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## Systems of $p$ -Laplacian differential inclusions with large diffusion <sup>☆</sup>

 Jacson Simsen <sup>a,\*</sup>, Cláudia B. Gentile <sup>b</sup>
<sup>a</sup> Departamento de Matemática e Computação, ICE, Universidade Federal de Itajubá, Av. BPS, 1303, Pinheirinho, 37500-903, Itajubá, Minas Gerais, Brazil

<sup>b</sup> Departamento de Matemática, Universidade Federal de São Carlos, Rodovia Washington Luis, Km 235, Postal Code 13565-905, São Carlos, Brazil

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### ABSTRACT

In this paper we consider coupled systems of  $p$ -Laplacian differential inclusions and we prove, under suitable conditions, that a homogenization process occurs when diffusion parameters become arbitrarily large. In fact we obtain that the attractors are continuous at infinity on  $L^2(\Omega) \times L^2(\Omega)$  topology, with respect to the diffusion coefficients, and the limit set is the attractor of an ordinary differential problem.

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## 1. Introduction

In this work we consider the following problem

$$(I) \quad \begin{cases} \frac{\partial u^{D_1}}{\partial t} - D_1 \Delta_p u^{D_1} + |u^{D_1}|^{p-2} u^{D_1} \in F(u^{D_1}, v^{D_2}) & \text{in } (0, T) \times \Omega, \\ \frac{\partial v^{D_2}}{\partial t} - D_2 \Delta_q v^{D_2} + |v^{D_2}|^{q-2} v^{D_2} \in G(u^{D_1}, v^{D_2}) & \text{in } (0, T) \times \Omega, \\ \frac{\partial u^{D_1}}{\partial n}(t, x) = \frac{\partial v^{D_2}}{\partial n}(t, x) = 0 & \text{in } (0, T) \times \partial\Omega, \\ u^{D_1}(0, x) = u_0^{D_1}(x), \quad v^{D_2}(0, x) = v_0^{D_2}(x) & \text{in } \Omega, \end{cases}$$

where  $D_1, D_2 \geq 1$  are positive constants,  $p, q > 2$ ,  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 1$ , with smooth boundary  $\partial\Omega$ ,  $u_0, v_0 \in H = L^2(\Omega)$  and  $F, G$  are bounded, positively sublinear and upper semicontinuous multivalued operators. The pair  $(F, G)$  of operators  $F, G : H \times H \rightarrow P(H)$ , which maps bounded subsets of  $H \times H$  into bounded subsets of  $H$ , is called positively sublinear if there exist  $a > 0$ ,  $b > 0$ ,  $c > 0$  and  $m_0 > 0$  such that for each  $(u, v) \in H \times H$  with  $\|u\| > m_0$  or  $\|v\| > m_0$  for which either there exists  $f_0 \in F(u, v)$  satisfying  $\langle u, f_0 \rangle > 0$  or there exists  $g_0 \in G(u, v)$  with  $\langle v, g_0 \rangle > 0$ , we have both

$$\|f\| \leq a\|u\| + b\|v\| + c$$

and

$$\|g\| \leq a\|u\| + b\|v\| + c$$

for each  $f \in F(u, v)$  and each  $g \in G(u, v)$ .
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<sup>\*</sup> Corresponding author.

 E-mail address: [jacson@unifei.edu.br](mailto:jacson@unifei.edu.br) (J. Simsen).

For each couple  $(D_1, D_2)$ , we can associate with  $(I)$  a generalized semiflow,  $\mathbb{G}(D_1, D_2)$ , which has a compact invariant global  $B$ -attractor  $\mathcal{A}_{(D_1, D_2)}$ , according to Section 2.3 in [29]. Our final objective is to prove that the family of attractors  $\{\mathcal{A}_{(D_1, D_2)}\}_{D_1, D_2 \geq 1}$  behaves continuously as  $\min\{D_1, D_2\}$  goes to infinity.

About twenty years ago there were several authors dealing with large diffusion semilinear problems, among them we mention [8,10,12,18–20], where is evidenced almost no spatial dependence in the asymptotic behavior. In [12], the precursor work proving this homogenization process, the principal theorem says that solutions of a certain reaction–diffusion system exponentially approach their own spatial average as diffusion and time both become arbitrarily large, and this is obtained as a consequence of the intrinsic linear structure of the considered problem. It is expected that huge diffusion must implies a quick homogenization of the concentrations and, in Laplacian problems, this is mathematically justified from the fact that, as the diffusion coefficients become large, there is a gap between the zero eigenvalue of the Neumann Laplacian and its first positive eigenvalue, which ensures that the space of constant functions is an exponentially attracting invariant manifold.

Problems involving the degenerate  $p$ -Laplacian operator in general presents similar phenomena than the correspondent Laplacian systems but, in spite of we can frequently enunciate almost the same theorems, their proves are rarely obtained by the same methods. The first work considering large diffusion for  $p$ -Laplacian problems is [30], where it is proved that there exists a positive time from which the spatial gradients of solutions go to zero as the diffusion goes to infinity and, as a simple consequence of the Poincaré–Wirtinger inequality, all the relevant elements to describe the asymptotic behavior are around their own spatial average if the diffusion is large enough. It is also proved that the attractors continuously approach the attractor of an ordinary equation. The main difference between this work and [30] is that here we consider a coupled system admitting non-globally Lipschitz perturbations of the  $p$ -Laplacian and, because of it, we have to consider multivalued systems.

The lack of the uniqueness was, as it is known, one of the most important delay factor in the understanding of the asymptotic properties of quasilinear problems. Today however, we have a very well-structured theory for multivalued dynamical systems which allow us to properly deal with problems admitting more than a unique solution for each initial date. Several authors have been dealing with multivalued problems and some efforts in this direction appeared more than fifty years ago [2,5–7,25–27,31]. The study of the global attractors for such kind of problems started only in the nineties and, in the beginning, most of works were concerned about conditions to obtain the existence of attractors [1,9,21,23,24] and it was still necessary to organize and complete the theory. As an example, to accomplish the results in this present work it was needed to know that attractors can be characterized as the union of all bounded complet orbits, and this simple fact can only be found in very recent texts [22,28].

For applied models with the  $p$ -Laplacian operator the reader can see, for example, [13–15] and references therein. In [13], a  $p$ -Laplacian differential inclusion is regarded as a climatological model and the authors deal with the sensitivity of the problem in long time with respect to small changes in the solar constant. In [14], the degenerate  $p$ -Laplacian appears in a climate model. The condition on  $F$  to be a bounded an upper semicontinuous multivalued operator also appears there (see  $(H_5)$ , p. 2067 in [14]). In [15], the one-dimensional  $p$ -Laplacian appears in a degenerate parabolic/hyperbolic system in glaciology.

Taken into account the work [30] we can say that a good candidate for the limit problem when diffusion coefficients in  $(I)$  go to infinity is

$$(II) \quad \begin{cases} \dot{u} + \phi_p(u) \in \tilde{F}(u, v), \\ \dot{v} + \phi_q(v) \in \tilde{G}(u, v), \\ u(0) = u_0, \quad v(0) = v_0, \end{cases}$$

where  $\phi_p(s) \doteq |s|^{p-2}s$ ,  $\tilde{F} \doteq F|_{\mathbb{R} \times \mathbb{R}}$ ,  $\tilde{G} \doteq G|_{\mathbb{R} \times \mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow P(\mathbb{R})$ .

In the next sections we are going to obtain the uniform estimates, the continuity of the flow and the necessary compactness to prove the upper semicontinuity of the attractors. Once the limit system is given by an ordinary problem whose solutions are also solutions of  $(I)$ , we also obtain the lower semicontinuity of the family of the global attractors in a trivial way since, in this case, the attractor  $\mathcal{A}^\infty$  of the limit problem  $(II)$  is contained in each attractor  $\mathcal{A}_{(D_1, D_2)}$  associated to the  $(I)$ .

## 2. Uniform estimates

In this section we obtain some estimates for the solutions  $(u^{D_1}, v^{D_2})$  of the problem  $(I)$ , uniformly on  $D_1, D_2 \geq 1$ .

