Weighted sharing of a small function by a meromorphic function and its derivative

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Received 13 June 2006; received in revised form 19 September 2006; accepted 11 October 2006

Abstract


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Keywords: Meromorphic function; Derivative; Small function; Weighted sharing

1. Introduction definitions and results

Let \(f\) and \(g\) be two nonconstant meromorphic functions defined in the open complex plane \(\mathbb{C}\). If, for some \(a \in \mathbb{C} \cup \{\infty\}\), \(f - a\) and \(g - a\) have the same set of zeros with the same multiplicities, we say that \(f\) and \(g\) share the value \(a\) CM (counting multiplicities), and if we do not consider the multiplicities, then \(f\) and \(g\) are said to share the value \(a\) IM (ignoring multiplicities).

A meromorphic function \(a\) is said to be a small function of \(f\) where \(T(r, a) = S(r, f)\), that is \(T(r, a) = o(T(r, f))\) as \(r \to \infty\), outside of a possible exceptional set of finite linear measure.

We denote by \(T(r)\) the maximum of \(T(r, f)\) and \(T(r, g)\). The notation \(S(r)\) denotes any quantity satisfying \(S(r) = o(T(r))\) as \(r \to \infty\), outside of a possible exceptional set of finite linear measure.

Also, we use \(I\) to denote any set of infinite linear measure in \(0 < r < \infty\).

In 1979, Mues and Steinmetz proved the following theorem.

**Theorem A** ([10]). Let \(f\) be a nonconstant entire function. If \(f\) and \(f'\) share two distinct values \(a, b\) IM, then \(f' = f\).

Considering the uniqueness problem of an entire function sharing one value with its derivative, the following result was proved in [3].

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Theorem B ([13]). Let \( f \) be a nonconstant entire function. If \( f \) and \( f' \) share the value 1 CM and if \( N(r, 0; f') = S(r, f) \), then \( \frac{f' - 1}{f - 1} \) is a nonzero constant.

Now it is a natural query whether the value 1 of Theorem B can be simply replaced by a small function \( a \) \((\neq 0, \infty)\). The following example shows that the answer is negative.

Example 1.1. Let \( f = 1 + e^z \) and \( a(z) = \frac{1}{1 - e^z} \).

By Lemma 2.6 of [4, p. 50] we know that \( a \) is a small function of \( f \). Also, it can be easily seen that \( f \) and \( f' \) share a CM and \( N(r, 0; f') = 0 \), but \( f - a \neq c (f' - a) \) for every nonzero constant \( c \). We note that \( f - a = e^{-z} (f' - a) \).

So, in order to replace the value 1 by a small function, some extra conditions are required.


Theorem C ([11]). Let \( f \) be a nonconstant entire function of finite order and let \( a \) \((\neq 0)\) be a finite constant. If \( f \), \( f^{(k)} \) share the value \( a \) CM, then \( \frac{f^{(k)} - a}{f - a} \) is a nonzero constant, where \( k \) \((\geq 1)\) is an integer.

Zhang [14] proved the following theorem.

Theorem D ([14]). Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share the value 1 CM, and if

\[
N(r, \infty; f) + N(r, 0; f') < (\lambda + o(1)) T(r, f')
\]

for some real constant \( \lambda \in (0, \frac{1}{2}) \), then \( \frac{f' - 1}{f - 1} \) is a nonzero constant.

Clearly for entire function condition (1.1) reduces to \( N(r, 0; f') < (2\lambda T(r, f) + S(r, f)) \), and hence the condition is weaker than that of Brück’s [3], and so it is an improvement of the result of Brück.

To state the next results, we require the following definition, known as weighted sharing of values, which measure how close a shared value is to be shared IM or to be shared CM.

Definition 1.1 ([5,6]). Let \( k \) be a nonnegative integer or infinity. For \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), then we say that \( f, g \) share the value \( a \) with weight \( k \).

The definition implies that if \( f, g \) share a value \( a \) with weight \( k \), then \( z_0 \) is an \( a \)-point of \( f \) with multiplicity \( m \) \((\leq k)\) if and only if it is an \( a \)-point of \( g \) with multiplicity \( n \) \((\leq k)\) and \( z_0 \) is an \( a \)-point of \( f \) with multiplicity \( m \) \((> k)\) if and only if it is an \( a \)-point of \( g \) with multiplicity \( n \) \((> k)\), where \( m \) is not necessarily equal to \( n \).

We write \( f, g \) share \( (a, k) \) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \( (a, k) \), then \( f, g \) share \( (a, p) \) for any integer \( p \), \( 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \( (a, 0) \) or \( (a, \infty) \) respectively.

If \( a \) is a small function, we accordingly define \( f \) and \( g \) share \( a \) IM or \( a \) CM or with weight \( l \) as \( f - a \) and \( g - a \) share \( (0, 0) \) or \( (0, \infty) \) or \( (0, l) \) respectively.

Though we use the standard notations and definitions of the value distribution theory available in [4], we explain some definitions and notations which are used in the paper.

