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Autonomous parts and decomposition of regular tournaments

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Abstract

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In this article we present the action of a decomposition criterion for regular tournaments, called W-decomposition on tournaments presenting autonomous parts, and especially those that are undecomposable with respect to W-decomposition.

0. Introduction

The definitions not given here can be found in [4, 9, 11].

A *tournament* is a directed graph in which every pair of vertices is joined by exactly one arc. In what follows, $T=(X, U)$ denotes a tournament where X is the set of vertices and U is the set of arcs. The *score* $s(x)$ of a vertex x is the number of vertices dominated by x, and we denote by $s(x, A)$ the number of vertices of A that are dominated by x. Similarly, $s^-(x)$ denotes the number of vertices dominating x, and $s^-(x, A)$ the number of vertices of A that dominate x. A tournament is *regular* if all vertices have equal scores. A tournament is called *rotational* if its vertices can be labelled $1, 2, \ldots, n$ in such a way that, for some subset S of $\{1, \ldots, n-1\}$, vertex i dominates vertex $i+j \pmod n$ if and only if $j \in S$. In this case, S is said to be the *symbol* of T. If x is any vertex, we let $O(x)=\{y \mid (x, y) \in U\}$ and $I(x)=\{z \mid (z, x) \in U\}$. The *cyclone* is the rotational tournament with symbol $\{1, 2, 3, \ldots, (n-1)/2\}$. It is unique up to isomorphism for n fixed.

An *automorphism* of a tournament is a permutation of the vertices which preserves the dominance relation. A tournament T is said to be *vertex-symmetric* if, for every

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pair of vertices x, y, there is an automorphism that sends x to y. The term *3-circuit* will denote a circuit of length 3.

Two tournaments $T=(X, U)$ and $T'=(X', U')$ are said to be *isomorphic* iff there is a bijection σ between X and X' such that for any arc (x, y) we have $(x, y) \in U \Leftrightarrow (\sigma(x), \sigma(y)) \in U'$.

In [5] it was defined a decomposition criterion for regular tournaments and an application to the isomorphism test of such graphs. In this paper we investigate more precisely the theoretical aspect of this method of decomposition.

1. Preliminaries: decomposition criterion

We are now going to recall the definition of the decomposition criterion for regular tournaments. The proofs of the results presented in this section can be found in [5, 7].

Definition 1.1. The weight $w(x, y)$ of any arc (x, y) of a regular tournament T of order n is the number of 3-circuits containing the arc (x, y).

This gives us a classification of the arcs, but it is better to classify the vertices; so, we now create two lists for each vertex x of T:

the in-weight list of x denoted by $\mathrm{IW}(x)$, giving the weight of all ingoing arcs of x, and

the out-weight list of x denoted by $\mathrm{OW}(x)$, giving the weight of all outgoing arcs of x.

These lists are ordered in nondecreasing order.

Remarks 1.2 (Kotzig [7]). The regularity of the tournament implies that, for any vertex x, we have

$$|\mathrm{IW}(x)| = |\mathrm{OW}(x)| = \frac{n-1}{2},$$

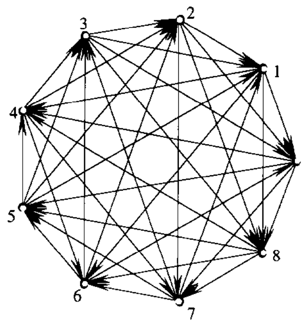
$$\sum_{y \in O(x)} w(x, y) = \tfrac{1}{8}(n-1)(n+1),$$

$$\text{for any } y \in O(x), \quad w(x, y) = s(y, I(x)) = \frac{n-1}{2} - s(y, O(x)),$$

$$\text{for any } z \in I(x), \quad w(z, x) = s^{-}(z, O(x)) = s(z, I(x)) + 1,$$

and, for any arc u of a regular tournament,

$$1 \leqslant w(u) \leqslant \frac{n-1}{2}.$$



W-classification:

IW-lists	OW-lists	W-classes
(1, 2, 3, 4)	(2, 2, 3, 3)	{8}
(1, 3, 3, 3)	(2, 2, 3, 3)	{2}
(2, 2, 3, 3)	(1, 2, 3, 4)	{3}
(2, 2, 3, 3)	(1, 3, 3, 3)	{9}
(2, 2, 3, 3)	(2, 2, 3, 3)	{1,4,5,6,7}

Fig. 1.

