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# The Functional Differential Equation $x'(t) = x(x(t))$*

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A classification of the solutions of the functional differential equation $x'(t) = x(x(t))$ is given and it is proved that every solution either vanishes identically or is strictly monotonic. For monotonically increasing solutions existence and uniqueness of the solution $x$ are proved with the condition $x(t_0) = x_0$ where $(t_0, x_0)$ is any given pair of reals in some specified subset of $\mathbb{R}^2$. Every monotonically increasing solution is thus obtained. It is analytic and depends analytically on $t_0$ and $x_0$. Only for $t_0 = x_0 = 1$ is the question of analyticity still open.

## INTRODUCTION

In many problems of physics and other sciences there occur retarded differential equations of the form

$$x'(t) = f(x(t_{\text{ret}}))$$

where the retarded time $t_{\text{ret}}$ is given as a function of $t$ in which the function $x$ itself may enter as well:

$$t_{\text{ret}} = F(t, x)$$

where $F$ is a functional with two arguments: a number $t$ and a function $x$.

Whereas for regular ordinary first-order differential equations there is, for every given value of $x(t_0)$ ($t_0$ is arbitrary but fixed), a unique solution $x$ of the equation, there is no similar general result for retarded differential equations. Consider, for example, the equation

$$x'(t) = x(t-1) \qquad \text{for} \quad t \leqslant 0.$$

If $\xi$ is any $C^\infty$-function mapping the closed interval $[-1, 0]$ into the set of reals $\mathbb{R}$ such that the $(n+1)$th derivative of $\xi$ at 0 is equal to the $n$th

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derivative of $\xi$ at $-1$, $\xi^{(n+1)}(0)=\xi^{(n)}(-1)$ for all nonnegative integers $n$ and if we define, for $t \leqslant 0$,

$x(t) := \xi^{(n)}(t+n)$ where $n$ is that nonnegative integer for which $-1 < t+n \leqslant 0$, then $x$ is a solution of the equation $x'(t)=x(t-1)$. There are retarded differential equations, however, where it is assumed that $|x(t)|$ does not grow too fast as $t \to -\infty$ (such a property can be contained in the retarded differential equation implicitly if it becomes singular when $|x(t)|$ grows too fast, e.g., when $|x'(t)| \geqslant c$, the speed of light, in electrodynamics). In some such cases it can be proved that there is a unique solution for any given "final value" $x(t_0)$ and, in the case of second-order equations, $x'(t_0)$ (see [1–3]) or for any given suitably defined "initial value at $t=-\infty$" (see [4]). We can make a coordinate transformation for $t$ and $x$ so that $t=-\infty$ is transformed into a finite value $s_0$ of a variable $s$. If the transformation connecting the variables $t$ and $s$ is such that $ds/dt < 0$ then the retarded differential equation is transformed into an advanced differential equation

$$y'(s) = g(y(s_{\text{adv}})) \qquad \text{with} \qquad s_{\text{adv}} = G(s, y).$$

The assumption about the growth of $|x(t)|$ becomes a local assumption about the behaviour of $y$ in the neighbourhood of $s_0$, e.g., a differentiability condition.

In this paper we consider the equation

$$x'(t) = x(x(t))$$

as an example of such an advanced differential equation. We shall prove existence, uniqueness, analyticity and analytic dependence of solutions on initial data.

## 1. Preliminaries and Monotonicity of Solutions

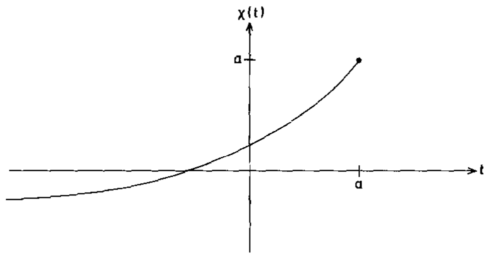
Definition 1. A solution of the functional differential equation

$$x' = x \circ x \tag{$*$}$$

is a function $x: A \to \mathbb{R}$ from an interval $A \subset \mathbb{R}$ (i.e., a connected subset of $\mathbb{R}$) into $\mathbb{R}$ such that $x'(t) = x(x(t))$ for all $t \in A$. ∎

This implies $x(A) \subset A$ for any solution $x$ of $(*)$.