Definition 1.2 ([8]). Let \( p \) be a positive integer and \( a \in \mathbb{C} \cup \{\infty\} \).

(i) \( N(r, a; f \mid \geq p)(\overline{N}(r, a; f \mid \geq p)) \) denotes the counting function (reduced counting function) of those \( a \)-points of \( f \) whose multiplicities are not less than \( p \).

(ii) \( N(r, a; f \mid \leq p)(\overline{N}(r, a; f \mid \leq p)) \) denotes the counting function (reduced counting function) of those \( a \)-points of \( f \) whose multiplicities are not greater than \( p \).

Definition 1.3 (6. Cf. [12]). For \( a \in \mathbb{C} \cup \{\infty\} \) and a positive integer \( p \) we denote by \( N_p(r, a; f) \) the sum

\[
\overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \ldots \overline{N}(r, a; f \mid \geq p).
\]

Clearly \( N_1(r, a; f) = \overline{N}(r, a; f) \).

Definition 1.4 ([7]). Let \( a, b \in \mathbb{C} \cup \{\infty\} \). We denote by \( N(r, a; f \mid g = b) \) the counting function of those \( a \)-points of \( f \), counted according to multiplicity, which are \( b \)-points of \( g \).
Definition 1.5 ([7]). Let \( a, b \in \mathbb{C} \cup \{\infty\} \). We denote by \( N(r, a; f \mid g \neq b) \) the counting function of those \( a \)-points of \( f \), counted according to multiplicity, which are not the \( b \)-points of \( g \).

Definition 1.6 ([15]). For a positive integer \( p \) and \( a \in \mathbb{C} \cup \{\infty\} \), we put
\[
\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}
\]
Clearly \( 0 \leq \delta(a; f) \leq \delta_p(a; f) \leq \delta_{p-1}(a; f) \ldots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f) \).

Definition 1.7 (Cf. [11], 2). Let \( f \) and \( g \) be two nonconstant meromorphic functions such that \( f \) and \( g \) share the value 1 IM. Let \( z_0 \) be a 1-point of \( f \) with multiplicity \( p \), and a 1-point of \( g \) with multiplicity \( q \). We denote by \( \overline{N}_L(r, 1; f) \) the counting function of those 1-points of \( f \) and \( g \) where \( p > q \), by \( \overline{N}_E^1(r, 1; f) \) the counting function of those 1-points of \( f \) and \( g \) where \( p = q = 1 \), and by \( \overline{N}_E^2(r, 1; f) \) the counting function of those 1-points of \( f \) and \( g \) where \( p = q \geq 2 \), each point in these counting functions is counted only once. In the same way, we can define \( \overline{N}_L(r, 1; g), \overline{N}_E^1(r, 1; g), \overline{N}_E^2(r, 1; g) \).

Definition 1.8 (Cf. [11], 2). Let \( k \) be a positive integer. Let \( f \) and \( g \) be two nonconstant meromorphic functions such that \( f \) and \( g \) share the value 1 IM. Let \( z_0 \) be a 1-point of \( f \) with multiplicity \( p \), and a 1-point of \( g \) with multiplicity \( q \). We denote by \( \overline{N}_{g=k}(r, 1; g) \) the reduced counting function of those 1-points of \( f \) and \( g \) such that \( p > q = k \). \( \overline{N}_{g=k}(r, 1; f) \) is defined analogously.

Definition 1.9 ([5,6]). Let \( f, g \) share a value a IM. We denote by \( \overline{N}_s(r, a; f, g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \).

Clearly, \( \overline{N}_s(r, a; f, g) \equiv \overline{N}_s(r, a; g, f) \) and \( \overline{N}_s(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g) \).

With the notion of weighted sharing of values, improving the results of Zhang [14], Lahiri–Sarkar [8] obtained the following two theorems.

Theorem E ([14]). Let \( f \) be a nonconstant meromorphic function and \( k \) be a positive integer. If \( f \) and \( f^{(k)} \) share (1, 2) and
\[
2\overline{N}(r, \infty; f) + N_2(r, 0; f^{(k)}) + N_2(r, 0; f) < (\lambda + o(1))T(r, f^{(k)})
\]
for \( r \in I \), where \( 0 < \lambda < 1 \), then \( \frac{f^{(k-1)}}{f-1} = c \) for some constant \( c \in \mathbb{C}/\{0\} \).

Theorem F ([14]). Let \( f \) be a nonconstant meromorphic function and \( k \) be a positive integer. If \( f \) and \( f^{(k)} \) share (1, 1) and
\[
2\overline{N}(r, \infty; f) + N_2(r, 0; f^{(k)}) + 2\overline{N}(r, 0; f) < (\lambda + o(1))T(r, f^{(k)})
\]
for \( r \in I \), where \( 0 < \lambda < 1 \), then \( \frac{f^{(k-1)}}{f-1} = c \) for some constant \( c \in \mathbb{C}/\{0\} \).

Recently Zhang [15] extended the result of Lahiri and Sarkar to a small function, and proved the following.