**Lemma 2.1.** *If  $(u^{D_1}, v^{D_2})$  is a solution of  $(I)$ , then there are positive constants  $r_0, t_0$  such that  $\|(u^{D_1}(t), v^{D_2}(t))\|_{H \times H} = \|u^{D_1}(t)\|_H + \|v^{D_2}(t)\|_H \leq r_0$ , for each  $t \geq t_0$  and  $D_1, D_2 \geq 1$ .*

**Proof.** Let  $(u^{D_1}, v^{D_2})$  be a solution of the problem  $(I)$ . Then, there are  $f, g \in L^1(0, T; H)$ , with

$$f(t) \in F(u^{D_1}(t), v^{D_2}(t)), \quad g(t) \in G(u^{D_1}(t), v^{D_2}(t)) \quad \text{a.e. in } (0, T),$$

and such that  $(u^{D_1}, v^{D_2})$  is a solution of the system:

$$(\tilde{I}) \quad \begin{cases} \frac{\partial u^{D_1}}{\partial t} - D_1 \Delta_p u^{D_1} + |u^{D_1}|^{p-2} u^{D_1} = f & \text{in } (0, T), \\ \frac{\partial v^{D_2}}{\partial t} - D_2 \Delta_q v^{D_2} + |v^{D_2}|^{q-2} v^{D_2} = g & \text{in } (0, T), \\ u^{D_1}(0) = u_0^{D_1}, \quad v^{D_2}(0) = v_0^{D_2}. \end{cases}$$

Doing the inner product of the first equation of  $(\tilde{I})$  with  $u^{D_1}(t)$  and the second equation of  $(\tilde{I})$  with  $v^{D_2}(t)$ , with the same arguments used in Theorem 2.8 in [29] we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u^{D_1}(t)\|_H^2 + \|v^{D_2}(t)\|_H^2) &\leq \frac{-1}{2(C_q)^q} (\|u^{D_1}(t)\|_H^{2q/2} + \|v^{D_2}(t)\|_H^{2q/2}) + C_1 \\ &\leq \frac{-1}{2^{1+q/2}(C_q)^q} (\|u^{D_1}(t)\|_H^2 + \|v^{D_2}(t)\|_H^2)^{q/2} + C_1 \end{aligned}$$

where  $C_q > 0$  is the immersion constant of  $W^{1,q}(\Omega)$  in  $H$  and  $C_1 = C_1(p, q, \Omega) > 0$  is a constant which does not depend of  $(D_1, D_2)$ . Thus, using Lemma 5.1 in [32], there exist positive constants  $r_0, t_0$  such that  $\|(u^{D_1}(t), v^{D_2}(t))\|_{H \times H} = \|u^{D_1}(t)\|_H + \|v^{D_2}(t)\|_H \leq r_0$ , for each  $t \geq t_0$  and  $D_1, D_2 \geq 1$ .  $\square$

**Remark 2.1.** The constants  $r_0, t_0$  in Lemma 2.1 are independent from the initial values and from the couples  $(D_1, D_2)$ .

**Remark 2.2.** For each fixed couple  $(D_1, D_2)$ , as an easy consequence of Gronwall–Bellman inequality, there is a positive constant  $K$  such that  $\|u^{D_1}(t)\|_H + \|v^{D_2}(t)\|_H \leq K, \forall t \in [0, t_0]$ , where  $K = K(u_0^{D_1}, v_0^{D_2}, t_0)$ . Furthermore, if the initial values are in a bounded subset of  $H \times H$ , then we have  $K$  uniform on  $(D_1, D_2)$  and, in this case we can consider  $t_0 = 0$  in Lemma 2.1.

**Lemma 2.2.** *There is a bounded set  $B_0$  in  $H \times H$  such that  $\mathcal{A}_{(D_1, D_2)} \subset B_0, \forall D_1, D_2 \geq 1$ .*

**Proof.** Let  $(x_{D_1}, y_{D_2}) \in \mathcal{A}_{(D_1, D_2)}$ . Since  $\mathcal{A}_{(D_1, D_2)} = T_{(D_1, D_2)}(t_0) \mathcal{A}_{(D_1, D_2)}$ , where  $T_{(D_1, D_2)}$  is the multivalued semigroup defined by  $\mathbb{G}(D_1, D_2)$ , then by Lemma 2.1,  $\|(x_{D_1}, y_{D_2})\|_{H \times H} \leq r_0$ .  $\square$

Now, using Lemma 2.1 and the fact that  $F$  and  $G$  maps bounded sets of  $H \times H$  in bounded sets of  $H$ , we can repeat the same arguments in the proof of Lemma 2.2 in [17] for each equation in  $(\tilde{I})$  and we obtain:

**Lemma 2.3.** *If  $(u^{D_1}, v^{D_2})$  is a solution of (I), then there exist positive constants  $r_1 > 0$  and  $t_1 > 0$  such that*

$$\|(u^{D_1}(t), v^{D_2}(t))\|_{W^{1,p} \times W^{1,q}} = \|u^{D_1}(t)\|_{W^{1,p}} + \|v^{D_2}(t)\|_{W^{1,q}} \leq r_1,$$

for each  $t \geq t_1$  and  $D_1, D_2 \geq 1$ , where  $t_0$  is the same as in Lemma 2.1.

As a consequence of Lemma 2.3 we have that  $\bigcup_{D_1, D_2 \geq 1} \mathcal{A}_{(D_1, D_2)}$  is a bounded subset of  $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$  and so we can conclude the following:

**Lemma 2.4.**  $\mathcal{A} \doteq \overline{\bigcup_{D_1, D_2 \geq 1} \mathcal{A}_{(D_1, D_2)}}$  is a compact subset of  $H \times H$ .

### 3. The limit problem and convergence properties

In order to obtain the limit problem we firstly prove

**Lemma 3.1.** *If  $(u^{D_1}, v^{D_2})$  is a solution of (I), then for each  $t > t_1$ , the sequences of real numbers  $\{\|\nabla u^{D_1}(t)\|_H\}_{D_1 \geq 1}$  and  $\{\|\nabla v^{D_2}(t)\|_H\}_{D_2 \geq 1}$  possess subsequences,  $\{\|\nabla u^{D_{1\ell}}(t)\|_H\}$  and  $\{\|\nabla v^{D_{2\ell}}(t)\|_H\}$  respectively, converging to zero as  $\ell \rightarrow +\infty$ . Here  $t_1$  is the positive constant in Lemma 2.3.*

**Proof.** Let  $T > 0, t \in (t_1, T)$ , and  $(u^{D_1}, v^{D_2})$  be a solution of problem (I). There are  $f, g \in L^1(0, T; H)$ , with

$$f(\tau) \in F(u^{D_1}(\tau), v^{D_2}(\tau)), \quad g(\tau) \in G(u^{D_1}(\tau), v^{D_2}(\tau)) \quad \text{a.e. in } (0, T),$$

and such that  $(u^{D_1}, v^{D_2})$  is a solution of the system:

$$(\tilde{I}) \quad \begin{cases} \frac{\partial u^{D_1}}{\partial t} - D_1 \Delta_p u^{D_1} + |u^{D_1}|^{p-2} u^{D_1} = f & \text{in } (0, T), \\ \frac{\partial v^{D_2}}{\partial t} - D_2 \Delta_q v^{D_2} + |v^{D_2}|^{q-2} v^{D_2} = g & \text{in } (0, T), \\ u^{D_1}(0) = u_0^{D_1}, \quad v^{D_2}(0) = v_0^{D_2}. \end{cases}$$

Doing the inner product of the first equation of  $(\tilde{I})$  with  $u^{D_1}(\tau)$ , it comes that

$$\frac{1}{2} \frac{d}{dt} \|u^{D_1}(\tau)\|_H^2 + D_1 \|\nabla u^{D_1}(\tau)\|_p^p + \|u^{D_1}(\tau)\|_p^p = \langle f(\tau), u^{D_1}(\tau) \rangle. \quad (3.1)$$

Analogously, we have that

$$\frac{1}{2} \frac{d}{dt} \|v^{D_2}(\tau)\|_H^2 + D_2 \|\nabla v^{D_2}(\tau)\|_q^q + \|v^{D_2}(\tau)\|_q^q = \langle g(\tau), v^{D_2}(\tau) \rangle. \quad (3.2)$$

