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NOTE

The Nevanlinna Functions of the Riemann Zeta-Function

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Both Nevanlinna theory and zeta-function theory have been studied for a rather long time. However, to my knowledge, there are no publications about the general distribution of the value sof the Riemann zeta-function in the context of Nevanlinna theory. Due to the recent development of finding analogies between number theory and Nevanlinna theory (e.g., [9, 10, 8]), it is natural to start working on the Riemann zeta-function in the light of Nevanlinna theory.

In this note, we are going to compute the Nevanlinna characteristic function, deficiencies, and counting functions of the Riemann zeta-function. Moreover, we generalize the Riemann–von Mandoldt formula which plays an important role in zeta-function theory. Since the Riemann zeta-function is related to the Euler gamma-function, computations of the Nevanlinna functions and all deficiencies of the Euler gamma-function are also included in the Appendix of this note. With these in hand, people could tackle other problems in Nevanlinna theory for the Riemann zeta-function, for instance, finding a precise structure of the error terms of the Riemann zeta-function in the sense of the second main theorem in Nevanlinna theory, as we have done in [5] for other classical functions such as the Euler gamma-function and the Weierstrass $\wp$-, ζ-, σ-, and ϑ-function. In fact, Goldberg [1] and Korenkov [3] computed the Nevanlinna deficiencies of the Weierstrass σ-function.

For the convenience of the general reader, we briefly give some definitions and notation of Nevanlinna theory and the Riemann zeta-function. Standard references for Nevanlinna theory and for the Riemann zeta-function are [2, 4, and 7], respectively.

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Let f be a meromorphic function in the complex plane $\mathbb{C}$ and $D_R = \{|z| < r\}$. Denote the number of poles of f in D_r by $n(f, \infty, r)$; and $n(f, a, r) = n(1/f - a, \infty, r)$ if $a \neq \infty$. We also let

$$N(f, a, r) = \int_0^r \frac{n(f, a, t) - n(f, a, 0)}{t}\, dt + n(f, a, 0) \log r.$$

This integrated function $N(f, a, r)$ occurs naturally in the main theorems of Nevanlinna theory. It measures the number of a-values of f in D_r.

The proximity function in Nevanlinna function is defined as

$$m(f, r) = \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| \frac{d\theta}{2\pi},$$

$$m(f, a, r) = m(1/f - a, r) \quad \text{for } a \in \mathbb{C},$$

where $\log^+ x = \max\{0, \log x\}$. This function measures how close f is to the value a on the boundary of D_r.

The characteristic function of f in Nevanlinna theory is defined by

$$T(f, r) = N(f, \infty, r) + m(f, r).$$

However, let $T(f, a, r) = N(f, a, r) + m(f, a, r)$; the first main theorem ([2, Theorem 1.2]) states, for any $a \in \mathbb{C}$,

$$T(f, r) = N(f, a, r) + m(f, a, r) + O(1).$$

The quantity

$$\delta(f, a) = \liminf_{r \to \infty} \frac{m(f, a, r)}{T(f, r)} = 1 - \limsup \frac{N(f, a, r)}{T(f, r)}$$

is called the deficiency of the value a of f. Obviously, $\delta(f, a)$ is positive only if there are relatively few roots of the equation $f(z) = a$ in $\mathbb{C}$. Moreover, the second main theorem ([2, Theorem 2.4]) in Nevanlinna theory implies the deficiency relation

$$\sum_{a \in C \cup \{\infty\}} \delta(f, a) \leq 2.$$

The Riemann zeta-function $\zeta(s)$ can be defined by a Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (s = \sigma + it); \tag{1}$$

or a Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (s = \sigma + it), \tag{2}$$

where p runs through all prime numbers. The notation $s = \sigma + it$ (σ, t real) in the Riemann zeta-function is traditional in this context. It is known that ζ can be analytically continued to a meromorphic function in the whole complex plane. In short, ζ has only one pole at $s = 1$, trivial simple zeros at $s = -2n$ ($n = 1, 2, \dots$), and no zeros in

$$\{s \in \mathbb{C}: \sigma < 0\} \cup \{s: \sigma > 1\} \backslash \{-2n \in \mathbb{R}: n = 1, 2, 3, \dots\}.$$

We also need the functional equation (Theorem 2.1 in [7])

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s), \tag{3}$$

and the Riemann–von Mangoldt formula ([7, Theorem 9.4])

$$N^*(R) = \frac{R}{2\pi} \log \frac{R}{2\pi} O(R), \tag{4}$$

where $R > 0$ and $N^*(R)$ is the number of zeros of ζ in the region $0 < \sigma < 1$, $0 < t < R$.