Theorem G ([15]). Let \( f \) be a nonconstant meromorphic function and \( k \) \((\geq 1), l \) \((\geq 0)\) be integers. Also let \( a \equiv a(z) \) \((\neq 0, \infty)\) be a meromorphic small function. Suppose that \( f - a \) and \( f^{(k)} - a \) share (0, 1). If \( l \) \((\geq 2)\) and
\[
2\overline{N}(r, \infty; f) + N_2(r, 0; f^{(k)}) + N_2(r, 0; (f/a)') < (\lambda + o(1))T(r, f^{(k)}) \tag{1.2}
\]
or \( l = 1 \) and
\[
2\overline{N}(r, \infty; f) + N_2(r, 0; f^{(k)}) + 2\overline{N}(r, 0; (f/a)') < (\lambda + o(1))T(r, f^{(k)}) \tag{1.3}
\]
or \( l = 0 \) and
\[
4N(r, \infty; f) + 3N_2(r, 0; f^{(k)}) + 2N_2(r, 0; (f/a)') < (\lambda + o(1)) T(r, f^{(k)})
\]  
for \( r \in I \), where \( 0 < \lambda < 1 \) then \( \frac{f^{(k)} - a}{f - a} = c \) for some constant \( c \in \mathbb{C}/\{0\} \).

In the present paper, we improve Theorem G by replacing the three given conditions (1.2)–(1.4) by three weaker ones. Also, we shall show that if \( l \geq k \), a better result can be obtained at the cost of considering \( a \) (\( \not\equiv 0, \infty \)) to be simply a constant.

Yu [13] recently considered the uniqueness problem of an entire or meromorphic function when it shares a small function with its derivative. Yu proved the following two theorems.

**Theorem H** ([13]). Let \( f \) be a nonconstant entire function and \( a \equiv a(z) (\not\equiv 0, \infty) \) be a meromorphic small function. If \( f - a \) and \( f^{(k)} - a \) share \( 0 \) CM and \( \delta(0; f) > \frac{3}{4} \), then \( f \equiv f^{(k)} \).

**Theorem I** ([13]). Let \( f \) be a nonconstant non-entire meromorphic function and \( a \equiv a(z) (\not\equiv 0, \infty) \) be a meromorphic small function. If

(i) \( f \) and \( a \) have no common poles.

(ii) \( f - a \) and \( f^{(k)} - a \) share the value \( 0 \) CM.

(iii) \( 4\delta(0; f) + 2(8 + k)\Theta(\infty; f) > 19 + 2k \)

then \( f \equiv f^{(k)} \) where \( k \) is a positive integer.

In the same paper, Yu [13] posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value?

(ii) Can the condition \( \delta(0; f) > \frac{3}{4} \) of Theorem H be further relaxed?

(iii) Can the condition (iii) in Theorem I be further relaxed?

(iv) Can in general the condition (i) of Theorem I be dropped?

In 2004, Lahiri and Sarkar [8] gave some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of \( a \). But they did not provide any definite answer corresponding to the question (i) of Yu as mentioned above. Rather, they confined their investigations of sharing to small functions up to weight 2.

Recently, Zhang [15] studied the problem of meromorphic or entire functions sharing one small function, and improved the results of Lahiri and Sarkar [8], answering all the four open questions of Yu.

Zhang proved the following theorems.

**Theorem J** ([15]). Let \( f \) be a nonconstant meromorphic function and \( k (\geq 1), l (\geq 0) \) be integers. Also let \( a \equiv a(z) (\not\equiv 0, \infty) \) be a meromorphic small function. Suppose that \( f - a \) and \( f^{(k)} - a \) share \((0, 1)\). If \( l (\geq 2) \) and

\[
(3 + k) \Theta(\infty, f) + 2\delta_{2+k}(0; f) > k + 4
\]  

or \( l = 1 \) and

\[
(4 + k) \Theta(\infty, f) + 3\delta_{2+k}(0; f) > k + 6
\]  

or \( l = 0 \) and

\[
(6 + 2k) \Theta(\infty, f) + 5\delta_{2+k}(0; f) > 2k + 10
\]  

then \( f \equiv f^{(k)} \).

In the present paper, we shall improve Theorem J by replacing the conditions (1.6) and (1.7) by two weaker ones, and thus provide a better answer to the first question of Yu than that of Zhang. However the author does not know whether the condition (1.5) can further be weakened.

The following three theorems are the main results of the paper.
Theorem 1.1. Let $f$ be a nonconstant meromorphic function, and $k \geq 1$, $l \geq 1$ be integers, and $a \not\equiv a(z)$ be a nonconstant meromorphic function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq k$ and
\[ 2N(r, \infty; f) + N_2(r, 0; f^{(k)}) + N(r, 0; f') < (\lambda + o(1)) \Theta(r, f^{(k)}) \] (1.8)
for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)} - a}{f - a}$ is a nonzero constant.