We consider  $\theta \doteq q/2$ ,  $s \doteq q/q'$  where  $\frac{1}{q} + \frac{1}{q'} = 1$ . Using the positive sublinearity of the couple  $(F, G)$  and the Young's inequality we prove that

$$\langle f(\tau), u^{D_1}(\tau) \rangle \leq \left(\frac{2}{q} + \frac{2}{q}\right) \|u^{D_1}(\tau)\|_H^q + \frac{1}{s'} \left(\frac{1}{q'} b^{q'}\right)^{s'} + \frac{1}{s} \|v^{D_2}(\tau)\|_H^q + \left(\frac{1}{\theta'} a^{\theta'} + \frac{1}{q'} c^{q'}\right) + C_1 m_0$$

and

$$\langle g(\tau), v^{D_2}(\tau) \rangle \leq \left(\frac{2}{q} + \frac{2}{q}\right) \|v^{D_2}(\tau)\|_H^q + \frac{1}{s'} \left(\frac{1}{q'} a^{q'}\right)^{s'} + \frac{1}{s} \|u^{D_1}(\tau)\|_H^q + \left(\frac{1}{\theta'} a^{\theta'} + \frac{1}{q'} c^{q'}\right) + C_1 m_0,$$

where  $C_1 > 0$  is a constant which does not depend on  $(D_1, D_2)$ , and  $a, b, c$  and  $m_0$  are the constants that appear in the definition of positive sublinearity of the couple  $(F, G)$  (see [29]).

Then, adding Eqs. (3.1) and (3.2), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u^{D_1}(\tau)\|_H^2 + \|v^{D_2}(\tau)\|_H^2) + D_1 \|\nabla u^{D_1}(\tau)\|_p^p + D_2 \|\nabla v^{D_2}(\tau)\|_q^q + \|u^{D_1}(\tau)\|_p^p + \|v^{D_2}(\tau)\|_q^q \\ & = \langle f(\tau), u^{D_1}(\tau) \rangle + \langle g(\tau), v^{D_2}(\tau) \rangle \\ & \leq \left(\frac{2}{q} + \frac{2}{q} + \frac{1}{s}\right) (\|u^{D_1}(\tau)\|_H^q + \|v^{D_2}(\tau)\|_H^q) + C_2, \end{aligned}$$

where  $C_2 > 0$  is a constant which does not depend on  $(D_1, D_2)$ . Using Lemma 2.1, there exists a constant  $C_3 > 0$  such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u^{D_1}(\tau)\|_H^2 + \|v^{D_2}(\tau)\|_H^2) + D_1 \|\nabla u^{D_1}(\tau)\|_p^p + D_2 \|\nabla v^{D_2}(\tau)\|_q^q \\ & + \|u^{D_1}(\tau)\|_p^p + \|v^{D_2}(\tau)\|_q^q \leq C_3, \quad \text{a.e. in } (t_1, T). \end{aligned}$$

As  $\|u^{D_1}(\tau)\|_p^p + \|v^{D_2}(\tau)\|_q^q \geq 0$ , we have in particular that

$$\frac{1}{2} \frac{d}{dt} (\|u^{D_1}(\tau)\|_H^2 + \|v^{D_2}(\tau)\|_H^2) + D_1 \|\nabla u^{D_1}(\tau)\|_p^p + D_2 \|\nabla v^{D_2}(\tau)\|_q^q \leq C_3, \quad (3.3)$$

a.e. in  $(t_1, T)$ .

Integrating the inequality (3.3) from  $t_1$  to  $T$ , we obtain

$$\begin{aligned} & \frac{1}{2} (\|u^{D_1}(T)\|_H^2 + \|v^{D_2}(T)\|_H^2) + D_1 \int_{t_1}^T \|\nabla u^{D_1}(\tau)\|_p^p d\tau + D_2 \int_{t_1}^T \|\nabla v^{D_2}(\tau)\|_q^q d\tau \\ & \leq \int_{t_1}^T C_3 d\tau + \frac{1}{2} (\|u^{D_1}(t_1)\|_H^2 + \|v^{D_2}(t_1)\|_H^2) \\ & \leq C_3 T + r_0^2 \doteq k(T). \end{aligned}$$

In particular

$$D_1 \int_{t_1}^T \|\nabla u^{D_1}(\tau)\|_p^p d\tau + D_2 \int_{t_1}^T \|\nabla v^{D_2}(\tau)\|_q^q d\tau \leq k(T),$$

which implies

$$\int_{t_1}^T \|\nabla u^{D_1}(\tau)\|_p^p d\tau \leq \frac{1}{D_1} k(T) \rightarrow 0 \text{ as } D_1 \rightarrow +\infty,$$

that means,

$$\|\nabla u^{D_1}(\tau)\|_p^p \Big\|_{L^1(t_1, T; \mathbb{R})} = \int_{t_1}^T \|\nabla u^{D_1}(\tau)\|_p^p d\tau \rightarrow 0 \text{ as } D_1 \rightarrow +\infty.$$

Therefore there exists a subsequence  $\{\|\nabla u^{D_{1\ell}}(\tau)\|_p^p\}$  such that

$$\|\nabla u^{D_{1\ell}}(\tau)\|_p^p \rightarrow 0 \text{ as } \ell \rightarrow +\infty, \tau\text{-a.e. in } (t_1, T),$$

and so there exists a subset  $J \subset (t_1, T)$  with Lebesgue measure  $m((t_1, T)/J) = 0$  such that

$$\|\nabla u^{D_{1\ell}}(\tau)\|_p^p \rightarrow 0 \text{ as } \ell \rightarrow +\infty, \forall \tau \in J.$$

Given  $t \in (t_1, T)$  we claim that there is at least one  $s \in J$  with  $s < t$ , on the contrary we would have  $(t_1, t) \cap J = \emptyset$ , so  $m((t_1, T)/J) > 0$  which is a contradiction. Now pick one  $s \in J$  with  $t_1 < s < t$  and let  $h = t - s$ . Let  $\varepsilon > 0$  and  $\ell_0 = \ell_0(\varepsilon) > 0$  be such that if  $\ell > \ell_0$  then

$$\|\nabla u^{D_{1\ell}}(s)\|_p^p < \frac{\varepsilon}{2}.$$

Now, we consider

$$\varphi^{D_{1\ell}}(v) \doteq \begin{cases} \frac{1}{p} [D_{1\ell} \int_{\Omega} |\nabla v|^p dx + \int_{\Omega} |v|^p dx], & v \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

We have that  $\varphi^{D_{1\ell}}$  is a convex, proper and l.s.c. map,  $A^{D_{1\ell}} = \partial\varphi^{D_{1\ell}}$  is maximal monotone in  $L^2(\Omega)$  and  $u^{D_{1\ell}}$  satisfies the equation

$$\frac{\partial u^{D_{1\ell}}}{\partial t} + A^{D_{1\ell}}(u^{D_{1\ell}}) = f$$

in  $(0, T)$  with  $f(\tau) \in F(u^{D_{1\ell}}(\tau), v^{D_{2\ell}}(\tau))$ , a.e. in  $(0, T)$ , therefore  $u^{D_{1\ell}}(\tau) \in D(A^{D_{1\ell}}) \subseteq W^{1,p}(\Omega)$  a.e. in  $(0, T)$ .

Using Lemma 2.1 and the hypothesis on  $F$  and  $G$  it follows that there exists a positive constant  $K$ , independent of  $(D_1, D_2)$ , such that  $\|f(\zeta)\|_H^2 \leq K, \forall \zeta \geq t_0$ .