We shall now classify the vertices of a regular tournament according to their IW- and OW-lists.

Definition 1.3. Let $T=(X, U)$ be a regular tournament. A W-class C is a subset of the set of vertices such that x and $x' \in C$ iff $\mathrm{IW}(x)=\mathrm{IW}(x')$ and $\mathrm{OW}(x)=\mathrm{OW}(x')$.

Example 1.4. See Fig. 1.

Proposition 1.5 (Dugat [5]). *Two vertices x and x' of a regular tournament are in the same W-class iff:*

(i) *the subtournaments $O(x)$ and $O(x')$ have the same score vector, as well as*

(ii) *the subtournaments $I(x)$ and $I(x')$ have the same score vector.*

Definition 1.6. The W-classification of a regular tournament $T=(X, U)$ is the partition of X in W-classes.

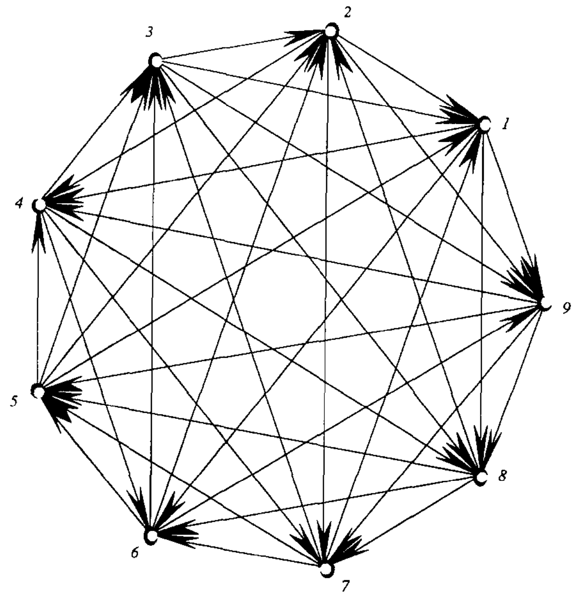
Remark 1.7. Two isomorphic tournaments have the same W-classification.

We present in Fig. 2 two tournaments that have the same W-classification with IW-lists and OW-lists equal to (2, 2, 3, 3), but they are not isomorphic (see [2]).

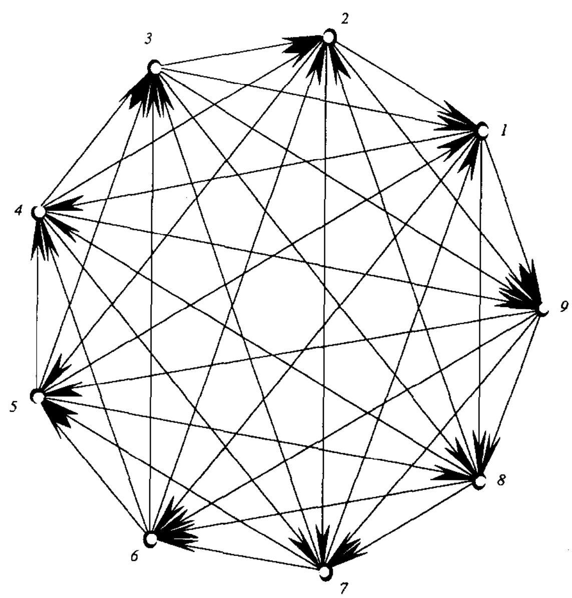
In [5] the W-classification was presented as the first step to compare two tournaments. Anyway, there exists nonisomorphic tournaments that have the same W-classification. Another problem is to deal with the tournaments that have only one W-class.

Proposition 1.8. *If T is a vertex-symmetric tournament, all the vertices have the same list as the IW-list and the OW-list; so, T has only one W-class.*

The converse is not true. There exists non-vertex-symmetric tournaments that have only one W-class. Figure 2 shows an example of such a situation: nh2 has only one



rotational
nh1



nh2

Fig. 2.

W-class, and it is not vertex-symmetric since it has three vertex orbits: $\{1,4,7\}$, $\{2,5,8\}$ and $\{3,6,9\}$. (In fact, this tournament is near homogeneous.)

Definition 1.9. We call pseudo-vertex-symmetric a tournament that has only one W-class.

We proved in [5] the following results.

