<td>2.1140</td>
<td>2.9904</td>
</tr>
<tr>
<td>Titanium 2</td>
<td>57.630</td>
<td>12.7845</td>
<td>8.1510</td>
<td>2.3177</td>
<td>3.6353</td>
</tr>
<tr>
<td>Aluminum alloy</td>
<td>44.965</td>
<td>9.6982</td>
<td>6.3526</td>
<td>2.2346</td>
<td>3.4116</td>
</tr>
</tbody>
</table>

It is shown that inverse problems of three unknowns (E, Y, and n) are well decomposed into two sub-problems of two variables:

(E', n) and (E' and Y). Although the first sub-problem was well studied in (Le (2008)), it is performed here to verify new explicit formulations related to n, as well as to compare inverse results of E' and H obtained from two sub-problems. Therefore, the inverse analysis algorithm as depicted in Fig. 8 is here adopted excepting that n is evaluated by using Eqs. (21) rather than Eq. (5c).

22 representative materials were chosen for the inverse analysis. Their mechanical properties are listed in Table 3. Poisson’s ratio is taken as 0.3 otherwise it is noted. The first 16 materials in Table 3 correspond to usual metals and engineering alloys, which have been investigated as representative materials in previous works (Bucaille et al., 2003; Swaddiwudhipong et al., 2005; Cao et al., 2007; Luo and Lin, 2007). The last 6 materials are rare groups of mystical materials with fixed Poisson’s ratios (material 17 and 18, v = 0.3) and with varied Poisson’s ratios (materials 19, 20, 21,

Table 5
Inverse results of the strain hardening exponent n and yield strength Y for representative materials.

<table>
<thead>
<tr>
<th>Materials</th>
<th>Original data</th>
<th>Inverse results</th>
</tr>
</thead>
<tbody>
<tr>
<td>n (MPa)</td>
<td>Y (MPa)</td>
<td>Deviation of Y (MPa)</td>
</tr>
<tr>
<td>Copper</td>
<td>0.5</td>
<td>0.10</td>
</tr>
<tr>
<td>Aluminum</td>
<td>0.15</td>
<td>0.20</td>
</tr>
<tr>
<td>Gold</td>
<td>0.22</td>
<td>0.38</td>
</tr>
<tr>
<td>Lead</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td>Zinc</td>
<td>0.27</td>
<td>0.60</td>
</tr>
<tr>
<td>Silicon</td>
<td>0.005</td>
<td>0.50</td>
</tr>
<tr>
<td>Material 17</td>
<td>0.12</td>
<td>0.23</td>
</tr>
<tr>
<td>Material 18</td>
<td>0.4</td>
<td>0.8030</td>
</tr>
<tr>
<td>Material 21</td>
<td>0.1</td>
<td>0.60</td>
</tr>
<tr>
<td>Material 22</td>
<td>0.122</td>
<td>0.30</td>
</tr>
</tbody>
</table>

All deviations were computed as \(\frac{X_{\text{original}} - X_{\text{predictive}}}{X_{\text{original}}}\), where X represents a variable.

Fig. 9. Indentation load–depth curves of materials 19, 20, 21 and 22.

Table 6
Inverse results of the reduced elastic modulus E’ for representative materials.