Theorem 1.2. Let $f$ be a nonconstant meromorphic function, and $k \geq 1$, $l \geq 0$ be integers, and $a \equiv a(z)$ be a nonconstant meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and
\[ 2N(r, \infty; f) + N_2(r, 0; f^{(k)}) + N_2(r, 0; (f/a)') - N(r, 0; (f/a)') \geq 3 < (\lambda + o(1)) T(r, f^{(k)}) \] (1.9)
or $l = 1$ and
\[ 2N(r, \infty; f) + N_2(r, 0; f^{(k)}) + 2N(r, 0; (f/a)') - N(r, 0; (f/a)') \geq 2 < (\lambda + o(1)) T(r, f^{(k)}) \] (1.10)
or $l = 0$ and
\[ 4N(r, \infty; f) + 2N_2(r, 0; f^{(k)}) + N(r, 0; f^{(k)} |\equiv 1) + 2N(r, 0; (f/a)') \] (1.11)
for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)} - a}{f - a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

Theorem 1.3. Let $f$ be a nonconstant meromorphic function and $k \geq 1, l \geq 0$ be integers, and $a \equiv a(z)$ be a nonconstant meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l = 1$ and
\[ \left(\frac{7}{2} + k\right) \Theta(\infty; f) + \frac{3}{2} \delta_2(0; f) + \delta_{2+k}(0; f) > k + 5 \] (1.12)
or $l = 0$ and
\[ (6 + 2k) \Theta(\infty; f) + 2 \Theta(0; f) + \delta_2(0; f) + \delta_{1+k}(0; f) + \delta_{2+k}(0; f) > 2k + 10, \] (1.13)
then $f \equiv f^{(k)}$.

From Theorem 1.3 we have the following corollary.

Corollary 1.1. Let $f$ be a nonconstant entire function, and $a \equiv a(z) \not\equiv 0, \infty$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share $(0, 1)$, then $\delta_{2+k}(0; f) > \frac{3}{5}$ or if $f - a$ and $f^{(k)} - a$ share $(0, 0)$ and $\delta_{2+k}(0; f) > \frac{4}{5} - \frac{1}{5} [2 \Theta(0; f) + \delta_2(0; f) + \delta_{1+k}(0; f) - 4 \delta_{2+k}(0; f)]$ then $f \equiv f^{(k)}$.

Clearly Corollary 1.1 provides a better answer corresponding to the first question of Yu than that given by Zhang. It also provides the answer corresponding to the second and third questions of Yu.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let $F, G$ be two nonconstant meromorphic functions. Henceforth, we shall denote by $H$ the following function.

\[ H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right). \] (2.1)

Lemma 2.1 ([15]). Let $f$ be a nonconstant meromorphic function and $p, k$ be positive integers; then
\[ N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + kN(r, \infty; f) + S(r, f). \]
Lemma 2.2 ([1]). If \( f, g \) be two nonconstant meromorphic functions such that they share \((1, 1)\), then
\[
2\overline{\nu}_L(r, 1; f) + 2\overline{\nu}_L(r, 1; g) + \overline{\nu}_E^2(r, 1; f) - \overline{\nu}_{f>2}(r, 1; g) \leq \overline{\nu}(r, 1; g) - \overline{\nu}(r, 1; g).
\]

Lemma 2.3 ([2]). Let \( f, g \) share \((1, 1)\). Then
\[
\overline{\nu}_{f>2}(r, 1; g) \leq \frac{1}{2}\overline{\nu}(r, 0; f) + \frac{1}{2}\overline{\nu}(r, \infty; f) - \frac{1}{2}\overline{\nu}_0(r, 0; f') + S(r, f).
\]

Lemma 2.4 ([2]). Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing \((1, 0)\). Then
\[
\overline{\nu}_L(r, 1; f) + 2\overline{\nu}_L(r, 1; g) + \overline{\nu}_E^2(r, 1; f) - \overline{\nu}_{f>1}(r, 1; g) - \overline{\nu}_{g>1}(r, 1; f) \leq \overline{\nu}(r, 1; g) - \overline{\nu}(r, 1; g).
\]

Lemma 2.5 ([2]). Let \( f, g \) share \((1, 0)\). Then
\[
\overline{\nu}_L(r, 1; f) \leq \overline{\nu}(r, 0; f) + \overline{\nu}(r, \infty; f) + S(r).
\]

Lemma 2.6 ([2]). Let \( f, g \) share \((1, 0)\). Then
\[
(i) \overline{\nu}_{f>1}(r, 1; g) \leq \overline{\nu}(r, 0; f) + \overline{\nu}(r, \infty; f) - \overline{\nu}_0(r, 0; f') + S(r, f)
\]
\[
(ii) \overline{\nu}_{g>1}(r, 1; f) \leq \overline{\nu}(r, 0; g) + \overline{\nu}(r, \infty; g) - \overline{\nu}_0(r, 0; f') + S(r, g).
\]

Lemma 2.7 ([9]). Let \( f \) be a nonconstant meromorphic function and let
\[
R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}
\]
be an irreducible rational function in \( f \) with constant coefficients \( \{a_k\} \) and \( \{b_j\} \) where \( a_n \neq 0 \) and \( b_m \neq 0 \). Then
\[
T(r, R(f)) = dT(r, f) + S(r, f),
\]
where \( d = \max\{n, m\} \).