THEOREM 1. (1) $T(\zeta, r) = (r/\pi) \log r + O(r)$.

(2) $\delta(\zeta, \infty) = 1$, *and* $\delta(\zeta, a) = 0$, *for any* $a \neq \infty$.

(3) *There exists a set* $E \subset \mathbb{R}$ *with finite Lebesgue measure such that, for any* $a \in \mathbb{C}$,

$$N(\zeta, a, r) = \frac{r}{\pi} \log r + O(r) \quad (r \notin E).$$

Remark. We have seen from (4) that the relationship between the Riemann–von Mangoldt formula and Nevanlinna counting functions is

$$N(\zeta, 0, r) = 2N(r) + O(r) = \frac{r}{\pi} \log r + O(r).$$

Moreover, part (3) of Theorem 1 tells us that

$$N(\zeta, a, r) = N(\zeta, 0, r) = \frac{r}{\pi} \log r + O(r) \quad (r \notin E)$$

for any $a \in \mathbb{C} \backslash \{0\}$. Thus, broadly speaking, the number of zeros of $\zeta - a$ is equal to the number of zeros of ζ up to a term $O(r)$. This generalizes the Riemann–von Mangoldt formula in the sense of Nevanlinna theory.

Proof of Theorem 1. For $r > 0$, the number of trivial zeros of ζ in D_r is $O(r)$. Thus (4) gives

$$n(\zeta, 0, r) \geq 2N\left(\sqrt{r^2 - 1}\right) + O(r) \geq \frac{\sqrt{r^2 - 1}}{\pi} \log \sqrt{r^2 - 1} + O(r),$$

and

$$n(\zeta, 0, r) \leq 2N(r) + O(r) \leq \frac{r}{\pi} \log r + O(r).$$

It follows that

$$N(\zeta, 0, r) = \frac{r}{\pi} \log r + O(r). \tag{5}$$

Let $\sigma_0 > 1$ be a fixed real number, $s = re^{i\theta} = \sigma + it$, and

$$\gamma_1(r, \sigma_0) = \left\{\theta \in [0, 2\pi]: \operatorname{Re}(re^{i\theta}) > \sigma_0\right\},$$
$$\gamma_2(r, \sigma_0) = \left\{\theta \in [0, 2\pi]: \operatorname{Re}(re^{i\theta}) < 1 - \sigma_0\right\},$$
$$\gamma_3(r, \sigma_0) = \left\{\theta \in [0, 2\pi]: 1 - \sigma_0 \leq \operatorname{Re}(re^{i\theta}) \leq \sigma_0\right\}.$$

In the sequel, we always write $s = re^{i\theta} = \sigma + it$.

For any s with $\sigma \geq \sigma_0$, we have from (1)

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{|\eta^s|} \leq \sum_{n=1}^{\infty} \frac{1}{\eta^{\sigma_0}}.$$

Consequently,

$$\int_{\gamma_1(r, \sigma_0)} \log^+ |\zeta(re^{i\theta})| \frac{d\theta}{2\pi} \leq O(1),$$

where $O(1)$ only depends on σ_0.

For any s with $\sigma \leq 1 - \sigma_0$, i.e., $\operatorname{Re}(1 - s) \geq \sigma_0$, we have from (3)

$$\log^+ |\zeta(s)| \leq \log^+ |\Gamma(1 - s)| + O(r).$$

Therefore, (10) in the Appendix of the note implies

$$\int_{\gamma_2(r,\sigma_0)} \log^+ \left| \zeta(re^{i\theta}) \right| \frac{d\theta}{2\pi} \le \int_{\gamma_2(r,\sigma_0)} \log^+ \left| \Gamma(1 - re^{i\theta}) \right| \frac{d\theta}{2\pi} + O(r)$$
$$\le \int_{-\pi/2}^{\pi/2} r \log r \cos \theta \frac{d\theta}{2\pi} + O(r)$$
$$= \frac{r}{\pi} \log r + O(r).$$