<table>
<thead>
<tr>
<th>Materials</th>
<th>Original data</th>
<th>Inverse results</th>
</tr>
</thead>
<tbody>
<tr>
<td>E’ (GPa)</td>
<td>E1’ (GPa)</td>
<td>E2’ (GPa)</td>
</tr>
<tr>
<td>Copper</td>
<td>140.66</td>
<td>137.10</td>
</tr>
<tr>
<td>Aluminum</td>
<td>76.92</td>
<td>76.97</td>
</tr>
<tr>
<td>Gold</td>
<td>86.81</td>
<td>86.26</td>
</tr>
<tr>
<td>Lead</td>
<td>17.58</td>
<td>17.82</td>
</tr>
<tr>
<td>Silver</td>
<td>91.21</td>
<td>90.60</td>
</tr>
<tr>
<td>Titanium 1</td>
<td>131.87</td>
<td>131.85</td>
</tr>
<tr>
<td>Steel 1</td>
<td>37.77</td>
<td>32.90</td>
</tr>
<tr>
<td>Nickel</td>
<td>227.47</td>
<td>227.59</td>
</tr>
<tr>
<td>Titanium 2</td>
<td>327.77</td>
<td>320.73</td>
</tr>
<tr>
<td>Aluminum alloy</td>
<td>120.88</td>
<td>121.43</td>
</tr>
<tr>
<td>Ti–6Al–4V</td>
<td>76.92</td>
<td>76.98</td>
</tr>
<tr>
<td>Zinc</td>
<td>9.89</td>
<td>9.95</td>
</tr>
<tr>
<td>Silicon</td>
<td>117.58</td>
<td>117.36</td>
</tr>
<tr>
<td>Material 17</td>
<td>114.01</td>
<td>113.08</td>
</tr>
<tr>
<td>Material 18</td>
<td>109.89</td>
<td>109.95</td>
</tr>
<tr>
<td>Material 19</td>
<td>131.22</td>
<td>131.79</td>
</tr>
<tr>
<td>Material 20</td>
<td>131.87</td>
<td>131.62</td>
</tr>
<tr>
<td>Material 21</td>
<td>130.35</td>
<td>129.34</td>
</tr>
<tr>
<td>Material 22</td>
<td>129.06</td>
<td>127.24</td>
</tr>
</tbody>
</table>

All deviations were computed as \(\frac{X_{\text{original}} - X_{\text{predictive}}}{X_{\text{original}}}\), where X represents a variable.
and 22) for the dual indenters ($\theta_1 = 70.3^\circ$ and $\theta_2 = 60^\circ$) according to Chen et al. (2007).

Indentation data are thus numerically generated by FEA and then used as input for the inverse analysis in order to extract the mechanical properties and hardness of representative materials. Main indentation parameters of these representative materials are shown in Table 4.

Table 5 shows in general good inverse results for $n$ and $Y$. The highest error in yield strength $Y$ (around 15%) appears for copper. Other cases exhibit in general good inverse results for yield strength $Y$. The strain hardening exponent $n$ exhibits high deviations for some very low strain hardening materials (lead, tungsten, zinc and silicon) and for mystical materials with varied Poisson’s ratios (materials 19, 21, and 22). Materials 17 and 18 were well discussed in Le (2008) for the cases of fixed Poisson’s ratios. Fig. 9 shows $P$–$h$ curves of four mystical materials with varied Poisson’s ratios. It is found that mystical materials with varied Poisson’s ratios are the most severe since their $P$–$h$ curves are visually identical. In fact, their corresponding indentation parameters exhibit very low deviation (within 1.5%). Hence, Poisson’s ratio has a strong effect in such cases.

It is noted that $n$ is accurately estimated here for mystical materials with $\nu = 0.3$ (materials 17, 18 and 20). Therefore, it can be seen that a good improvement in inverse results is made if effect of Poisson’s ratio is taken into account.

Inverse results of reduced elastic modulus and hardness, which are obtained from two sub-problems ($E_1/C_1$ and $H_1$; and $E_2/C_2$ and $H_2$), are fairly different as clearly shown in Tables 6 and 7, respectively. However, it is found that the reduced elastic modulus $E$ and hardness $H$, which are obtained by averaging the corresponding solutions in two sub-problems, exhibit overall lower errors than those estimated in each separated one. Errors in all considered cases, including 6 mystical materials, appear within 2% and 6% for $E$ and $H$, respectively.

<table>
<thead>
<tr>
<th>Table 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse results of the hardness $H$ for representative materials.</td>
</tr>
<tr>
<td>Materials</td>
</tr>
<tr>
<td>$H$ (GPa)</td>
</tr>
<tr>
<td>Copper</td>
</tr>
<tr>
<td>Aluminum</td>
</tr>
<tr>
<td>Gold</td>
</tr>
<tr>
<td>Lead</td>
</tr>
<tr>
<td>Silver</td>
</tr>
<tr>
<td>Tungsten</td>
</tr>
<tr>
<td>Iron</td>
</tr>
<tr>
<td>Titanium 1</td>
</tr>
<tr>
<td>Steel 1</td>
</tr>
<tr>
<td>Nickel</td>
</tr>
<tr>
<td>Steel 2</td>
</tr>
<tr>
<td>Titanium 2</td>
</tr>
<tr>
<td>Aluminum alloy</td>
</tr>
<tr>
<td>Zinc</td>
</tr>
<tr>
<td>Silicon</td>
</tr>
<tr>
<td>Material 17</td>
</tr>
<tr>
<td>Material 18</td>
</tr>
<tr>
<td>Material 19</td>
</tr>
<tr>
<td>Material 20</td>
</tr>
<tr>
<td>Material 21</td>
</tr>
<tr>
<td>Material 22</td>
</tr>
</tbody>
</table>

All deviations were computed as $\frac{X_{prediction} - X_{FEA}}{X_{FEA}}$, where $X$ represents a variable.