Lemma 2.8 (p. 68, [4]). Suppose that \( f \) is meromorphic and transcendental in the plane and that
\[
f^n P = Q
\]
where \( P \) and \( Q \) are differential polynomials in \( f \) and the degree of \( Q \) is at most \( n \). Then
\[
m[r, P] = S(r, f) \quad \text{as} \quad r \to +\infty
\]

3. Proofs of the theorems

Proof of Theorem 1.1. Let \( F = \frac{f}{a} \) and \( G = \frac{f^{(k)}}{a} \). It follows that \( F \) and \( G = F^{(k)} \) share \((1, l)\). It is clear that \( F \) does not possess any 1-point with multiplicity greater than \( k \). Since the set of zeros of \( F - 1 \) and \( F^{(k)} - 1 \) are equal up to multiplicity \( l \) (\( \geq k \)), it follows that \( F \) and \( F^{(k)} \) practically share \((1, \infty)\). Hence \( \overline{\nu}_a(r, 1; F, G) = 0 \).

Case 1. Let \( H \neq 0 \).

From (2.1), it can be easily calculated that the possible poles of \( H \) occur at: (i) multiple zeros of \( F \) and \( G \), (ii) those 1 points of \( F \) and \( G \) whose multiplicities are different (iii) those poles of \( F \) and \( G \) whose multiplicities are different, (iv) zeros of \( F'G' \) which are not the zeros of \( F(F-1)(G(G-1)) \).

Since \( H \) has only simple poles, we get
\[
\overline{\nu}(r, \infty; H) \leq \overline{\nu}(r, \infty; F) + \overline{\nu}_a(r, 1; F, G) + \overline{\nu}(r, 0; F |\geq 2) + \overline{\nu}(r, 0; G |\geq 2) + \overline{\nu}_0(r, 0; F') + \overline{\nu}_0(r, 0; G').
\]

(3.1)
where \( N_0(r, 0; F') \) is the reduced counting function of those zeros of \( F' \) which are not the zeros of \( F(F - 1) \), and \( \overline{N}_0(r, 0; G') \) is similarly defined.

Let \( z_0 \) be a simple zero of \( F - 1 \). Then, by a simple calculation, we see that \( z_0 \) is a zero of \( H \), and hence

\[
N_E^{(3)}(r, 1; F) = N(r, 1; F \mathcal{H} 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, F).
\]

By the second fundamental theorem, we get

\[
T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1; G) - N_0(r, 0; G') + S(r, G).
\]

So from (3.1)–(3.3), and noting that \( N(r, 1; F \mathcal{H} 1) = N(r, 1; G \mathcal{H} 1) \), we get

\[
T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1; G \mathcal{H} 1) + \overline{N}(r, 1; G \mathcal{H} 2) - N_0(r, 0; G') + S(r, G)
\]

\[
\leq 2\overline{N}(r, \infty; F) + N_2(r, 0; G) + \overline{N}(r, 0; F \mathcal{H} 2) + \overline{N}(r, 1; F \mathcal{H} 2) + \overline{N}_0(r, 0; F') + S(r, f).
\]

We note that

\[
\overline{N}(r, 0; F \mathcal{H} 2) + \overline{N}(r, 1; F \mathcal{H} 2) + \overline{N}_0(r, 0; F') \leq \overline{N}(r, 0; F').
\]

So, from (3.4), we get

\[
T \left( r, f^{(k)} \right) \leq 2\overline{N}(r, \infty; f) + N_2(r, 0; f^{(k)}) + \overline{N}(r, 0; (f/a)') + S(r, f),
\]

which contradicts (1.8).

Case 2. Let \( H \equiv 0 \).

On integration, we get from (2.1)

\[
\frac{1}{F - 1} \equiv \frac{C}{G - 1} + D,
\]

where \( C, D \) are constants and \( C \neq 0 \). If \( z_0 \) is a pole of \( f \) with multiplicity \( p \), then it is a pole of \( G \) with multiplicity \( p + k \) respectively. This contradicts (3.5). It follows that \( F \) and \( G \) have no pole, and so \( F \) and \( G \) are entire functions here.

Let \( D \neq 0 \). Then from (3.5), we get

\[
\frac{f^{(k)}}{a} = \frac{(C - D)f^- a + D + 1 - C}{-Df^- a + D + 1}.
\]

Therefore,

\[
-Dff^{(k)} = a((C - D)f + a(D + 1 - C)) - a(D + 1)f^{(k)}.
\]

Hence, by Lemma 2.8, we obtain

\[
m(r, f^{(k)}) = T(r, f^{(k)}) = S(r, f).
\]

So using Lemma 2.7 from (3.5) we get \( T(r, f) = S(r, f) \), which is absurd. Hence \( D = 0 \), and so \( \frac{G - 1}{F - 1} = C \) or \( \frac{f^{(k)} - a}{f - a} = C \). This proves the theorem.