We have that

$$\frac{d}{d\tau} \varphi^{D_{1\ell}}(u^{D_{1\ell}}(s + \tau)) = \langle \partial\varphi^{D_{1\ell}}(u^{D_{1\ell}}(s + \tau)), u_{\tau}^{D_{1\ell}}(s + \tau) \rangle, \text{ a.e. in } (0, T).$$

Now, repeating the same arguments used in the proof of Lemma 3.1 in [30] we obtain

$$\|\nabla u^{D_{1\ell}}(t)\|_H \rightarrow 0 \text{ as } \ell \rightarrow +\infty.$$

Analogously we conclude that

$$\|\nabla v^{D_{2\ell}}(t)\|_H \rightarrow 0 \text{ as } \ell \rightarrow +\infty. \quad \square$$

**Remark 3.1.** If  $(u^{D_1}, v^{D_2})$  is a solution of the problem (I) in  $(0, t_1)$ , then for each  $t \in [0, t_1]$ , the sequences of real numbers  $\{\|\nabla u^{D_1}(t)\|_p^p\}_{D_1 \geq 1}$  and  $\{\|\nabla v^{D_2}(t)\|_q^q\}_{D_2 \geq 1}$  remain limited as  $D_1, D_2 \rightarrow +\infty$  if the initial values are in a bounded subset of  $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ . If, for all  $D_1, D_2 \geq 1$ , the initial data are equal to a same constant, that is, if  $(u^{D_1}(0), v^{D_2}(0)) = (u_0, v_0) \in \mathbb{R} \times \mathbb{R}, \forall D_1, D_2 \geq 1$ , then for each  $t \in [0, t_1]$ , the sequences of real numbers  $\{\|\nabla u^{D_1}(t)\|_p^p\}_{D_1 \geq 1}$  and  $\{\|\nabla v^{D_2}(t)\|_q^q\}_{D_2 \geq 1}$  converge to zero as  $D_1, D_2 \rightarrow +\infty$ .

In fact, let  $(u^{D_1}, v^{D_2})$  be a solution of problem (I) in  $(0, t_1)$ . Therefore, there are

$$f^{(D_1, D_2)}, g^{(D_1, D_2)} \in L^1(0, t_1; H),$$

with

$$f^{(D_1, D_2)}(t) \in F(u^{D_1}(t), v^{D_2}(t)), \quad g^{(D_1, D_2)}(t) \in G(u^{D_1}(t), v^{D_2}(t)) \quad \text{a.e. in } (0, t_1),$$

and such that  $(u^{D_1}, v^{D_2})$  is a solution of the system  $(\bar{I})$  below:

$$(\bar{I}) \quad \begin{cases} \frac{\partial u^{D_1}}{\partial t} - D_1 \Delta_p u^{D_1} + |u^{D_1}|^{p-2} u^{D_1} = f^{(D_1, D_2)} & \text{in } (0, t_1), \\ \frac{\partial v^{D_2}}{\partial t} - D_2 \Delta_q v^{D_2} + |v^{D_2}|^{q-2} v^{D_2} = g^{(D_1, D_2)} & \text{in } (0, t_1), \\ u^{D_1}(0) = u_0^{D_1}, \quad v^{D_2}(0) = v_0^{D_2}. \end{cases}$$

Doing the inner product of the first equation with  $\frac{\partial u^{D_1}}{\partial t}(t)$ , we obtain

$$\begin{aligned} \left\| \frac{\partial u^{D_1}}{\partial t}(t) \right\|_H^2 + \frac{d}{dt} \varphi^{D_1}(u^{D_1}(t)) &= \left\langle f^{(D_1, D_2)}(t), \frac{\partial u^{D_1}}{\partial t}(t) \right\rangle \\ &\leq \|f^{(D_1, D_2)}(t)\|_H \left\| \frac{\partial u^{D_1}}{\partial t}(t) \right\|_H \\ &\leq \frac{1}{2} \|f^{(D_1, D_2)}(t)\|_H^2 + \frac{1}{2} \left\| \frac{\partial u^{D_1}}{\partial t}(t) \right\|_H^2. \end{aligned} \tag{3.4}$$

Therefore

$$\frac{1}{2} \left\| \frac{\partial u^{D_1}}{\partial t}(t) \right\|_H^2 + \frac{d}{dt} \varphi^{D_1}(u^{D_1}(t)) \leq \frac{1}{2} \|f^{(D_1, D_2)}(t)\|_H^2.$$

In particular,

$$\frac{d}{dt} \varphi^{D_1}(u^{D_1}(t)) \leq \frac{1}{2} \|f^{(D_1, D_2)}(t)\|_H^2. \tag{3.5}$$

Using Remark 2.2 and the fact that  $F$  and  $G$  map bounded sets of  $H \times H$  in bounded sets of  $H$ , it follows that there exists a positive constant  $C$  such that  $\|f^{(D_1, D_2)}(t)\|_H^2 \leq C, \forall t \in [0, t_1]$  and  $\forall D_1, D_2 \geq 1$ . Integrating from 0 to  $\tau, \tau \in [0, t_1]$  in (3.5), we obtain

$$\begin{aligned} \frac{1}{p} (D_1 \|\nabla u^{D_1}(\tau)\|_p^p + \|u^{D_1}(\tau)\|_p^p) &= \varphi^{D_1}(u^{D_1}(\tau)) \\ &\leq \varphi^{D_1}(u_0^{D_1}) + \frac{1}{2} \int_0^\tau \|f^{(D_1, D_2)}(t)\|_H^2 dt \\ &\leq \varphi^{D_1}(u_0^{D_1}) + \frac{1}{2} \int_0^{t_1} \|f^{(D_1, D_2)}(t)\|_H^2 dt \\ &\leq \varphi^{D_1}(u_0^{D_1}) + \frac{1}{2} C t_1 \\ &\leq \frac{1}{p} (D_1 \|\nabla u_0^{D_1}\|_p^p + \|u_0^{D_1}\|_p^p) + \frac{1}{2} C t_1, \end{aligned} \tag{3.6}$$

$\forall \tau \in [0, t_1]$  and  $\forall D_1 \geq 1$ . Therefore,

$$\|\nabla u^{D_1}(\tau)\|_p^p \leq \|\nabla u_0^{D_1}\|_p^p + \frac{1}{D_1} \left( \|u_0^{D_1}\|_p^p + \frac{p}{2} C t_1 \right), \quad \forall \tau \in [0, t_1] \text{ and } \forall D_1 \geq 1. \tag{3.7}$$

Analogously we prove that

$$\|\nabla v^{D_2}(\tau)\|_q^q \leq \|\nabla v_0^{D_2}\|_q^q + \frac{1}{D_2} \left( \|v_0^{D_2}\|_q^q + \frac{q}{2} C t_1 \right), \quad \forall \tau \in [0, t_1] \text{ and } \forall D_2 \geq 1. \quad \square \tag{3.8}$$

Lemma 3.1 confirms that Eq. (II) is a good candidate for the limit problem.

**Lemma 3.2.** *The problem (II) has a global solution.*

**Proof.** Consider  $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  given by  $\phi_p(v) \doteq |v|^{p-2}v$  and  $\psi(v) \doteq \int_0^v |s|^{p-2}s ds$ , respectively. We have that  $\phi_p$  is the subdifferential of the non-negative, convex, proper and l.s.c. map,  $\psi$ , defined on the Hilbert space  $\mathbb{R}$  with  $\psi(0) = 0$  (see Lemma 3.2 in [30]). Consequently, we conclude that  $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$  is a maximal monotone operator with  $D(\phi_p) = \mathbb{R}$ . Applying Theorem 2.4 in [29] we obtain the existence of a local strong solution for the problem (II).

As the couple  $(F, G)$  is positively sublinear in  $H \times H$ , the couple  $(\tilde{F}, \tilde{G})$  is positively sublinear in  $\mathbb{R} \times \mathbb{R}$ , and so we can prove the existence of global solution by standard arguments, as it is done in [29].  $\square$

**Theorem 3.1.** *The problem (II) defines a generalized semiflow  $\mathbb{G}^\infty$  which has a global B-attractor  $\mathcal{A}^\infty$ .*

**Proof.** Let  $D(u_0, v_0)$  be the set of the solutions of (II) with initial values  $(u_0, v_0)$  and consider  $\mathbb{G}^\infty \doteq \bigcup_{(u_0, v_0) \in \mathbb{R} \times \mathbb{R}} D(u_0, v_0)$ . Note that  $A = \phi_p, B = \phi_q$  are univalued operators, which are subdifferentials of non-negatives, convex, proper and l.s.c. maps,  $\psi_A, \psi_B$ , respectively, defined in a real Hilbert space  $H = \mathbb{R}$ ,  $\psi_A(0) = \psi_B(0) = 0$ , with  $A$  and  $B$  generating compact semigroups. So, we can apply the abstract results in [29].