<table>
<thead>
<tr>
<th>Table 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case study for sensitive analysis of representative materials.</td>
</tr>
<tr>
<td>Changes in the input data</td>
</tr>
<tr>
<td>Case 1</td>
</tr>
<tr>
<td>Case 2</td>
</tr>
<tr>
<td>Case 3</td>
</tr>
<tr>
<td>Case 4</td>
</tr>
</tbody>
</table>

Fig. 10. Sensitivity study for common metals and alloys: (a) yield strength $Y$, and (b) hardness $H$. M.-Q. Le / International Journal of Solids and Structures 46 (2009) 2988–2998 2995
common metals and alloys, and in Fig. 11 for 6 mystical materials. The maximum variation of deviations in absolute values in each case are depicted in Fig. 10 for uncertainties of input data trend to decrease when $E/Y$ of $Y$ and $E/Y$ is found to reach 52%, 55%, 85%, and ±1% errors in cases 1, 2, 3, and 4, respectively. High variation of $Y$ was also found by Swaddiwudhipong et al. (2005) for high strain hardening materials with large values of $E/Y$. The authors reported that due to ±2% errors in $C_{60}$ and $C_{70}$, and ±1% errors in $(W_p/W_t)_{60}$ and $(W_p/W_t)_{70}$, the maximum variation of $Y$ can reach approximately 70% for such materials.

For other materials with very low yield strength-elastic modulus ratio, $Y/E < 0.001$ (aluminum 1, gold, lead, and silver), the maximum variation of $Y$ fall within 32%, 35%, 56%, and 60% in cases 1, 2, 3, and 4, respectively. These four materials exhibit low and medium strain hardenings. Errors in $Y$ for the last materials ($Y/E > 0.001$) appear within 25%, 27%, 38%, and 41% in cases 1, 2, 3, and 4, respectively.

Mystical materials are defined by low deviations between their indentation data of an interested material may probably provide inverse results matching to another material. These two materials may become candidates for a pair of mystical materials, which must exhibit low deviations in their hardness. This correlates to low error sensitivity in hardness (within 16% in the most severe case) as shown in Fig. 10b and 11b. It should be emphasized that Vickers hardness estimated by the contact area can reach errors up to 15% due to the typical friction between the indenter and specimen (Mata and Alcalà, 2004).

### 5.4. Sensitivity analysis

Sensitivity analysis for $E/Y$ and $n$ was well performed in Le (2008) and hence is not repeated here. Four cases of perturbations of the input data are considered in this work as tabulated in Table 8. Sensitivity analysis was carried out for 22 representative materials. Maximum deviations in absolute values in each case are depicted in Fig. 10 for common metals and alloys, and in Fig. 11 for 6 mystical materials.

Overall, it is seen in Fig. 10a and 11a that errors in $Y$ due to uncertainties of input data trend to decrease when $E/Y$ and $n$ decrease. The maximum variation of $Y$ is very high for copper with $n = 0.5$ and exceptionally low value of $Y/E$. Errors in $Y$ reach 52%, 55%, 85%, and 89% for copper in cases 1, 2, 3, and 4, respectively. High variation of $Y$ was also found by Swaddiwudhipong et al. (2005) for high strain hardening materials with large values of $E/Y$. The authors reported that due to ±2% errors in $C_{60}$ and $C_{70}$, and ±1% errors in $(W_p/W_t)_{60}$ and $(W_p/W_t)_{70}$, the maximum variation of $Y$ can reach approximately 70% for such materials.