**Proof of Theorem 1.2.** Let \( F = \frac{f}{a} \) and \( G = \frac{f^{(k)}}{a} \). Then \( F - 1 = \frac{f - a}{a} \) and \( G - 1 = \frac{f^{(k)} - a}{a} \). Since \( f - a \) and \( f^{(k)} - a \) share \((0, 1)\), it follows that \( F, G \) share \((1, l)\) except the zeros and poles of \( a(z) \). Now we consider the following cases.

Case 1. Let \( H \neq 0 \).

Subcase 1.1. \( l \geq 1 \)

From (2.1), we get

\[
N(r, \infty; H) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 1; F \mathcal{H} l + 1) + \overline{N}(r, 0; F \mathcal{H} 2) + \overline{N}(r, 0; G \mathcal{H} 2) + \overline{N}_0(r, 0; F')
\]

\[
+ \overline{N}_0(r, 0; G') + \overline{N}(r, 0; a) + \overline{N}(r, \infty; a).
\]
Let $z_0$ be a simple zero of $F - 1$, but $a(z_0) \neq 0, \infty$. Then $z_0$ is a simple zero of $G - 1$ and a zero of $H$. So

$$N(r, 1; F | = 1) \leq N(r, 0; H) + N(r, \infty; a) + N(r, 0; a) \leq N(r, \infty; H) + S(r, f).$$  \hfill (3.10)

Hence

$$\overline{N}(r, 1; G) \leq N(r, 1; F | = 1) + \overline{N}(r, 1; F | \geq 2) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 1; F | \geq l + 1) + \overline{N}(r, 0; F | \geq 2)$$

$$+ \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, 1; F | \geq 2) + \overline{N}(r, 0; F') + \overline{N}(r, 0; G') + S(r, f).$$  \hfill (3.11)

By the second fundamental theorem, (3.11), and noting that $\overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) + S(r, f)$, we get

$$T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1; G) - N_0(r, 0; G') + S(r, G)$$

$$\leq 2\overline{N}(r, \infty; F) + \overline{N}(r, 0; G) + \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, 0; F | \geq 2)$$

$$+ \overline{N}(r, 1; F | \geq l + 1) + \overline{N}(r, 1; F | \geq 2) + \overline{N}_0(r, 0; F') + S(r, f).$$  \hfill (3.12)

While $l \geq 2$

$$\overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 1; F | \geq l + 1) + \overline{N}(r, 1; F | \geq 2) + \overline{N}_0(r, 0; F')$$

$$\leq N_2(r, 0; F') - \overline{N}(r, 0; F | \geq 3).$$  \hfill (3.13)

So

$$T(r, G) \leq 2\overline{N}(r, \infty; F) + N_2(r, 0; G) + N_2(r, 0; F') - \overline{N}(r, 0; F | \geq 3) + S(r, f)$$

that is,

$$T \left( r, f^{(k)} \right) \leq 2\overline{N}(r, \infty; f) + N_2(r, 0; f^{(k)}) + N_2 \left( r, 0; (f/a)^{\prime} \right) - \overline{N} \left( r, 0; (f/a) | \geq 3 \right) + S(r, f).$$

which contradicts (1.9).

While $l = 1$ (3.13) changes to

$$\overline{N}(r, 0; F | \geq 2) + 2 \overline{N}(r, 1; F | \geq 2) + \overline{N}_0(r, 0; F')$$

$$\leq 2 \overline{N}(r, 0; F') - \overline{N}(r, 0; F | \geq 2).$$

Similarly as above, we have

$$T \left( r, f^{(k)} \right) \leq 2\overline{N}(r, \infty; f) + N_2 \left( r, 0; f^{(k)} \right) + 2\overline{N} \left( r, 0; (f/a)^{\prime} \right) - \overline{N} \left( r, 0; (f/a) | \geq 2 \right) + S(r, f),$$

which contradicts (1.10).

**Subcase 1.2.** $l = 0$.

In this case, $F$ and $G$ share $(1, 0)$ except the zeros and poles of $a(z)$. In this case, we have

$$N(r, \infty; H) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}_L(r, 1; F)$$

$$+ \overline{N}_L(r, 1; G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f).$$  \hfill (3.14)

Let $z_0$ be a zero of $F - 1$ with multiplicity $p$, and a zero of $G - 1$ with multiplicity $q$. It is easy to see that

$$N_1^E (r, 1; F) = \overline{N}_1^E (r, 1; G) + S(r, f)$$

$$\overline{N}_2^E (r, 1; F) = \overline{N}_2^E (r, 1; G) + S(r, f)$$

and

$$N_1^E (r, 1; F) \leq N(r, \infty; H) + S(r, f).$$  \hfill (3.15)
By the second fundamental theorem, we get, using (3.14) and (3.15)
\[
T(r, G) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + N^{1}_{E}(r, 1; F) + \overline{N}_{L}(r, 1; F) + \overline{N}_{E}^{2}(r, 1; F) \\
+ \overline{N}_{L}(r, 1; G) - \overline{N}_{0}(r, 0; F') + S(r, f) \\
\leq N(r, 0; G \mid = 1) + \overline{N}(r, 0; G \mid \geq 2) + 2\overline{N}(r, \infty; F) + \overline{N}(r, 0; F \mid \geq 2) + \overline{N}(r, 0; G \mid \geq 2) \\
+ 2\overline{N}_{L}(r, 1; F) + 2\overline{N}_{L}(r, 1; G) + \overline{N}_{0}(r, 0; G') + S(r, f) \\
\leq N(r, 0; G \mid = 1) + 2\overline{N}(r, 0; G') + 2\overline{N}(r, 0; F') - \overline{N}(r, 0; F \mid \geq 2) + 2\overline{N}(r, \infty; F) + S(r, f).
\]