The dissipativity can be obtained as it is done in Theorem 2.8 [29]. It follows from Theorem 2.7 in [29] that  $\mathbb{G}^\infty$  is asymptotically compact. Then, Theorem 9 in [28] guarantees that  $\mathbb{G}^\infty$  has a global B-attractor  $\mathcal{A}^\infty$ .  $\square$

Now we prove that (II) is in fact the limit problem for (I), as  $D_1, D_2 \rightarrow +\infty$ .

**Theorem 3.2.** *Let  $(u^{D_{1n}}, v^{D_{2n}})$  be a solution of the problem (I). Suppose that the initial values  $(u^{D_{1n}}(0), v^{D_{2n}}(0)) = (u_0^{D_{1n}}, v_0^{D_{2n}}) \rightarrow (u_0, v_0) \in \mathbb{R} \times \mathbb{R}$  in  $H \times H$  as  $n \rightarrow +\infty$ . Then there exists a solution  $(u, v)$  for (II) satisfying  $(u(0), v(0)) = (u_0, v_0)$  and a subsequence  $\{(u^{D_{1n_j}}, v^{D_{2n_j}})\}_j$  of  $\{(u^{D_{1n}}, v^{D_{2n}})\}_n$  such that, for each  $T > 0, u^{D_{1n_j}} \rightarrow u, v^{D_{2n_j}} \rightarrow v$  in  $C([0, T]; H)$  as  $j \rightarrow +\infty$ .*

**Proof.** Let  $T > 0$  be arbitrarily large. Let  $(u^{D_{1n}}, v^{D_{2n}})$  be a solution for (I) with  $(u^{D_{1n}}(0), v^{D_{2n}}(0)) = (u_0^{D_{1n}}, v_0^{D_{2n}}) \rightarrow (u_0, v_0) \in \mathbb{R} \times \mathbb{R}$  in  $H \times H$  as  $n \rightarrow +\infty$ . Therefore, there are  $f_n, g_n \in L^1(0, T; H)$ , with

$$f_n(t) \in F(u^{D_{1n}}(t), v^{D_{2n}}(t)), \quad g_n(t) \in G(u^{D_{1n}}(t), v^{D_{2n}}(t)) \quad \text{a.e. in } (0, T),$$

and such that  $(u^{D_{1n}}, v^{D_{2n}})$  is a solution of system  $(P_n^1)$  below:

$$(P_n^1) \quad \begin{cases} \frac{\partial u^{D_{1n}}}{\partial t} - D_{1n} \Delta_p u^{D_{1n}} + |u^{D_{1n}}|^{p-2} u^{D_{1n}} = f_n & \text{in } (0, T), \\ \frac{\partial v^{D_{2n}}}{\partial t} - D_{2n} \Delta_q v^{D_{2n}} + |v^{D_{2n}}|^{q-2} v^{D_{2n}} = g_n & \text{in } (0, T), \\ u^{D_{1n}}(0) = u_0^{D_{1n}}, \quad v^{D_{2n}}(0) = v_0^{D_{2n}}. \end{cases}$$

We denote  $u^{D_{1n}}(\cdot) \doteq I(u_0^{D_{1n}})f_n(\cdot)$  and  $v^{D_{2n}}(\cdot) \doteq I(v_0^{D_{2n}})g_n(\cdot)$  and also denote by  $z^{D_{1n}}(\cdot) \doteq I(u_0)f_n(\cdot)$  and  $w^{D_{2n}}(\cdot) \doteq I(v_0)g_n(\cdot)$  the solutions of the problems

$$(P_{f_n, u_0}) \quad \begin{cases} \frac{\partial z^{D_{1n}}}{\partial t} - D_{1n} \Delta_p z^{D_{1n}} + |z^{D_{1n}}|^{p-2} z^{D_{1n}} = f_n, \\ z^{D_{1n}}(0) = u_0 \end{cases}$$

and

$$(P_{g_n, v_0}) \quad \begin{cases} \frac{\partial w^{D_{2n}}}{\partial t} - D_{2n} \Delta_q w^{D_{2n}} + |w^{D_{2n}}|^{q-2} w^{D_{2n}} = g_n, \\ w^{D_{2n}}(0) = v_0, \end{cases}$$

respectively.

Doing the inner product of the first equation in  $(P_n^1)$  with  $u^{D_{1n}}$  and integrating from 0 to  $t, t \leq T$ , we obtain

$$\frac{1}{2} \|u^{D_{1n}}(t)\|_H^2 \leq \frac{1}{2} \|u_0^{D_{1n}}\|_H^2 + \int_0^t \langle f_n(s), u^{D_{1n}}(s) \rangle ds.$$

As  $\{u_0^{D_{1n}}\}$  is a convergent sequence we have that there exists a positive constant  $R$  such that  $\|u_0^{D_{1n}}\|_H^2 \leq R^2$ . Thus,

$$\frac{1}{2} \|u^{D_{1n}}(t)\|_H^2 \leq \frac{1}{2} R^2 + \int_0^t \langle f_n(s), u^{D_{1n}}(s) \rangle ds.$$

Now we use the positive sublinearity of the pair  $(F, G)$ . Consider the constants  $a, b, c, m_0 > 0$  given in the introduction. Once

$$f_n(t) \in F(u^{D_{1n}}(t), v^{D_{2n}}(t)), \quad g_n(t) \in G(u^{D_{1n}}(t), v^{D_{2n}}(t)) \quad \text{a.e. in } (0, T),$$

we can consider the measurable subset  $D \subset [0, T]$  defined in the following way:  $s \in D$  iff

$$\|u^{D_{1n}}(s)\|_H \leq m_0 \quad \text{and} \quad \|v^{D_{2n}}(s)\|_H \leq m_0$$

or

$$\langle u^{D_{1n}}(s), f_n(s) \rangle \leq 0 \quad \text{and} \quad \langle v^{D_{2n}}(s), g_n(s) \rangle \leq 0.$$

Consider also the following two measurable subsets  $\tilde{D} \doteq D \cap (0, t)$  and  $\tilde{\tilde{D}} \doteq D^c \cap (0, t)$ . So, there is a constant  $M_0 > 0$  such that

$$\int_{\tilde{\tilde{D}}} \langle u^{D_{1n}}(s), f_n(s) \rangle ds \leq M_0.$$

From the positive sublinearity of the pair  $(F, G)$  we have that for  $s \in \tilde{\tilde{D}}$

$$\langle u^{D_{1n}}(s), f_n(s) \rangle \leq \|u^{D_{1n}}(s)\|_H \|f_n(s)\|_H \leq \|u^{D_{1n}}(s)\|_H [a\|u^{D_{1n}}(s)\|_H + b\|v^{D_{2n}}(s)\|_H + c].$$

Then,

$$\begin{aligned} \|u^{D_{1n}}(t)\|_H^2 &\leq R^2 + 2 \int_{\tilde{D} \cup \tilde{\tilde{D}}} \langle f_n(s), u^{D_{1n}}(s) \rangle ds \\ &\leq R^2 + 2 \int_{\tilde{D}} \langle f_n(s), u^{D_{1n}}(s) \rangle ds + 2 \int_{\tilde{\tilde{D}}} \langle f_n(s), u^{D_{1n}}(s) \rangle ds \\ &\leq R^2 + 2M_0 + 2 \int_{\tilde{\tilde{D}}} [a\|u^{D_{1n}}(s)\|_H + b\|v^{D_{2n}}(s)\|_H + c] \|u^{D_{1n}}(s)\|_H ds \\ &\leq R^2 + 2M_0 + 2a \int_{\tilde{\tilde{D}}} \|u^{D_{1n}}(s)\|_H^2 ds + 2b \int_{\tilde{\tilde{D}}} \|v^{D_{2n}}(s)\|_H \|u^{D_{1n}}(s)\|_H ds + 2c \int_{\tilde{\tilde{D}}} \|u^{D_{1n}}(s)\|_H ds \\ &\leq R^2 + 2M_0 + 2a \int_0^t \|u^{D_{1n}}(s)\|_H^2 ds + 2b \int_0^t \|v^{D_{2n}}(s)\|_H \|u^{D_{1n}}(s)\|_H ds + 2c \int_0^t \|u^{D_{1n}}(s)\|_H ds. \end{aligned}$$

So,

$$\frac{1}{2} \|u^{D_{1n}}(t)\|_H^2 \leq \frac{1}{2} C^2 + \int_0^t [a\|u^{D_{1n}}(s)\|_H + b\|v^{D_{2n}}(s)\|_H + c] \|u^{D_{1n}}(s)\|_H ds,$$

where  $C$  is a positive constant.