Theorem 1.10. *Let T be a regular tournament that has only one W-class with the IW-list equal to the OW-list equal to $(w_1, \ldots, w_{(n-1)/2})$, and let x be any vertex of T. Then there exists a labelling of the vertices of $O(x)=\{y_1, \ldots, y_{(n-1)/2}\}$ and $I(x)=\{z_1, \ldots, z_{(n-1)/2}\}$ such that*

$$w_i = w(x, y_i) = w(z_i, x),$$

$$s(y_i, I(x)) = w_i = \frac{n-1}{2} - s(z_i, O(x)) - 1,$$

$$s(y_i, O(x)) = \frac{n-1}{2} - w_i,$$

$$s(z_i, I(x)) = w_i - 1,$$

for $i = 1, 2, \ldots, (n-1)/2$.

Theorem 1.11 (Dugat [5]). *Let $T=(X, U)$ be a regular tournament of order n. There exists a vertex x of X such that $\mathrm{IW}(x)=\mathrm{OW}(x)=(1,2,3,\ldots,k)$ iff T is a rotational tournament with symbol $S=\{1,2,3,\ldots,k\}$, where $k=(n-1)/2$.*

A particular case of pseudo-vertex-symmetric tournaments are homogeneous and near-homogeneous tournaments. These notions have been defined by Kotzig [8] and Tabib [13]. We give here the same definitions, but using the weights language.

Definition 1.12 (Kotzig [8]). A tournament T is homogeneous iff, for any arc (x, y), $w(x,y)=(n+1)/4$, so, $\forall x \in X$, $\mathrm{IW}(x)=\mathrm{OW}(x)=(w, \ldots, w)$, where $w=(n+1)/4$ and $|\mathrm{IW}(x)|=|\mathrm{OW}(x)|=(n-1)/2$.

Definition 1.13 (Tabib [13]). A tournament $T=(X, U)$ is near homogeneous iff $\forall x \in X$, $\mathrm{IW}(x)=\mathrm{OW}(x)=(w, \ldots, w, w+1, \ldots, w+1)$, where $w=(n-1)/4$, $|\mathrm{IW}(x)|=|\mathrm{OW}(x)|=(n-1)/2$ and there are $(n-1)/4$ terms of value w and $(n-1)/4$ terms of value $w+1$.

Our aim is now to study the W-classification of tournaments that have autonomous parts.

2. New results: weights and autonomous parts

Definition 2.1 (Fried Laskar [6]). Let $T=(X, U)$ be any tournament of order n, and S be a part of X; S is an *autonomous part* of T if $\forall x\in X-S$ we have either that S dominates x or S is dominated by x. It T has no trivial autonomous part S (that is, $S\neq X$ and $\mathrm{card}(S)\geqslant 2$), T is said to be simple.

Notation 2.2. If $T=(X, U)$ is a regular tournament and A is a part of X, we denote by $w_A(u)$ the weight of the arc u in the subtournament induced by A.

Proposition 2.3. *If $T=(X, U)$ is a regular tournament and S is an autonomous part of T we have the following $\forall u\in T_S$, the subtournament induced by S, $w(u)$, only depends on the structure of T_S, that is to say, $w_T(u)=w_{T_S}(u)$.*

Proof. Let $R_1=\{x\in X-S$ such that $z\to x$, $\forall z\in S\}$ and $R_2=\{x\in X-S$ such that $x\to z$, $\forall z\in S\}$ (cf. Fig. 3). In [3] we found the following result: if $\mathrm{card}(S)=s$ then $\mathrm{card}(R_1)=\mathrm{card}(R_2)=(n-s)/2$; moreover, the subtournament induced by S is regular. Now let u be any arc of S, say $u=(z_1, z_2)$; we see in Fig. 4 that $\forall x\in R_1\cup R_2$, the calculus of the weight of u is independent of x. □

Proposition 2.4. *With the previous notations,*

(i) *Let r_1 be any vertex of R_1; then $\forall z\in S$, $w(z, r_1)=s(r_1, R_2)$.*
(ii) *Let r_2 be any vertex of R_2; we have $\forall z\in S$, $w(r_2, z)=s^-(r_2, R_1)$.*
(iii) *$\forall z\in S$, and $r_1\in R_1$, $(s+1)/2\leqslant w(z, r_1)\leqslant(n-s)/2$.*
Similarly, $\forall z\in S$, and $r_2\in R_2$, $(s+1)/2\leqslant w(r_2, z)\leqslant(n-s)/2$.