For other materials with very low yield strength-elastic modulus ratio, $Y/E < 0.001$ (aluminum 1, gold, lead, and silver), the maximum variation of $Y$ fall within 32%, 35%, 56%, and 60% in cases 1, 2, 3, and 4, respectively. These four materials exhibit low and medium strain hardenings. Errors in $Y$ for the last materials ($Y/E > 0.001$) appear within 25%, 27%, 38%, and 41% in cases 1, 2, 3, and 4, respectively.

Mystical materials are defined by low deviations between their corresponding indentation parameters (Chen et al., 2007). Therefore, features of mystical materials should be explored to deeply interpret the sensitivity characteristics of inverse results as made in Le (2008) because several percent of indentation error is very common in practice. Le (2008) showed that low sensitivity in elastic modulus is related to the fact that elastic modulus of mystical materials must be fairly different. Since, differences in corresponding indentation parameters and elastic modulus are low for mystical materials, their hardness must be also fairly different according to Eqs. (25). This is clearly seen in Table 7 as hardness estimated by FEA exhibit deviations within 10% for 6 mystical materials (materials 17 and 18; and materials 19, 20, 21 and 22). Consequently, not only their elastic modulus but also their hardness can be accurately estimated even when their Poisson ratio are varied.

In the other hand, it can be supposed that the perturbation of indentation data of an interested material may probably provide inverse results matching to another material. These two materials may become candidates for a pair of mystical materials, which must exhibit low deviations in their hardness. This correlates to low error sensitivity in hardness (within 16% in the most severe case) as shown in Fig. 10b and 11b. It should be emphasized that Vickers hardness estimated by the contact area can reach errors up to 15% due to the typical friction between the indenter and specimen (Mata and Alcalà, 2004).

### 6. Conclusions

In the present work, Le’s approach (Le, 2008) is extended to investigate several fundamental issues in instrumented sharp indentation. The yield strength and hardness are especially focused. The main results are summarized as follows:

- It is demonstrated that C can be used not only instead of the reduced elastic modulus $E/Y$ but also instead of the yield strength $Y$ to formulate useful dimensionless functions associating material elasto-plastic properties with characteristics parameters of $P-h$ curves of single indenters. Relationships between corresponding parameters are further developed for dual indenters with respect to $E/Y$. Linear features of formulated functions in normal scale at a given strain hardening exponent $n$ as indicated in Le (2008) are also found in logarithmic scale at a given yield strength-elastic modulus ratio $Y/E$.

- The dimensionless indentation parameter $E_p/(CH)^{0.5}$ is found to correlate linearly to each of three characteristic parameters of $P-h$ curves $(S/(Ch_a), W_t/W_r$, and $h_m/h_b)$. As a result, $E_p/(CH)^{0.5}$ exhibit similar features of these three dimensionless parameters. Hardness is functionally related to the loading curvature $C$ under explicit forms, which are alternatively expressed as functions of $E/Y$ and $n$, or of $E/Y$ and $Y$.

- Although indentation response is governed by all material properties ($E/Y$, $n$, and $E/Y$), the present approach allows formulating dimensionless functions, in which at least one material variable is absent. It leads to a decomposition of inverse problems with three unknowns ($E/Y$, $n$, and $E/Y$) into two sub-problems of two unknowns: ($E/Y$ and $n$), and ($E/Y$ and $Y$). $n$ is explicitly formulated as functions of indentation parameters of dual indenters. An inverse analysis procedure based on dual indenters is suggested for material characterization, giving good inverse results for experimental data from the literature and various representative materials.

- It is shown that mystical materials exhibit not only fair differences in their elastic modulus (Chen et al., 2007) but also in their hardness. These two features are related to low variations of elastic modulus and hardness due to uncertainties of input data. By considering small deviations in corresponding indentation parameters of such materials as perturbation, inverse solution can be obtained in such severe cases without any special treatment. Overall, comprehensive sensitivity analyses show that the proposed method is quite robust and can be applied to a wide range of materials.

### Acknowledgements

The author is grateful to the Alexander von Humboldt foundation of Germany for the award of a post doctoral fellowship and Hanoi University of Technology, Viet nam for granting study leave.
Appendix A

See Table A1.

Table A1
The material elasto-plastic properties used in the computations.