Using the Gronwall's inequality we obtain

$$\|u^{D_{1n}}(t)\|_H \leq C + cT + \int_0^t [a\|u^{D_{1n}}(s)\|_H + b\|v^{D_{2n}}(s)\|_H] ds.$$

So, there is a positive constant  $M$  independent of  $t \in [0, T]$  such that

$$\|u^{D_{1n}}(t)\|_H \leq M + \int_0^t [a\|u^{D_{1n}}(s)\|_H + b\|v^{D_{2n}}(s)\|_H] ds.$$



Analogously, there exists a positive constant  $\tilde{M}$  independent of  $t \in [0, T]$  such that

$$\|v^{D_{2n}}(t)\|_H \leq \tilde{M} + \int_0^t [b\|u^{D_{1n}}(s)\|_H + a\|v^{D_{2n}}(s)\|_H] ds.$$

Adding this two inequalities and denoting by  $N \doteq M + \tilde{M}$  and  $\rho \doteq a + b$  we have

$$\|u^{D_{1n}}(t)\|_H + \|v^{D_{2n}}(t)\|_H \leq N + \rho \int_0^t [\|u^{D_{1n}}(s)\|_H + \|v^{D_{2n}}(s)\|_H] ds$$

and so it follows from the Gronwall–Bellman’s inequality that

$$\|u^{D_{1n}}(t)\|_H + \|v^{D_{2n}}(t)\|_H \leq Ne^{\rho T},$$

for all  $t \in [0, T]$  and for all  $n \in \mathbb{N}$ .

Therefore there exists  $L > 0$  such that

$$\|f_n(t)\|_H \leq L \quad \text{and} \quad \|g_n(t)\|_H \leq L, \quad \text{for all } t \in [0, T] \text{ and for all } n \in \mathbb{N}.$$

So, we conclude that there exists a positive constant  $\tilde{L}$  such that

$$\|f_n\|_{L^2(0,T;H)} \leq \tilde{L} \quad \text{and} \quad \|g_n\|_{L^2(0,T;H)} \leq \tilde{L}, \quad \text{for all } n \in \mathbb{N}.$$

As  $L^2(0, T; H)$  is a reflexive Banach space, there are  $f, g \in L^2(0, T; H)$  and subsequences  $\{f_{n_j}\}$  and  $\{g_{n_j}\}$  such that  $f_{n_j} \rightharpoonup f$  and  $g_{n_j} \rightharpoonup g$  in  $L^2(0, T; H)$ . Consequently  $f_{n_j} \rightarrow f$  and  $g_{n_j} \rightarrow g$  in  $L^1(0, T; H)$ .

Consider  $K \doteq \{f_n; n \in \mathbb{N}\}$ ,  $\tilde{K} \doteq \{g_n; n \in \mathbb{N}\}$ ,  $M(K) \doteq \{z^{D_{1n}}; n \in \mathbb{N}\}$  and  $M(\tilde{K}) \doteq \{w^{D_{2n}}; n \in \mathbb{N}\}$ . Since  $K$  and  $\tilde{K}$  are bounded sets in  $H$ , it is easy to see that they are uniformly integrable subsets in  $L^1(0, T; H)$ .

Given  $t \in (0, T]$  and  $h > 0$  such that  $t - h \in (0, T]$ , we consider the operator  $T_h : M(K)(t) \rightarrow H$  defined by  $T_h z^{D_{1n}}(t) = S^{D_{1n}}(h)z^{D_{1n}}(t - h)$ , where  $S^{D_{1n}}$  is the semigroup generated by the operator  $A^{D_{1n}}$  in  $H$  with  $A^{D_{1n}}(\theta) \doteq -D_{1n}\Delta_p\theta + |\theta|^{p-2}\theta$ . For details about the operator  $A^{D_{1n}}$  see [29].

**Statement 1.** The operator  $T_h : M(K)(t) \rightarrow H$  is compact.

The proof is completely analogous to the demonstration of Statement 1, p. 10 in [29].

Then, by Theorem 3.2 in [29], the set  $M(K)$  is relatively compact in  $C([0, T]; H)$  and so there are  $z \in C([0, T]; H)$  and a subsequence  $\{z^{D_{1n_j}}(\cdot)\}$  of  $\{z^{D_{1n}}(\cdot)\}$  such that  $z^{D_{1n_j}} \rightarrow z$  in  $C([0, T]; H)$ .

As each  $z^{D_{1n_j}}$  is a solution of  $(P_{f_{n_j}, u_0})$  in  $(0, T)$ , then by Proposition 3.6 in [3],  $z^{D_{1n_j}}$  verifies

$$\frac{1}{2} \|z^{D_{1n_j}}(t) - \theta\|^2 \leq \frac{1}{2} \|z^{D_{1n_j}}(s) - \theta\|^2 + \int_s^t \langle f_{n_j}(\tau) - y_j, z^{D_{1n_j}}(\tau) - \theta \rangle d\tau \tag{3.9}$$

for all  $\theta \in \mathcal{D}(A^{D_{1n_j}}) \subset W^{1,p}(\Omega) \subset H$ ,  $y_j = A^{D_{1n_j}}(\theta) \doteq -D_{1n_j}\Delta_p\theta + |\theta|^{p-2}\theta$  and  $0 \leq s \leq t \leq T$ .

Analogously, we can show that there exists  $w \in C([0, T]; H)$  and there exists a subsequence  $\{w^{D_{2n_j}}(\cdot)\}$  of  $\{w^{D_{2n}}(\cdot)\}$  such that  $w^{D_{2n_j}} \rightarrow w$  in  $C([0, T]; H)$ , verifying

$$\frac{1}{2} \|w^{D_{2n_j}}(t) - \theta\|^2 \leq \frac{1}{2} \|w^{D_{2n_j}}(s) - \theta\|^2 + \int_s^t \langle g_{n_j}(\tau) - y_j, w^{D_{2n_j}}(\tau) - \theta \rangle d\tau \tag{3.10}$$

for all  $\theta \in \mathcal{D}(B^{D_{2n_j}}) \subset W^{1,q}(\Omega) \subset H$ ,  $y_j = B^{D_{2n_j}}(\theta) \doteq -D_{2n_j}\Delta_q\theta + |\theta|^{q-2}\theta$  and  $0 \leq s \leq t \leq T$ .

**Statement 2.**  $u^{D_{1n_j}} \rightarrow z$  and  $v^{D_{2n_j}} \rightarrow w$  in  $C([0, T]; H)$  and moreover  $f(t) \in F(z(t), w(t))$  and  $g(t) \in G(z(t), w(t))$  a.e. in  $[0, T]$ .

In fact, let  $t \in [0, T]$ . We have

$$\|u^{D_{1n_j}}(t) - z(t)\|_H \leq \|u^{D_{1n_j}}(t) - z^{D_{1n_j}}(t)\|_H + \|z^{D_{1n_j}}(t) - z(t)\|_H.$$

Then,

$$\begin{aligned} \sup_{t \in [0, T]} \|u^{D_{1n_j}}(t) - z(t)\|_H &\leq \sup_{t \in [0, T]} \|I(u_0^{D_{1n_j}})f_{n_j}(t) - I(u_0)f_{n_j}(t)\|_H + \sup_{t \in [0, T]} \|z^{D_{1n_j}}(t) - z(t)\|_H \\ &\leq \|u_0^{D_{1n_j}} - u_0\|_H + \sup_{t \in [0, T]} \|z^{D_{1n_j}}(t) - z(t)\|_H \rightarrow 0 \end{aligned}$$

as  $j \rightarrow +\infty$ .