Now consider the case when $1 - \sigma_0 \le \sigma \le \sigma_0$. Since the order of the entire function $(s-1)\zeta(s)$ is 1 (see [7, Theorem 2.12 and formula (2.12.6)]), we have

$$|\zeta(s)| \le C \exp(r^{3/2}) \quad \text{for } r > 2,$$

where C is a positive absolute constant. Noting the Lebesgue measure $|\gamma_3(r, \sigma_0)| \le O(1)/r$, we have

$$\int_{\gamma_3(r,\sigma_0)} \log^+ \left| \zeta(re^{i\theta}) \right| \frac{d\theta}{2\pi} \le O(r^{1/2}).$$

It turns out

$$m(\zeta, r) = \left(\int_{\gamma_1(r,\sigma_0)} + \int_{\gamma_2(r,\sigma_0)} + \int_{\gamma_3(r,\sigma_0)} \right) \log^+ \left| \zeta(re^{i\theta}) \right| \frac{d\theta}{2\pi}$$
$$\le \frac{r}{\pi} \log r + O(r).$$

Hence, we obtain from the definition of $T(\zeta, 0, r)$, the first main theorem, and the fact $N(\zeta, \infty, r) = O(r)$ that

$$N(\zeta, 0, r) \le T(\zeta, 0, r) = N(\zeta, \infty, r) + m(\zeta, r) + O(1)$$
$$\le \frac{r}{\pi} \log r + O(r).$$

Combining this with (5), we prove the first part of the theorem and

$$\delta(\zeta, \infty) = 1 - \lim \frac{N(\zeta, \infty, r)}{T(\zeta, r)} = 1.$$

From (1), there exists $\sigma_* > 2$ such that, for $\mathrm{Re}(s) = \sigma > \sigma_*$,

$$|\zeta'(s)| = \left| \frac{\log 2}{2^s} + \sum_{n=3}^{\infty} \frac{\log n}{n^s} \right| \geq \frac{\log 2}{2|2^s|}.$$

Therefore, when $\mathrm{Re}(s) = \sigma \geq \sigma_*$,

$$\frac{1}{2\pi} \int_{\gamma_1(r,\sigma_*)} \log^+ \frac{1}{|\zeta'(s)|} \leq |s| \log 2 + O(1) = O(r).$$

When $1 - \sigma_* \leq \mathrm{Re}(s) \leq \sigma_*$, we write $\zeta'(s) = g(s)/(s-1)^2$ where g is an entire function with order 1. By ([6, Theorem 8.71]), there is a set E of finite Lebesgue measure such that $|g(s)| \geq \exp(-r^{3/2})$ for $|s| = r \notin E$. Therefore,

$$\log^+ |1/\zeta'(s)| \leq \log^+ \frac{(|s|+1)^2}{|g(s)|} \leq r^{3/2} + 2 \log r \tag{6}$$

for all large r with $|s| = r \notin E$. Thus, noting $|\gamma_3(r, \sigma_*)| = O(1)/r$,

$$\frac{1}{2\pi} \int_{\gamma_3(r,\sigma_*)} \log^+ \frac{1}{|\zeta'(s)|} \leq |\gamma_3(r,\sigma_*)| \big(r^{3/2} + O(\log r) \big) \leq O(r).$$

When $\mathrm{Re}(s) < 1 - \sigma_*$, we let

$$\beta_1(r) = \{\theta \in [\pi/2, 3\pi/2] : \mathrm{Re}(s) < 1 - \sigma_*, |\mathrm{Im}\, s| > 1\},$$
$$\beta_2(r) = \{\theta \in [\pi/2, 3\pi/2] : \mathrm{Re}(s) < 1 - \sigma_*, |\mathrm{Im}\, s| \leq 1\}.$$

Clearly, $\gamma_2(r, \sigma_*) = \beta_1(r) \cup \beta_2(r)$. When $s = re^{i\theta}$ with $\theta \in \beta_1$ and $r \notin E$ and $r > r_0$, (6) yields

$$\frac{1}{2\pi} \int_{\beta_2(r)} \log^+ \frac{1}{|\zeta'(s)|} \leq |\beta_2(r)| 2r^{3/2} = O(T(\zeta, r))$$

since $|\beta_2(r)| \leq O(1)/r$.