Consider intervals $A$ of the form $]-\infty, a]$. If $0 < a \leqslant 1$ then $(*)$ together with the initial condition $x(a) = a$ will be, in a neighbourhood of $a$, an advanced differential equation which can be transformed into a retarded one by transforming $a$ to $-\infty$ on the $t$-axis (see the Introduction).




LEMMA 1 (Monotonicity of Solutions). *If $A \subset \mathbb{R}$ is some interval and $x: A \to \mathbb{R}$ obeys a differential equation of the form*

$$x' = f \circ x \qquad \text{where } f \text{ is } C^1$$

*then* sign $\circ\, x'$ *is constant.* ∎

*Proof.* Otherwise there would be a $t_0 \in A$ such that $x'(t_0) = 0$. Thus $f(x(t_0)) = 0$. So, by the uniqueness theorem for ordinary differential equations, $x(t) \equiv x(t_0)$, and hence $x' \equiv 0$.

DEFINITION 2 ($\mathbb{R}_\infty$). Let $\mathbb{R}_\infty$ be the set $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ with the linear ordering $\leqslant$ defined as

$$r \leqslant s :\Leftrightarrow (r = -\infty \text{ or } s = \infty \text{ or } (r, s \in \mathbb{R} \text{ and } r \leqslant_{\text{in } \mathbb{R}} s)).$$

An *open set* in $\mathbb{R}_\infty$ is either an ordinary open set in $\mathbb{R}$ or is the union of such a set with $]r, \infty]$ and/or $[-\infty, s[$ for some real $r, s$. Let $A$ be an interval in $\mathbb{R}$ and $x: A \to \mathbb{R}$ be a continuous function. If $x$ has a continuous extension $y$: $\bar{A}^{\mathbb{R}_\infty} \to \mathbb{R}_\infty$, and $t \in \bar{A}^{\mathbb{R}_\infty}$, then let $x(t)$ be defined as $x(t) := y(t)$. ∎

THEOREM 1 (Monotonicity of Solutions). *Let $x$ be a solution of $(*)$ in an interval $A$. Then* $\operatorname{sign}(x'(t))$ *is independent of $t$ for $t \in \bar{A}$. (Closure taken in $\mathbb{R}$, not in $\mathbb{R}_\infty$!)* ∎

*Proof.* Obviously, $x$ is $C^\infty$. Setting, in Lemma 1, $f = x$ gives the constancy of sign $\circ\, x'$ in $A$. This implies that $x$ is monotonic, and therefore also $x' = x \circ x$ is monotonic.

Thus $x'$ has a continuous extension to a function from $\bar{A}^{\mathbb{R}_\infty}$ to $\bar{A}^{\mathbb{R}_\infty}$. If sign $\circ\, x'$ were not constant in $\bar{A}$ then there would exist an $a \in \bar{A} \backslash A$ such that $x'(a) = 0$ but $x'$ nonzero in $A$. Then $x$ could be extended to a differentiable function $y$ on $A \cup \{a\}$ with $y'(a) = 0$. So $y' = y \circ y$, and $y$ is $C^\infty$. Using Lemma 1 again, for $y$ instead of $x$ and $f$, yields a contradiction. ∎

## 2. Maximal and Inextendable Solutions

Definition 3. A solution $x$ of $(*)$ on an open interval $A$ is *maximal* iff it is not a proper restriction of a solution on an open interval.

An extension $y$ of $x$ to an open interval $B \subset \mathbb{R}$ is a $(*)$-*extension* iff, for any open interval $C$ with $A \subset C \subset B$, $y|_C$ is a solution of $(*)$.[1]

$x$ is *inextendable* iff it has no proper $(*)$-extension. ∎

Every maximal solution of $(*)$ is inextendable. Every solution of $(*)$ is a restriction of a maximal solution of $(*)$ by Zorn's lemma.

Lemma 2 (Values at Endpoints). *Let $x$: $]a, b[ \to \mathbb{R}$ be an inextendable solution of* $(*)$. *Then*

$$\begin{array}{llll} a = -\infty & \textit{and} & b = \infty & \textit{if } x' \equiv 0, \\ (a = -\infty \textit{ or } x(a) = a) & \textit{and} & (b = \infty \textit{ or } x(b) = b) & \textit{if } x' > 0, \\ (a = -\infty \textit{ or } x(a) = b) & \textit{and} & (b = \infty \textit{ or } x(b) = a) & \textit{if } x' < 0. \end{array}$$

