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Spectral Characterization of Some Graphs

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The lexicographic product (or composition) of a cycle of length r with a totally disconnected graph on n vertices is shown to be a graph characterized by the eigenvalues of its adjacency matrix for all odd r.

INTRODUCTION

0.1. Definitions not given here can be found in [1]. We shall consider only finite undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote respectively the vertex set and edge set of the graph G. Let $D(x)$ be the set of vertices adjacent to x. The distance between two vertices x and y is denoted by $d(x, y)$. We shall use the following notations:

C_r: cycle with r vertices;

S_n: independent set or null graph with n vertices;

$C_r[S_{n_1},..., S_{n_r}]$: generalized composition of the r null graphs $S_{n_1},..., S_{n_r}$ with the graph C_r [9, p. 167]. This graph is formed by taking r independent sets $S_{n_1},..., S_{n_r}$ and then joining every vertex of S_{n_i} with every vertex of $S_{n_{i-1}}$ and of $S_{n_{i+1}}$ (the subscript i is taken modulo r). In the case $n_i = n$ $(i = 1, 2,..., r)$ we write simply $C_r[S_n]$. This graph is also called the lexicographic product of C_r by S_n and was studied for its chromatic properties in [6, 7]. For examples, the graph $C_4[S_2]$ is depicted in Fig. 1, and $C_3[S_n]$ is also called $K_{n,n,n}$.

0.2. The adjacency matrix $A(G)$ of the graph G with r vertices labeled $\{1,..., r\}$ is the $r \times r$ matrix $A(G) = (a_{ij})$ whose entries a_{ij} are given by $a_{ij} = 1$ if $d(i, j) = 1$ and by $a_{ij} = 0$, otherwise.

The characteristic polynomial of G is $P_G(x) = \det(xI - A(G))$ (where I is the unit $r \times r$ matrix). The spectrum of G, denoted by Spec G, is the set of the r (nonnecessary distinct) eigenvalues of $A(G)$.

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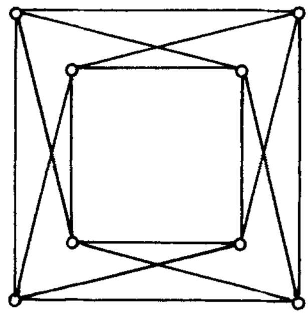

FIGURE 1

Details on the spectral properties of graphs can be found in [2, 3, 5, 10]. In particular, by [3, theorem 3.2] the graph C_r is characterized by its spectrum.

0.3. In this paper we prove the following theorem:

THEOREM. *If r is odd, the graphs $C_r[S_n]$ are characterized by their spectra.*

The proof is given in Section 1. It is, if $r \geqslant 5$, an easy consequence of a combinatorial characterization of $C_r[S_n]$ (Proposition 1 2) In Section 2, we announce without proofs partial results in the case r even and give a conjecture.

1. PROOF OF THE THEOREM

1.1. *Case $r = 3$.*

Since $C_3[S_n]$ is the complement of the union of three cliques K_n, it is characterized by its spectrum in [4, Proposition 3.2].

PROPOSITION 1.2. *Let $r \geqslant 5$. Then the following three properties are equivalent if r is odd*:

(1) *G is isomorphic to $C_r[S_n]$;*

(2) $P_G(x) = x^{r(n-1)} \prod_{j=1}^{r} (x - 2n \cos(2\pi/r)j)$;

(3) *G is a graph satisfying*:

  (i) $|V(G)| \leqslant rn$,

  (ii) *G is regular of degree $d \geqslant 2n$,*

  (iii) *The length of the shortest cycle of odd length in G is r.*

Proof. (1) *implies* (2). Indeed, we have Spec $C_r = \{2\cos(2\pi/r)j;\ 1 \leqslant j \leqslant r\}$ [9, p. 170]. By [9, Theorem 7] if $G = C_r[S_n]$, $P_G(x) = n^r x^{r(n-1)} P_{C_r}(x/n)$.

(2) *implies* (3). Assume that the characteristic polynomial of a graph G is given by (2). Then G has rn vertices and is regular of degree $2n$, so it has properties (i) and (ii) of (3). In [8], Sachs proved that if G is a graph, the characteristic polynomial of which is $P_G(x) = x^m + a_1 x^{m-1} + \cdots + a_m$, then the length r of the shortest cycle of an odd length in G is equal to the index of the first of the coefficients $a_3, a_5, a_7 \cdots$ which is different from zero. Since C_r has the property (iii) and $P_G(x) = x^{r(n-1)} P_{C_r}(x/n)$, so does the graph G.