Proof. Let $a_1=s(r_1, R_2)$, $a_2=s^-(r_1, R_2)$, $b_1=s(r_1, R_1)$, $b_2=s^-(r_1, R_1)$ and z a vertex of S. The regularity of the tournament gives the following equations:

$$a_1+a_2=\frac{n-s}{2},\qquad b_2+a_2=\frac{n-1}{2}-s,\qquad a_1+b_1=\frac{n-1}{2},$$

$$b_1+b_2=\frac{n-s}{2}-1.$$

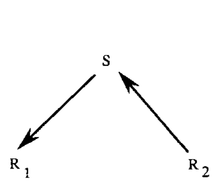


Fig. 3.

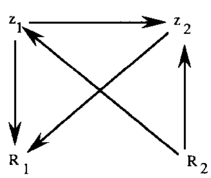


Fig. 4.

(i) The arc (z, r_1) forms a 3-circuit only with the vertices of R_2 that are dominated by r_1; so, $w(z, r_1) = a_1 = s(r_1, R_2)$, which is independent of z.

(ii) Same as the previous one.

(iii) Let us establish the proof for an element of R_1; it is the same for R_2. We have $a_1 = s(r_1, R_2)$ and $|R_2| = (n-s)/2$; so $a_1 \leqslant (n-s)/2$. On the other hand, $b_1 \leqslant ((n-s)/2) - 1$ and $b_1 = ((n-1)/2) - a_1$; so, $((n-1)/2) - a_1 \leqslant ((n-s))/2 - 1 \Leftrightarrow (s+1)/2 \leqslant a_1$. Hence, $(s+1)/2 \leqslant a_1 \leqslant (n-s)/2$. □

Corollary 2.5. *Let T be a regular tournament, S be an autonomous part of T and $z \in S$, $\mathrm{IW}(z) = (p_1, p_2, \ldots, p_\sigma, q_1, \ldots, q_\rho)$, where $p_1, \ldots, p_\sigma$ are the weights of the ingoing arcs of z starting in S, and $q_1, \ldots, q_\rho$ the weights of the arcs going from R_2 to z, with $\rho = (n-s)/2$ and $\sigma = (s-1)/2$; then we have $\sum_{i=1}^{\sigma} p_i = \frac{1}{8}(s-1)(s+1)$ and $\frac{1}{8}(s-1)(s+1) + \sum_{i=1}^{\rho} q_i = \frac{1}{8}(n-1)(n+1)$. Similarly, $\mathrm{OW}(z) = (p'_1, p'_2, \ldots, p'_\sigma, q'_1, \ldots, q'_\rho)$. The part $q_1, \ldots, q_\rho$ belongs to all the lists $\mathrm{IW}(z)$, whatever is z belonging to S, and the part $q'_1, \ldots, q'_\rho$ belongs to all the lists $\mathrm{OW}(z)$, $\forall z \in S$.*

Proof. Let S be an autonomous part of a regular tournament $T = (X, U)$, with $|S| = s$ and $|X| = n$. Let u be any arc of S; we have $1 \leqslant w(u) \leqslant (s-1)/2$ according to Proposition 2.3. We deduce by using Proposition 2.4 that, for any arc v from S to R_1, $w(u) < w(v)$. Hence, in the IW-lists and OW-lists of any vertex z of S, the weights corresponding to the arcs of S are at the head of the list. Let us consider the IW-list of a vertex z of S. Denote by p_i, $1 \leqslant i \leqslant (s-1)/2$, the weights of the arcs of S going to z, and by q_j, $1 \leqslant j \leqslant (n-s)/2$, the weights of the other arcs. We have $\mathrm{IW}(z) = (p_1, p_2, \ldots, p_\sigma, q_1, \ldots, q_\rho)$. And $\sum_{i=1}^{\sigma} p_i = \frac{1}{8}(s-1)(s+1)$ according to Proposition 2.3: $(p_1, p_2, \ldots, p_\sigma)$ is the IW-list of z in the subtournament induced by S. Moreover, we have $\frac{1}{8}(s-1)(s+1) + \sum_{i=1}^{\rho} q_i = \frac{1}{8}(n-1)(n+1)$ by definition of an IW-list in T. Of course, we can establish a result similarly for the OW-list. □

Theorem 2.6. *If T is a pseudo-vertex-symmetric tournament with an autonomous part S of order s, then we have the following:*

(i) *T_S, the subtournament induced by S, is pseudo-vertex-symmetric, and*

(ii) *s divides the order n of T.*

Proof. (i) Let S be an autonomous part of T, a pseudo-vertex-symmetric regular tournament. Let z be a vertex of S and R_1, R_2 as in Fig. 3.