So  $u^{D_{1n_j}} \rightarrow z$  in  $C([0, T]; H)$  as  $j \rightarrow +\infty$ . Analogously we show that  $v^{D_{2n_j}} \rightarrow w$  in  $C([0, T]; H)$  as  $j \rightarrow +\infty$ .

Then, by Theorem 3.3 in [16],  $f(t) \in F(z(t), w(t))$  and  $g(t) \in G(z(t), w(t))$  a.e. in  $[0, T]$ .

Now consider  $\bar{\theta} \in \mathbb{R} \subset H$  and let  $\bar{h} \doteq \phi_p(\bar{\theta}) \in \mathbb{R} \subset H$ . We consider  $y_j \doteq A^{D_{1n_j}}(\bar{\theta})$ . It follows from (3.9) that

$$\frac{1}{2} \|z^{D_{1n_j}}(t) - \bar{\theta}\|^2 \leq \frac{1}{2} \|z^{D_{1n_j}}(s) - \bar{\theta}\|^2 + \int_s^t \langle f_{n_j}(\tau) - \bar{h}, z^{D_{1n_j}}(\tau) - \bar{\theta} \rangle d\tau.$$

Taking the limit as  $j \rightarrow +\infty$ , we obtain

$$\frac{1}{2} \|z(t) - \bar{\theta}\|^2 \leq \frac{1}{2} \|z(s) - \bar{\theta}\|^2 + \int_s^t \langle f(\tau) - \bar{h}, z(\tau) - \bar{\theta} \rangle d\tau \quad (3.11)$$

for all  $\bar{\theta} \in \mathbb{R}$ ,  $\bar{h} \doteq \phi_p(\bar{\theta})$  and for all  $0 \leq s \leq t \leq T$ .

In the same way we can show that

$$\frac{1}{2} \|w(t) - \bar{\theta}\|^2 \leq \frac{1}{2} \|w(s) - \bar{\theta}\|^2 + \int_s^t \langle g(\tau) - \bar{h}, w(\tau) - \bar{\theta} \rangle d\tau$$

for all  $\bar{\theta} \in \mathbb{R}$ ,  $\bar{h} \doteq \phi_q(\bar{\theta})$  and for all  $0 \leq s \leq t \leq T$ .

**Statement 3.**  $z(t)$  and  $w(t)$  are independents on  $x$ , for each  $t > 0$ .

In fact, let  $t > 0$ . We already know that  $z^{D_{1n_j}}(t) \rightarrow z(t)$  in  $H$ . Since  $z^{D_{1n}}(0) = u_0$ ,  $\forall n \in \mathbb{N}$ , then by Remark 3.1 and Lemma 3.1 we have that  $\|\nabla z^{D_{1n_j}}(t)\|_H \rightarrow 0$  as  $j \rightarrow +\infty$ . We also have that  $z^{D_{1n_j}}(t) \in \mathcal{D}(A^{D_{1n_j}}) \subset W^{1,p}(\Omega) \subset W^{1,2}(\Omega)$ . Then, by the Poincaré–Wirtinger's inequality (see [4])

$$\|z^{D_{1n_j}}(t) - \overline{z^{D_{1n_j}}(t)}\|_H \leq C \|\nabla z^{D_{1n_j}}(t)\|_H \rightarrow 0 \quad \text{as } j \rightarrow +\infty.$$

Then

$$\begin{aligned} \|z(t) - \overline{z(t)}\|_H &= \|z(t) - z^{D_{1n_j}}(t) + z^{D_{1n_j}}(t) - \overline{z^{D_{1n_j}}(t)} + \overline{z^{D_{1n_j}}(t)} - \overline{z(t)}\|_H \\ &\leq \|z(t) - z^{D_{1n_j}}(t)\|_H + \|z^{D_{1n_j}}(t) - \overline{z^{D_{1n_j}}(t)}\|_H + \|\overline{z^{D_{1n_j}}(t)} - \overline{z(t)}\|_H \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \end{aligned}$$

So  $z(t) = \overline{z(t)} \doteq \frac{1}{|\Omega|} \int_{\Omega} z(t)(x) dx$ .

Analogously, we show that  $w(t) = \overline{w(t)}$ .

We already show in Statement 2 that  $f(t) \in F(z(t), w(t))$  and  $g(t) \in G(z(t), w(t))$  a.e. in  $(0, T)$ . Therefore  $f(t)$  and  $g(t)$  are independents on  $x$ , a.e. in  $(0, T)$ .

Thus, from (3.11)

$$\frac{1}{2} |z(t) - \bar{\theta}|^2 |\Omega| \leq \frac{1}{2} |z(s) - \bar{\theta}|^2 |\Omega| + \int_s^t \int_{\Omega} (f(\tau) - \bar{h})(z(\tau) - \bar{\theta}) dx d\tau.$$

So

$$\frac{1}{2} |z(t) - \bar{\theta}|^2 \leq \frac{1}{2} |z(s) - \bar{\theta}|^2 + \int_s^t (f(\tau) - \bar{h})(z(\tau) - \bar{\theta}) d\tau$$

for all  $\bar{\theta} \in \mathbb{R}$ ,  $\bar{h} \doteq \phi_p(\bar{\theta})$  and for all  $0 \leq s < t \leq T$ .

If  $t = s = 0$ , we have  $z(0) = \lim_{j \rightarrow +\infty} z^{D_{1n_j}}(0) = \lim_{j \rightarrow +\infty} u_0 = u_0$ . Therefore  $\frac{1}{2}|z(0) - \bar{\theta}|^2 = \frac{1}{2}|u_0 - \bar{\theta}|^2$ .  
So

$$\frac{1}{2}|z(t) - \bar{\theta}|^2 \leq \frac{1}{2}|z(s) - \bar{\theta}|^2 + \int_s^t (f(\tau) - \bar{h})(z(\tau) - \bar{\theta}) d\tau$$

for all  $\bar{\theta} \in \mathbb{R}$ ,  $\bar{h} \doteq \phi_p(\bar{\theta})$  and for all  $0 \leq s \leq t \leq T$ .

In the same way,

$$\frac{1}{2}|w(t) - \bar{\theta}|^2 \leq \frac{1}{2}|w(s) - \bar{\theta}|^2 + \int_s^t (g(\tau) - \bar{h})(w(\tau) - \bar{\theta}) d\tau$$

for all  $\bar{\theta} \in \mathbb{R}$ ,  $\bar{h} \doteq \phi_q(\bar{\theta})$  and for all  $0 \leq s \leq t \leq T$ .

So by Proposition 3.6 in [3], we conclude that  $(z, w)$  is a weak solution of problem (II) with  $(z(0), w(0)) = (u_0, v_0)$  (see Definition 2.10 in [29]), but as  $f, g \in L^2(0, T; H)$  we have in fact that  $(z, w)$  is a strong solution of problem (II).  $\square$

**Remark 3.2.** Theorem 3.2 continues valid without the hypothesis  $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$ , whenever  $(u_0^{D_{1n}}, v_0^{D_{2n}}) \in \mathcal{A}_{(D_{1n}, D_{2n})}$ ,  $\forall n \in \mathbb{N}$ , because in this case we prove, analogously as was done in Lemma 4.1 in [30], that  $u_0$  and  $v_0$  are independents on  $x$ .

### 3.1. Continuity of attractors

In this section we prove that the family of attractors behaves continuously as the diffusion parameter increases to infinity. We start by proving the upper semicontinuity, and it is done by constructing a complete bounded orbit through the limit of any sequence of points in the attractors.

**Theorem 3.3.** *The family of attractors  $\{\mathcal{A}_{(D_1, D_2)}\}_{D_1, D_2 \geq 1}$  associated with problem (I) is upper semicontinuous on infinity, on the topology of  $H \times H$ .*

**Proof.** Let  $\{(u_0^{D_{1j}}, v_0^{D_{2j}})\}_{D_{1j}, D_{2j} \geq 1}$  be an arbitrary sequence with

$$(u_0^{D_{1j}}, v_0^{D_{2j}}) \in \mathcal{A}_{(D_{1j}, D_{2j})}, \quad \forall D_{1j}, D_{2j} \geq 1 \text{ and } D_{1j}, D_{2j} \rightarrow +\infty \text{ as } j \rightarrow +\infty.$$

By Lemma 2.4, there exists a subsequence, that we still denote in the same way, such that  $(u_0^{D_{1j}}, v_0^{D_{2j}}) \rightarrow (u_0, v_0)$  in  $H \times H$  as  $j \rightarrow +\infty$ . By [11], it is enough to prove that  $(u_0, v_0) \in \mathcal{A}^\infty$ .