From Lemma 2.1 for \( p = 1, k = 1 \), we get
\[
\overline{N}(r, 0; G') \leq N_{2}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G).
\]
So
\[
T(r, G) \leq N(r, 0; G \mid = 1) + 2N_{2}(r, 0; G) + 2\overline{N}(r, 0; F') - \overline{N}(r, 0; F \mid \geq 2) + 4\overline{N}(r, \infty; F) + S(r, f),
\]
that is,
\[
T\left(r, f^{(k)}\right) \leq 4\overline{N}(r, \infty; f) + 2N_{2}\left(r, 0; f^{(k)}\right) + N\left(r, 0; f^{(k)} \mid = 1\right) + 2\overline{N}\left(r, 0; (f/a)\right) \\
- \overline{N}\left(r, 0; (f/a) \mid \geq 2\right).
\]
This contradicts (1.11).

Case 2. Let \( H \equiv 0 \).

Integrating (2.1), we get (3.5). If there exists a pole \( z_{0} \) of \( f \) with multiplicity \( p \) which is not a pole and zero of \( a(z) \), then \( z_{0} \) is the pole of \( F \) with multiplicity \( p \) and the pole of \( G \) with multiplicity \( p + k \). This contradicts (3.5). So,
\[
N(r, \infty; f) \leq N(r, 0; a) + N(r, \infty; a) = S(r, f),
\]
and hence
\[
N(r, \infty; f^{(k)}) = S(r, f).
\]

Suppose \( D \neq 0 \).

Now noting that here \( a \) is a small function, using Lemma 2.8, and proceeding in the same way as done in Case 2 of Theorem 1.1, we can first deduce
\[
T\left(r, f^{(k)}\right) = m\left(r, f^{(k)}\right) + N\left(r, \infty; f^{(k)}\right) = S(r, f).
\]
Hence, using Lemma 2.7 from (3.5), we get
\[
T(r, f) = T\left(r, f^{(k)}\right) + S(r, f) = S(r, f),
\]
which is absurd and so \( D = 0 \). So \( \frac{f^{(k)} - a}{f - a} = c \). This completes the proof of the theorem. 

Proof of Theorem 1.3. Let \( F \) and \( G \) be defined as in Theorem 1.2. Then \( F \) and \( G \) share \((1, l)\), except the zeros and poles of \( a(z) \).

Case 1. Let \( H \neq 0 \).

Subcase 1.1, \( l = 1 \)

By the second fundamental theorem, we see that
\[
T(r, F) + T(r, G) \leq \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, 0; G) \\
+ \overline{N}(r, 1; F) + \overline{N}(r, 1; G) - N_{0}(r, 0; F') - N_{0}(r, 0; G') \\
+ S(r, F) + S(r, G).
\]

(3.16)
Using Lemmas 2.2 and 2.3, (3.9) and (3.10) we get
\[
\begin{align*}
\overline{N}(r, 1; F) + \overline{N}(r, 1; G) &\leq N(r, 1; F = 1) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F) + \overline{N}(r, 1; G) \\
&\leq N(r, 1; F = 1) + N(r, 1; G) - \overline{N}_L(r, 1; F) - \overline{N}_L(r, 1; G) + \overline{N}_{F>2}(r, 1; G) \\
&\quad + \overline{N}_n(r, 1; F, G) + T(r, G) - m(r, 1; G) + O(1) \\
&\quad - \overline{N}_L(r, 1; F) - \overline{N}_L(r, 1; G) + \frac{1}{2} \overline{N}(r, 0; F) \\
&\quad + \frac{1}{2} \overline{N}(r, \infty; F) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, F) + S(r, G).
\end{align*}
\] (3.17)

Combining (3.16) and (3.17), we get
\[
T(r, F) \leq \frac{7}{2} \overline{N}(r, \infty; F) + N_2(r, 0; F) + N_2(r, 0; G) + \frac{1}{2} \overline{N}(r, 0; F) + S(r, f)
\]
\[
\leq \frac{7}{2} \overline{N}(r, \infty; F) + \frac{3}{2} N_2(r, 0; f) + N_2(r, 0; G) + S(r, f),
\]
that is
\[
T(r, f) \leq \left(\frac{7}{2} + k\right) \overline{N}(r, \infty; f) + \frac{3}{2} N_2(r, 0; f) + N_2(r, 0; f) + S(r, f).
\]

By Lemma 2.1 for \( p = 2 \), we get
\[
T(r, f) \leq \left(\frac{7}{2} + k\right) \Theta(\infty; f) + \frac{3}{2} \delta_2(0; f) + \delta_2(0; f) \leq k + 5,
\]
which contradicts with (1.12).