Notation. We denote by $\mathrm{IW}_A(x)$ and $\mathrm{OW}_A(x)$ the weights lists of the vertex x in the subtournament induced by the subset A of the set of vertices.

$\forall z \in S$, ($\mathrm{IW}(z)$ contains $\sigma = (s-1)/2$ weights of the ingoing arcs of z in S and of $\rho = (n-s)/2$ arcs coming from R_2. Similarly, $\mathrm{OW}(z)$ is formed of σ weights of S and of $(n-s)/2$ weights of the arcs going in R_1. But $\forall r_1 \in R_1$, $w(z, r_1)$ is a constant $\forall z \in S$, and $\forall r_2 \in R_2$, $w(r_2, z)$ is a constant $\forall z \in S$. So $\mathrm{IW}(z)$ is a constant list iff $\mathrm{IW}_S(z)$ is a constant

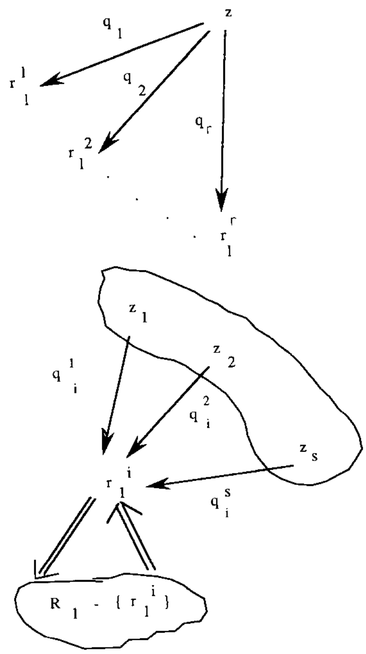


Fig. 5.

list according to Proposition 2.4 and Corollary 2.5. We can say the same thing about the $\mathrm{OW}(z)$ and $\mathrm{OW}_S(z)$. We deduce that the regular subtournament induced by S is pseudo-vertex-symmetric.

(ii) Let us consider the structure of an OW-list. Let z be a vertex of the autonomous part S, and suppose that $\mathrm{OW}_S(z)=(p_1,p_2,\dots,p_\sigma)$. Let us study the arcs going from z to R_1. Let q_i be the weight of the arc (z,r_1^i), where r_1^i is any vertex of R_1. So, $\mathrm{OW}(z)=(p_1,p_2,\dots,p_\sigma,q_1,\dots,q_\rho)$ and $p_i<q_i$ according to Corollary 2.5. Similarly, $\mathrm{IW}(z)=(p'_1,p'_2,\dots,p'_\sigma,q'_1,\dots,q'_\rho)$, where q'_i is the weight of the arc (r_2^i,z), if r_2^i is any vertex of R_2. Let us now study the IW- and OW-lists of any vertex r_1^i of R_1.

According to Proposition 2.4, $w(z,r_1^i)$ is independent of the $z\in S$ considered. Hence, $q_i^1=q_i^2=\cdots=q_i^s=q_i$, which depends only on the r_1^i considered. So, $\mathrm{IW}(r_1^i)=(x,x,\dots,x,\ \ q_i,q_i,q_i,\dots,q_i,\ \ x,x,\dots,x)$, with q_i appearing s times (cf. Proposition 2.3), and where the x represent the other weights, unknown for the moment. We now have $\forall r_1^i$ a vertex of R_1, $\mathrm{IW}(r_1^i)=\mathrm{IW}(z)$ $\forall z\in S$, since T is pseudo-vertex-symmetric. For the same reason, $\mathrm{IW}(r_1^i)=\mathrm{IW}(r_1^j)$ $\forall i\neq j$. If we suppose that all the q_i are different then, for all i, $\mathrm{IW}(r_1^i)$ has the structure $(x,\dots,x,q_1,\dots,q_1,$ $q_2,\dots,q_2,\dots,q_\rho,\dots,q_\rho,x,\dots,x)$, where each q_i is present s times.