(3) *implies* (1). Assume that a graph G satisfies the properties (i)–(iii). Let $(x_1, ..., x_r, x_1)$ be a cycle of length r in G.

We break the proof in six parts. All the subscripts written in this section are to be taken modulo r.

(a) *If $j - i$ and $i - j$ are different from* 0 *and* 2 *modulo r, then* $D(x_i) \cap D(x_j) = \varnothing$.

Indeed suppose that there exists a $t \in D(x_i) \cap D(x_j)$ with $j > i$. If $j - i$ is odd, the cycle $(t, x_i, x_{i+1} \cdots x_j, t)$ is of odd length $j - i + 2 < r$ (since $j - i \neq r - 2, 0$), contradicting (iii). If $j - i$ is even, the cycle $(t, x_i, x_{i-1} \cdots x_1, x_r, x_{r-1} \cdots x_j, t)$ is of odd length $r - (j - i) + 2 < r$ (since $j - i \neq 0, 2$), contradicting (iii).

(b) *Let $F_i = D(x_{i-1}) \cap D(x_{i+1})$. Then $F_i \cap F_j = \varnothing$ $(1 \leqslant i < j \leqslant r)$.*

If $j - i \neq 2$, $r - 2$, according to (a), we have $D(x_{i-1}) \cap D(x_{j-1}) = \varnothing$.

If $j - i = 2$, then $j + 1 - (i - 1) = 4 \neq r - 2$ (since r is odd, $r \geqslant 5$), thus $D(x_{j+1}) \cap D(x_{i-1}) = \varnothing$.

If $j - i = r - 2$, then $j - i - (i + 1) = r - 4 \neq 2$ and $D(x_{i+1}) \cap D(x_{j-1}) = \varnothing$.

So we have proved that $F_i \cap F_j = \varnothing$ in all cases.

(c) *Let $\delta_i = | F_i |$ $(1 \leqslant i \leqslant r)$. Then $\sum_{i=1}^{r} \delta_i \leqslant | V(G)| \leqslant rn$.*

This result follows from (b) and condition (i).

(d) $V(G) = \bigcup_{i=1}^{r} F_i$.

In order to prove (d), we break the proof into three cases depending upon the congruent class of r modulo 3.

Case 1: $r = 3t$. Consider the sets $E_i = D(x_{3_i+1}) \cup D(x_{3_i+3})$ $(0 \leqslant i \leqslant t - 1)$. From (a), we deduce that the sets E_i are pairwise disjoint, so that we have by (i):

$$\sum_{i=0}^{t-1} | E_i | \leqslant | V(G)| \leqslant 3nt.$$

From condition (ii), it follows that $|E_i| = 2d - \delta_{3i+2} \geqslant 4n - \delta_{3i+2}$, where d is the degree of the graph.

Thus,

$$3nt \geqslant \sum_{i=0}^{t-1} |E_i| \geqslant 4nt - \sum_{i=0}^{t-1} \delta_{3i+2}.$$

Hence

$$\sum_{i=0}^{t-1} \delta_{3i+2} \geqslant nt.$$

By symmetry of x_1, x_2, x_3, we also have

$$\sum_{i=0}^{t-1} \delta_{3i} \geqslant nt \qquad \text{and} \qquad \sum_{i=0}^{t-1} \delta_{3i+1} \geqslant nt.$$

Adding the last three inequalities we obtain

$$\sum_{i=1}^{r} \delta_i \geqslant 3nt = nr.$$

Now from the inequality (c) we obtain

$$|V(G)| = rn = \sum_{i=1}^{r} \delta_i.$$

Therefore

$$V(G) = \bigcup_{i=1}^{n} F_i.$$

Case 2: $r = 3t + 1$. The proof is similar to the precedent and left to the reader. It suffices to consider the sets

$$E_i = D(x_{3i+1}) \cup D(x_{3i+3}), \qquad (0 \leqslant i \leqslant t - 2),$$
$$E_{t-1} = D(x_{3t-2}), \qquad \text{and} \qquad E_t = D(x_{3t+1}).$$

Case 3: $r = 3t + 2$. Since r is odd, we can assume that $t = 2t' + 1$. Similarly to case 1, we can prove that $V(G) = \bigcup_{i=1}^{r} F_i$ by considering the sets

$$E_{2p} = D(x_{6p+1}) \cup D(x_{6p+3}), \qquad (0 \leqslant p \leqslant t'),$$
$$E_{2p+1} = D(x_{6p+2}) \cup D(x_{6p+4}), \qquad (0 \leqslant p \leqslant t' - 1),$$
$$E_t = D(x_{3t-1}).$$

(e) Define a relationship $\sim$ on $V(G)$ to mean $x \sim y$ if $D(x) = D(y)$. Then $\sim$ is an equivalence relation and the equivalence class of x denoted by $S(x)$ is an independent set. We shall prove that $F_i = S(x_i)$ $(i = 1,..., r)$.