<table>
<thead>
<tr>
<th>$E$ (GPa)</th>
<th>$Y$ (MPa)</th>
<th>$Y/E$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>8</td>
<td>5.0E-5</td>
<td>0.3, 0.35, 0.4, 0.45, 0.5, 0.6</td>
</tr>
<tr>
<td>200</td>
<td>15</td>
<td>7.5E-5</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>1.0E-4</td>
<td>0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6</td>
</tr>
<tr>
<td>100</td>
<td>25</td>
<td>2.5E-4</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>100</td>
<td>5.0E-4</td>
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<tr>
<td>100</td>
<td>75</td>
<td>7.5E-4</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
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<td>3120</td>
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<td>10</td>
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<td></td>
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<tr>
<td>50</td>
<td>2000</td>
<td>0.04</td>
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<tr>
<td>100</td>
<td>5000</td>
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<td></td>
</tr>
<tr>
<td>100</td>
<td>6000</td>
<td>0.06</td>
<td></td>
</tr>
</tbody>
</table>

Poisson’s ratio is fixed at 0.3, resulting a total of 159 different cases.

It is noted that $y = \ln(E'/Y)$. The coefficients in Eqs. (13) for $\theta = 60^\circ$:

\[
\begin{align*}
&k_s = 1.5843; G_s = 0.003329y^2 - 0.08649y + 0.3774; \\
&k_u = 1.0959; G_u = 0.008224y^2 - 0.16387y - 0.0739; \\
&k_o = 1.1212; G_o = 0.011676y^2 - 0.21682y + 0.1355; \\
\end{align*}
\]

and for $\theta = 70.3^\circ$:

\[
\begin{align*}
&k_s = 2.4485; G_s = 0.002907y^2 - 0.085y - 0.3497; \\
&k_u = 1.7011; G_u = 0.008344y^2 - 0.17407y + 0.0187; \\
&k_o = 1.6446; G_o = 0.012663y^2 - 0.24363y + 0.3137; \\
\end{align*}
\]

The coefficients in Eq. (14):

\[
\begin{align*}
D_{11}' &= -0.009335y^4 + 0.031705y^4 - 0.416863y^4 + 2.65862y^4 - 8.3153y + 11.7252; \\
D_{12}' &= 0.003591y^4 - 0.120617y^4 + 1.56169y^4 - 9.7068y^4 + 29.124y - 36.605. \\
\end{align*}
\]

The coefficients in Eqs. (15):

\[
\begin{align*}
D_{11}' &= 0.00020747y^4 - 0.0085146y^4 + 0.141935y^4 - 1.2286y^4 + 5.83388y^2 - 14.5582y + 16.5259; \\
D_{12}' &= -0.00021811y^4 - 0.0091307y^4 - 0.154224y^4 - 1.3423y^4 - 6.3165y^4 + 15.2935y - 15.3867; \\
D_{13}' &= 0.00012035y^4 - 0.005849y^4 + 0.087454y^4 - 0.782311y^4 + 3.84029y^4 - 9.8914y + 11.8802; \\
D_{21}' &= -0.0003122y^4 - 0.001539y^4 - 0.029809y^4 + 0.28969y^4 - 1.4783y + 3.7169y - 3.6294; \\
D_{22}' &= 0.00010927y^4 - 0.0044428y^4 + 0.07365y^4 - 0.63571y^4 + 3.0087y^4 - 7.4352y + 8.7867; \\
D_{23}' &= -0.00008225y^4 + 0.003395y^4 - 0.057046y^4 + 0.49716y^4 - 2.34941y^4 + 5.6387y - 5.3108; \\
\end{align*}
\]

The coefficients in Eq. (20): $a_0 = -2.6554; a_i = -1.3635; b_0 = -2.4096; b_1 = 2.1515; c_0 = 0.84866; b_0 = 1.0769; g = 0.92896; \text{ for } \theta = 60^\circ; c_0 = 0.66456; b_0 = 1.0593; g = 0.88432; \text{ for } \theta = 70.3^\circ.$

The coefficients in Eqs. (26) for $\theta = 70.3^\circ$:

\[
\begin{align*}
K_{11} &= 1.7216n^2 - 4.3917n + 6.3351; K_{12} = -1.567n^2 + 3.2533n - 1.8145; \\
G_{11} &= -0.001955y^3 + 0.027714y^2 - 0.003591y - 1.90925; \\
G_{12} &= 0.019041y^3 - 0.376089y^2 + 3.05565y + 0.4841. \\
\end{align*}
\]
References