From [6, p. 151, eq. (2)], we have, for $-\pi < \theta < \pi$,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} - \int_0^{\infty} \frac{[x] - x + 1/2}{(x+s)^2}\, dx.$$

Since $r|\sin\theta| > 1$ for $s = re^{i\theta}$ with $\theta \in \beta_1(r)$ and $-\pi/2 < \arctan x < \pi/2$, we have

$$\left|\int_0^\infty \frac{[x] - x + 1/2}{(x+s)^2}\,dx\right| \leq \int_0^\infty \frac{dx}{(r\cos\theta + x)^2 + (r\sin\theta)^2}$$
$$\leq \frac{\pi}{r|\sin\theta|} = O(1).$$

Thus, for $s = re^{i\theta}$ with $\theta \in \beta_1(r)$,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(1). \tag{7}$$

When $s = re^{i\theta}$ with $\theta \in \beta_1(r)$, $\mathrm{Re}(1-s) > \sigma_*$. Therefore, for $s = re^{i\theta}$ with $\theta \in \beta_1(r)$, taking logarithms and differentiating (2), we get

$$\left|\frac{\zeta'}{\zeta}(1-s)\right| = \left|\sum_p \frac{\log p}{p^{1-s}}\left(1 - \frac{1}{p^{1-s}}\right)^{-1}\right|$$
$$= \left|\sum_p \log p \sum_{m=1}^{\infty} \frac{1}{p^{m(1-s)}}\right|$$
$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{|n^{1-s}|} \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{|n^{\sigma_*}|} = O(1), \tag{8}$$

where $\Lambda(n) = \log p$ if n is p or a power of p, and otherwise $\Lambda(n) = 0$.

Since (3) is equivalent to (see [7, p. 22, eq. (2.6.4)])

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{9}$$

taking the logarithmic derivative of (9) gives

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{2}\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} - \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \log\pi - \frac{\zeta'(1-s)}{\zeta(1-s)}.$$

Plugging (7) and (8) into this equation, we obtain

$$\log^+ \frac{|\zeta(s)|}{|\zeta'(s)|} = O(1)$$

for $s = re^{i\theta}$ with $\theta \in \beta_1(r)$. Consequently,

$$\log^+ \frac{1}{|\zeta'(s)|} \le \log^+ \frac{1}{|\zeta(s)|} + \log^+ \frac{|\zeta(s)|}{|\zeta'(s)|} \le \log^+ \frac{1}{|\zeta(s)|} + O(1).$$

Therefore, (3) gives

$$\begin{aligned} \int_{\beta_1(r)} \log^+ \frac{1}{|\zeta'(s)|} \frac{d\theta}{2\pi} &\le \int_{\beta_1(r)} \log^+ \frac{1}{|\Gamma(1-s)|} \frac{d\theta}{2\pi} \\ &\quad + \int_{\beta_1(r)} \log^+ \frac{1}{|\sin(s\pi/2)|} \frac{d\theta}{2\pi} \\ &\quad + \int_{\beta_1(r)} \log^+ \frac{1}{|\zeta(1-s)|} \frac{d\theta}{2\pi} + O(1) \\ &\le \int_{\beta_1(r)} \log^+ \frac{1}{|\Gamma(1-s)|} \frac{d\theta}{2\pi} + O(r). \end{aligned}$$

The last inequality holds because (see [7, eq. (3.6.5)])

$$\frac{1}{|\zeta(1-s)|} \le C \log^7 r \quad \text{for } s \in \beta_1(r).$$

It turns out from (10) in the Appendix that

$$\begin{aligned} \int_{\beta_1(r)} \log^+ \frac{1}{|\zeta'(s)|} \frac{d\theta}{2\pi} &\le \int_{-\pi/2}^{\pi/2} \log^+ |z\phi_1(z)| \frac{d\theta}{2\pi} \\ &\quad + m(\phi_2, r) + m(\phi_3, r) + m(\phi_4, r) + O(r) \\ &= O(r) = o(T(\zeta, r)), \end{aligned}$$

where ϕ_i's are defined in the Appendix. Thus we have proved that there is a set-E finite Lebesgue measure such that

$$\begin{aligned} m(1/\zeta', r) &= \left(\int_{\gamma_1(r,\sigma_*)} + \int_{\gamma_3(r,\sigma_*)} + \int_{\beta_1(r)} + \int_{\beta_2(r)} \right) \log^+ \frac{1}{|\zeta'(s)|} \frac{d\theta}{2\pi} \\ &= O(r) \end{aligned}$$

for $r \notin E$. Note that the set E only depends on ζ'.