**Subcase 1.2.** \( l = 0 \). Using Lemmas 2.4–2.6 and (3.14) and (3.15), we get
\[
\begin{align*}
\overline{N}(r, 1; F) + \overline{N}(r, 1; G) &\leq N_E^1(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F) + \overline{N}(r, 1; G) \\
&\leq N_E^1(r, 1; F) + N(r, 1; G) - \overline{N}_L(r, 1; G) + \overline{N}_{F>1}(r, 1; G) + \overline{N}_{G>1}(r, 1; F) \\
&\quad + \overline{N}_n(r, 1; F, G) + T(r, G) - m(r, 1; G) + O(1) \\
&\quad - \overline{N}_L(r, 1; G) - \overline{N}_{F>1}(r, 1; G) + \overline{N}_{G>1}(r, 1; F) \\
&\quad + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f). \\
&\leq N_2(r, 0; F) + N_2(r, 0; G) + 4\overline{N}(r, \infty; F) \\
&\quad + \overline{N}(r, 0; F) + T(r, G) - m(r, 1; G) + S(r, f).
\end{align*}
\] (3.18)

Combining (3.16) and (3.18), we get
\[
T(r, F) \leq 6\overline{N}(r, \infty; F) + N_2(r, 0; F) + 2\overline{N}(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, 0; G) + S(r, f),
\]
that is
\[
T(r, f) \leq 6\overline{N}(r, \infty; F) + N_2(r, 0; f) + 2\overline{N}(r, 0; f) + N_2(r, 0; f) + N_2(r, 0; f) + S(r, f).
\]

By Lemma 2.1 for \( p = 2 \) and for \( p = 1 \) respectively, we get
\[
T(r, f) \leq (6 + 2k) \overline{N}(r, \infty; f) + N_2(r, 0; f) + 2\overline{N}(r, 0; f) + N_2(r, 0; f) + N_1(r, 0; f) + S(r, f).
\]
we get
\((6 + 2k) \Theta(\infty; f) + \delta_2(0; f) + 2\Theta(0; f) + \delta_{2+k}(0; f) + \delta_{1+k}(0; f) \leq 10 + 2k,\)

which contradicts (1.13).

**Case 2.** Let \(H \equiv 0.\)

Integrating (2.1) we get (3.5). In this case, we also have \(N(r, \infty; f) = S(r, f),\) and \(N(r, \infty; f^{(k)}) = S(r, f).\)

So \(\Theta(\infty; f) = 1.\) From (1.12) and (1.13) we know, respectively,

\[
\frac{3}{2} \delta_2(0; f) + \delta_{2+k}(0; f) > \frac{3}{2}
\]

and

\[
\delta_2(0; f) + 2 \Theta(0; f) + \delta_{1+k}(0; f) + \delta_{2+k}(0; f) > 4.
\]

We first assume that \(D \neq 0.\) Now proceeding in the same manner as in the proof of **Case 2** of Theorem 1.2, we can deduce \(T(r, f) = S(r, f)\), which is a contradiction. So \(D = 0\), and so from (3.5), we get

\[G - 1 \equiv C(F - 1).\]

If \(C \neq 1,\)

\[G \equiv C \left( F - 1 + \frac{1}{C} \right)\]

and

\[\overline{N}(r, 0; G) = \overline{N}\left(r, 1 - \frac{1}{C}; F\right)\]

By the second fundamental theorem, and noting that \(\overline{N}(r, \infty; F) = S(r, f)\), we get

\[
T(r, F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}\left(r, 1 - \frac{1}{C}; F\right) + S(r, G)
\]

\[
\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, f).
\]

By Lemma 2.1, for \(p = 1\) we have

\[
T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}\left(r, 0; f^{(k)}\right) + S(r, f) \leq \overline{N}(r, 0; f) + N_{1+k}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f)
\]

\[
\leq \overline{N}(r, 0; f) + N_{1+k}(r, 0; f) + S(r, f).
\]

Hence

\[
\Theta(0; f) + \delta_{1+k}(0; f) \leq 1.
\]

So,

\[
\frac{1}{2} \delta_2(0; f) + \delta_2(0; f) + \delta_{2+k}(0; f) \leq \frac{1}{2} \delta_2(0; f) + \Theta(0; f) + \delta_{1+k}(0; f) \leq \frac{3}{2}
\]

and

\[
\Theta(0; f) + \delta_{2+k}(0; f) + \Theta(0; f) + \delta_{1+k}(0; f) + \delta_2(0; f) \leq 2\{\Theta(0; f) + \delta_{1+k}(0; f)\} + \delta_2(0; f) \leq 3.
\]

This contradicts (3.19) and (3.20). Hence \(C = 1\) and so \(F \equiv G,\) that is \(f \equiv f^{(k)}.\) This completes the proof of the theorem. \(\square\)
Acknowledgement

The author is thankful to the referee for his/her valuable comments and suggestions towards the improvement of the paper.

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