According to the definition of $S(x_i)$ and from $d(x_i, x_{i-1}) = d(x_i, x_{i+1}) = 1$ it follows that $S(x_i) \subset F_i$.

Suppose that $S(x_i) \neq F_i$. Then there exists a $t \in F_i$, with $D(t) \neq D(x_i)$, and since G is regular, we can find $u \in D(t)$, $u \notin D(x_i)$. From (d), it follows that $u \in F_j$, for some $j \in \{1,..., r\}$. Now, by hypothesis (iii), G has no triangle $(r \geqslant 5)$, therefore F_i is an independent set. As $t \in F_i$ and $u \in D(t)$, then $u \notin F_i$ and $j \neq i$. But $u \notin D(x_i)$, hence $u \notin F_{i-1} \cup F_{i+1}$, so $j \neq i - 1, i + 1$. Now, consider the two following cycles:

$$(u, t, x_{i+1}, x_{i+2}, ..., x_{j-1}, u, t),$$
$$(u, t, x_{i-1}, x_{i-2}, ..., x_{,+1}, u, t).$$

At least one of them is of odd length $l < r$, contradicting (iii). We have proved that $S(x_i) = F_i$, then, by (d), $V(G) = \bigcup_{i=1}^{r} S(x_i)$ and therefore G is isomorphic to the graph $C_r[S(x_1),..., S(x_r)]$.

(f) $\delta_i = |S(x_i)| = n, \quad (i = 1,..., r)$.

Since G is regular of degree d, we have $\delta_i + \delta_{i+2} = d$, and then $\delta_{i+4} = \delta_i$.

If $r = 4t + 1$, we get $\delta_{4t} = \delta_4 = d - \delta_2$; $\delta_{4t+1} = \delta_1$; $\delta_{4t} + \delta_1 = \delta_{4t+1} + \delta_2 = d$. Hence, $\delta_1 = \delta_2 = \frac{1}{2}d$ and $\delta_i = \frac{1}{2}d$ for every $i \in \{1,..., r\}$.

If $r = 4t + 3$, similar proof works.

By inequality (c) and condition (ii), we get $\frac{1}{2}d = n$, so that $|S(x_i)| = n$ $(i = 1,..., r)$ and G is isomorphic to $C_r[S_n]$.

Remark 1.3. Property (3) does not imply (1) for $r = 3$. As a counter-example, consider the graph G, the complement of which $\bar{G}$ is the disjoint union of C_4 and C_5. The graph G has nine vertices, is regular of degree 6 and contains C_3, but is not $C_3[S_3]$ (or $K_{3,3,3}$) since $\bar{G}$ is not the disjoint union of three cycles C_3.

However, $C_3[S_n]$ is characterized by the following properties:

(i) The number of vertices is $3n$.

(ii) The graph is connected, regular of degree $2n$.

(iii) If $d(x, y) = 2$, then $D(x) = D(y)$.

2. Case r Even

In the case r even, the graphs $C_r[S_n]$ being bipartite we cannot have a proposition similar to proposition 1.2. However we conjecture:

Conjecture. The graphs $C_r[S_n]$ are characterized by their spectra for all r.

This conjecture is true for $r = 4, 6, 8$. Indeed, for $r = 4$, since $C_4[S_n]$ is $K_{2n,2n}$ this graph is characterized by its spectrum as an easy consequence of [4, Proposition 3.1].

For $r = 6$ and $r = 8$, it is an easy consequence of the following propositions, the proofs of which are too long and technical to take place here.

PROPOSITION 2.1. *Let G be a connected, bipartite, regular graph of degree d with three positive eigenvalues. Then, either $d = 2n$ and G is isomorphic to $C_6[S_n]$ or G is isomorphic to $C_8[S_{n_1},..., S_{n_8}]$ with $n_i + n_{i+2} = d$ and $n_{i+4} = n_i$ $(i = 1,..., 4)$.*

PROPOSITION 2.2. *If the graphs $C_r[S_{n_1},..., S_{n_r}]$* and *$C_r[S_n]$ have the same spectrum, they are isomorphic.*

These propositions are proved in [11].

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