So, we have at least $s\cdot(n-s)/2$ elements in the list whose length is $k=(n-1)/2$. We deduce that $s\cdot(n-s)/2\leqslant(n-1)/2 \Leftrightarrow -s^2+n\cdot s+1-n\leqslant 0$. There are two solutions: $s=1$ or $s=n-1$, which is excluded since S is a nontrivial autonomous part. Hence, we must suppose that there exists some equal q_i for different vertices of R_1. Let us look for the maximum number λ of different q_i. Of course, we have $s\cdot\lambda\leqslant k \Rightarrow 2\cdot s\cdot\lambda\leqslant n-1$. Moreover, $\mathrm{IW}(r_1^i)=\mathrm{IW}(z) \Rightarrow p_j'\in\mathrm{IW}(r_1^i)$, $1\leqslant j\leqslant\sigma$. Hence,

$$\mathrm{IW}(r_1^i)=\mathrm{IW}(z)=(\boxed{p_1',\ldots,p_\sigma'},\boxed{q_1},\boxed{q_2},\ldots,\boxed{q_\lambda}),$$

where each q_i bloc is of length s. Moreover, we must have $\sigma+\lambda\cdot s=k \Leftrightarrow 2\cdot s\cdot\lambda=n-s$, and λ is an integer, so, s must divide $(n-s)/2$ and $n=s\cdot(2\cdot\lambda+1)$. We deduce that s divides n and that the quotient must be odd. n and s are odd; so, it suffices that s divides n. The result obtained for the IW-lists is the same as that for the OW-lists. If the IW-lists have λ different q_i and the OW-lists λ' different q_i', we have $\lambda=\lambda'$. Now if we let $m=n/s=2\cdot\lambda+1$ then $n=m\cdot s$ and $m=2\lambda+1 \Rightarrow \mathrm{card}(R_1)=\mathrm{card}(R_2)=\lambda\cdot s$. □

Corollary 2.7. *If T is pseudo-vertex-symmetric of order n prime then T is simple.*

Proof. It is a trivial consequence of Theorem 2.6(ii). □

Lemma 2.8. *Let T be a regular tournament, u any arc of T and S an autonomous part of T. If u belongs to at least k 3-circuits, with k an integer, then $|S|\geqslant 2 \Rightarrow |S|\geqslant 2k+1$.*

Proof. Let x, y be two vertices of S. Let z be a vertex of T such that x, y, z form a 3-circuit. By definition of an autonomous part, z must belong to S. Since the arc (x, y) belongs to at least k 3-circuits, there are at least k vertices in S (different from x and y). Now in [7] it is said that an arc of a regular tournament belonging to k 3-circuits belongs to $k-1$ transitive triples. Thus, here if $x,. y, t$ form such a transitive triple, t must belong to S too. Finally, there are at least $k+2+k-1=2k+1$ vertices in S. □

Corollary 2.9. (i) *A homogeneous tournament is simple.*
(ii) *A near-homogeneous tournament is simple.*

Proof. *T is homogeneous*: Let S be a nontrivial autonomous part of T. Denote by $4k-1$ the order of T and by s the cardinality of S. s must divide $4k-1$ because of Theorem 2.6(ii); so, $s\neq 4k-1 \Rightarrow s<2k$. But we saw in Lemma 2.8 that $s\geqslant 2k+1$. Thus, $s=4k-1$.
[*Note.* Müller has established this result by using another method in [10]. He proved that the homogeneous tournaments have the maximal simplicity number (the simplicity number is the minimum number of arcs that must be inverted to obtain an autonomous part in a simple tournament).]

T is near homogeneous: Let $4k+1$ be the order of T; s must divide $4k+1$. The greatest dividers of $4k+1$ are less than or equal to $2k+\frac{1}{2}$. So, $S=T$. □

Theorem 2.10. *Let T be a pseudo-vertex-symmetric tournament of order n. T has an autonomous part of order s such that $n=3s$ iff T is the wreath product of the 3-circuit with pseudo-vertex-symmetric tournaments of order s and these tournaments have the same W-classification.*

Proof. The notations are the same as in Theorem 2.6. If $n=3\cdot s$, we have $m=3$ and $\lambda=1$. So, T has three autonomous parts of order s according to [3]. We establish that $|R_1|=|R_2|=s$, $s(r_1^i, R_2)=s$. Hence, S, R_1, R_2 have autonomous parts inducing tournaments that have the same W-decomposition. On the other hand, we remark that if all the q_i are equal to an integer q, we have $\lambda=1$ and $m=3$; so, $n=3\cdot s$. □

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