Using the invariance of the attractors, Lemma 3.1 and Poincaré–Wirtinger’s inequality, we can prove analogously to Lemma 4.1 in [30], that  $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$ .

For each  $j \in \mathbb{N}$ , consider  $t_j > j$ ,  $t_1 < t_2 < \dots < t_j < \dots$ . By invariance of the attractors, there are  $(x_j, y_j) \in \mathcal{A}_{(D_{1j}, D_{2j})}$  and solutions  $\varphi^{(D_{1j}, D_{2j})} = (\varphi_1^{D_{1j}}, \varphi_2^{D_{2j}}) \in \mathbb{G}(D_{1j}, D_{2j})$  with  $\varphi^{(D_{1j}, D_{2j})}(0) = (x_j, y_j)$  such that  $\varphi^{(D_{1j}, D_{2j})}(t_j) = (u_0^{D_{1j}}, v_0^{D_{2j}}) \rightarrow (u_0, v_0)$  in  $H \times H$  as  $j \rightarrow +\infty$ . Note that

$$\varphi^{(D_{1j}, D_{2j})}(t_j) \in T_{(D_{1j}, D_{2j})}(t_j)(x_j, y_j) \in \mathcal{A}_{(D_{1j}, D_{2j})}, \quad \forall j \in \mathbb{N}.$$

Using condition  $(H_2)$  on the definition of generalized semiflow, for each  $j \in \mathbb{N}$ , the translates  $(\varphi^{(D_{1j}, D_{2j})})^{t_j}$  are also solutions, and we have  $(\varphi^{(D_{1j}, D_{2j})})^{t_j}(0) \rightarrow (u_0, v_0)$  in  $H \times H$  as  $j \rightarrow +\infty$ .

Using Theorem 3.2, we obtain that there exist a solution  $g_0$  of the limit problem (II) with  $g_0(0) = (u_0, v_0)$  and a subsequence of  $\{(\varphi^{(D_{1j}, D_{2j})})^{t_j}\}_j$ , that we still denote the same, such that

$$(\varphi^{(D_{1j}, D_{2j})})^{t_j}(t) \rightarrow g_0(t) \quad \text{in } H \times H \text{ as } j \rightarrow +\infty, \quad \forall t \geq 0.$$

Now we consider the sequence  $\{\varphi^{(D_{1j}, D_{2j})}(t_j - 1)\}$ . Note that

$$\varphi^{(D_{1j}, D_{2j})}(t_j - 1) \in T_{(D_{1j}, D_{2j})}(t_j - 1)(x_j, y_j) \subset \bigcup_{D_1, D_2 \geq 1} \mathcal{A}_{(D_1, D_2)}$$

that is a precompact subset in  $H \times H$ , then, passing to a subsequence if necessary,

$$(\varphi^{(D_{1j}, D_{2j})})^{(t_j - 1)}(0) = \varphi^{(D_{1j}, D_{2j})}(t_j - 1) \rightarrow z_1 \quad \text{in } H \times H \text{ as } j \rightarrow +\infty.$$

As for each  $j \in \mathbb{N}$ ,  $\varphi^{(D_{1j}, D_{2j})}$  is a solution beginning on the attractor  $\mathcal{A}_{(D_{1j}, D_{2j})}$ , we obtain by the invariance of the attractors that the sequence of initial values

$$\varphi^{(D_{1j}, D_{2j})}(t_j - 1) \in \mathcal{A}_{(D_{1j}, D_{2j})}, \quad \forall j \in \mathbb{N}.$$

So using Remark 3.2 and Theorem 3.2, we obtain that there exist a solution  $g_1$  of the limit problem (II) with  $g_1(0) = z_1$  and a subsequence of  $\{(\varphi^{(D_{1j}, D_{2j})})^{(t_j-1)}\}_j$ , that we still denote in the same way, such that

$$(\varphi^{(D_{1j}, D_{2j})})^{(t_j-1)}(t) \rightarrow g_1(t) \quad \text{in } H \times H \text{ as } j \rightarrow +\infty, \quad \forall t \geq 0.$$

Now note that  $g_1^1 = g_0$ , since for each  $t \geq 0$ , we have

$$g_1^1(t) = g_1(t+1) = \lim_{j \rightarrow +\infty} (\varphi^{(D_{1j}, D_{2j})})^{(t_j-1)}(t+1) = \lim_{j \rightarrow +\infty} (\varphi^{(D_{1j}, D_{2j})})^{t_j}(t) = g_0(t).$$

Proceeding so inductively, we find for each  $r = 0, 1, 2, \dots$ , a solution  $g_r \in \mathcal{G}^\infty$  with  $g_r(0) = z_r$  such that  $g_{r+1}^1 = g_r$ . Given  $t \in \mathbb{R}$ , we define  $g(t)$  as the common value of  $g_r(t+r)$  for  $r > -t$ . Then we have that  $g$  is a complete orbit for  $\mathcal{G}^\infty$  with  $g(0) = g_0(0) = (u_0, v_0)$ .

Note that for each  $t \geq 0$ ,  $r = 0, 1, 2, \dots$ , we have that each

$$g_r(t) = \lim_{j \rightarrow +\infty} (\varphi^{(D_{1j}, D_{2j})})^{(t_j-r)}(t) \quad \text{and} \quad (\varphi^{(D_{1j}, D_{2j})})^{(t_j-r)}(t) \in \mathcal{A}_{(D_{1j}, D_{2j})}, \quad \forall j \in \mathbb{N}.$$

Working with the coordinated functions and using the invariance of the attractors, Lemma 3.1 and the Poincaré–Wirtinger’s inequality, we can prove, analogously to Lemma 4.1 in [30], that each  $g_r(t)$  independents on  $x$ . Consequently, we obtain that  $g(t)$  is a constant function on  $x$ . As  $\mathcal{A}_{(D_{1j}, D_{2j})} \subset \bigcup_{D_1, D_2 \geq 1} \mathcal{A}_{(D_1, D_2)}$ ,  $\forall j \in \mathbb{N}$ , we obtain that there exists a constant  $C > 0$  such that  $\|g_r(t)\|_{H \times H} \leq C$ ,  $\forall t \geq 0$ ,  $r = 0, 1, 2, \dots$ . So, in particular, we have that  $g(t)$  is bounded in  $H \times H$ . Then, there exists a constant  $\tilde{C} > 0$  such that

$$\|g(t)\|_{\mathbb{R} \times \mathbb{R}} = \frac{1}{|\Omega|^{1/2}} \|g(t)\|_{H \times H} \leq \tilde{C}, \quad \forall t \in \mathbb{R}.$$

So, we conclude that  $g : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  is a complete bounded orbit for  $\mathcal{G}^\infty$  through  $(u_0, v_0)$ .

Using Theorem 15 in [28], we obtain that  $(u_0, v_0) \in \mathcal{A}^\infty$ .  $\square$

**Remark 3.3.** Note that each complete bounded orbit of the limit problem (II) is a complete bounded orbit of problem (I). So  $\mathcal{A}^\infty \subset \mathcal{A}_{(D_1, D_2)}$ ,  $\forall D_1, D_2 \geq 1$ . Consequently we obtain that the family of attractors  $\{\mathcal{A}_{(D_1, D_2)}\}_{D_1, D_2 \geq 1}$  associated with problem (I) is lower semicontinuous on infinity, on the topology of  $H \times H$ , that means,

$$\sup_{x \in \mathcal{A}^\infty} \text{dist}_{H \times H}(x, \mathcal{A}_{(D_1, D_2)}) \rightarrow 0 \quad \text{as } D_1, D_2 \rightarrow +\infty.$$

So, using Theorem 3.3 and Remark 3.3, we obtain that the family of attractors  $\{\mathcal{A}_{(D_1, D_2)}\}_{D_1, D_2 \geq 1}$  associated with the problem (I) is continuous on infinity, on the topology of  $H \times H$ .

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