The logarithmic derivative lemma ([1, Theorem 2.3]) gives

$$m(\zeta'/(\zeta - a), r) = O(\log r).$$

It follows for any $a \neq \infty$ and for all large r with $r \notin E$ that

$$m(\zeta, a, r) \leq m(\zeta'/(\zeta - a), r) + m(1/\zeta', r) = O(r).$$

Therefore, $\delta(\zeta, a) = 0$ for any $a \neq \infty$. So the second statement of the theorem is proved.

Since, for any $a \in \mathbb{C}$,

$$N(\zeta, a, r) = T(\zeta, r) - m(\zeta, a, r) + O(1) = \frac{r}{\pi} \log r + O(r),$$

the third statement of the theorem follows immediately. Thus Theorem 1 is proved completely.

APPENDIX
THE NEVANLINNA FUNCTIONS OF THE EULER GAMMA-FUNCTION

The Euler gamma-function $\Gamma(z)$ is given by

$$\Gamma(z) = z^{-1} e^{-\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k},$$

where $\gamma = \lim_{n \to \infty}(\sum_{k=1}^{n} k^{-1} - \log n)$ is Euler's constant. Clearly, $\Gamma(z)$ is a meromorphic function with simple poles $\{-k\}_{k=0}^{+\infty}$, and $\Gamma(z) \neq 0$ for any $z \in \mathbb{C}$.

THEOREM 2. (1) $T(\Gamma, r) = (1 + o(1))(r/\pi)\log r$.

(2) $\delta(\Gamma, 0) = \delta(\Gamma, \infty) = 1$; $\delta(\Gamma, a) = 0$, for $a \neq 0, \infty$.

Proof. For any $z = re^{i\theta}$, there is an integer n_0 with $n_0 < r \leq n_0 + 1$ such that

$$\begin{aligned} \frac{1}{\Gamma(z)} &= z\left\{\exp\left(\gamma z - \sum_{n<2r} \frac{z}{n}\right)\right\} \prod_{n=1}^{n_0}\left(1 + \frac{z}{n}\right) \\ &\quad \times \prod_{n=n_0+1}^{[2r]}\left(1 + \frac{z}{n}\right) \prod_{n=[2r]+1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-(z/n)} \\ &\equiv z\phi_1(z)\phi_2(z)\phi_3(z)\phi_4(z). \end{aligned}$$

Noting that $\gamma - \sum_{n<2r} 1/n = -\log(2r) + o(1)$, we obtain

$$\log\left|z\phi_1(z)\right| = \log r + (-\log 2r + o(1))r\cos\theta = -r\log r\cos\theta + O(r). \tag{10}$$

Also after a little computation, we have

$$m(\phi_2, r) = O(r), \qquad m(\phi_3, r) = O(r), \qquad m(\phi_4, r) = O(r).$$

Therefore,

$$\begin{aligned} m(\Gamma, 0, r) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|\Gamma(re^{i\theta})|}\, d\theta \\ &\leq m(z\phi_1, r) + m(\phi_2, r) + m(\phi_3, r) + m(\phi_4, r) \\ &= -r\log r \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \cos\theta\, d\theta + O(r) \\ &= (1 + o(1))\frac{1}{\pi} r\log r. \end{aligned}$$

The first main theorem and the fact $\Gamma(z) \neq 0$ give

$$\begin{aligned} T(\Gamma, r) &= T(\Gamma, 0, r) + O(1) \\ &= m(\Gamma, 0, r) + O(1) = (1 + o(1))\frac{1}{\pi} r\log r. \end{aligned}$$

Hence, the first part of the theorem is proved. Furthermore,

$$\delta(\Gamma, 0) = \liminf_{r\to\infty} \frac{m(\Gamma, 0, r)}{T(\Gamma, r)} = 1.$$

Since $N(\Gamma, \infty, r) = o(T(\Gamma, r))$,

$$\delta(\Gamma, \infty) = 1 - \limsup_{r\to\infty} \frac{N(\Gamma, \infty, r)}{T(\Gamma, r)} = 1.$$

It follows from the deficiency relation that $\delta(\Gamma, a) = 0$ for any $a \neq 0, \infty$. Thus the theorem is proved completely. ∎

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