*Proof.* If $a > -\infty$ then $x(a) \notin ]a, b[$, as otherwise the existence theorem for ordinary differential equations would yield a solution $y$ of $y' = x \circ y$ and $y(a) = x(a)$ in a neighbourhood of $a$. By the uniqueness theorem, $x$ and $y$ coincide where they both are defined, in contradiction to the inextendability of $x$. So $x(a) = a$ or $x(a) = b$ if $a > -\infty$. The same is true for $x(b)$ if $b < \infty$. From $x(]a, b[) \subset ]a, b[$ follows the assertion. ∎

## 3. Classification of Inextendable Solutions

Theorem 2 (Classification of Solutions). *Let $x$: $]a, b[ \to \mathbb{R}$ be an inextendable solution of* $(*)$. *Then one and only one of the following statements is true*:

(1) $x \equiv 0 \wedge a = -\infty \wedge b = \infty$

(2) $x' > 0 \wedge 0 < b < \infty \wedge x(b) = b \wedge$

(i) $0 < a < 1 < b \wedge x(a) = a$

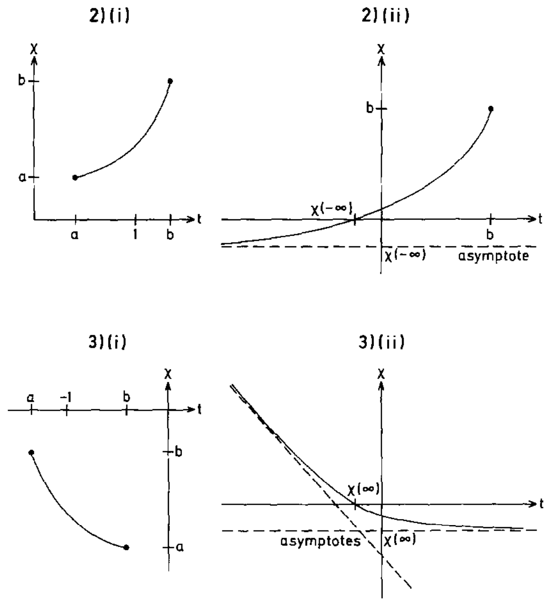
(ii) $a = -\infty \wedge -\infty < x(-\infty) < 0 \wedge x(x(-\infty)) = 0$

[1] Note that, in contrast to the case of ordinary differential equations, the restriction of a solution $x$ of $(*)$ to a subinterval of its domain need not be a solution of $(*)$; the subinterval might fail to be invariant under $x$.

(3) $x' < 0 \wedge$

(i) $-\infty < a < -1 < b < 0 \wedge x(a) = b \wedge x(b) = a$

(ii) $a = -\infty \wedge b = \infty \wedge x(-\infty) = \infty \wedge -\infty < x(\infty) < 0$
$\wedge\, x(x(\infty)) = 0.$ ∎




All asymptotes are approached exponentially

*Proof.* By Theorem 1 we have three cases: $x' \equiv 0$, $x' > 0$ and $x' < 0$.

*Case* 1: $x' \equiv 0$. Then $x \circ x = x' \equiv 0$. Since $x$ is constant, we have $x \equiv 0$. By the inextendability of $x$, $a = -\infty$ and $b = \infty$.

*Case* 2: $x' > 0$. Then $b < \infty$. For, assume $b = \infty$. Then $x'' = (x' \circ x) \cdot x' = (x \circ x \circ x) \cdot (x \circ x)$ is positive and monotonically increasing. Thus $x$ grows at least quadratically. So, for large enough $t$,

$$t + 1 < x(t)$$

and

$$x(t+1) > x(t) + x'(t) > x'(t) = x(x(t))$$

in contradiction to the monotonicity of $x$. So we have $b < \infty$, and $b \geqslant x(x(t)) = x'(t) > 0$. By Lemma 2, $x(b) = b$.

If $a > -\infty$ then, by Lemma 2, $x(a) = a$. $a = x(a) = x(x(a)) = x'(a)$. This is $>0$ by Theorem 1. As $x'$ is increasing and $(x(b) - x(a))/(b - a) = 1$, we have $x'(a) < 1 < x'(b)$. But $a = x'(a)$ and $b = x'(b)$. Thus $a < 1 < b$.

If $a = -\infty$ then $x(x(-\infty)) = x'(-\infty) \geqslant 0$. So $x(-\infty) > -\infty$, and thus $x'(-\infty) = 0$. So $x(x(-\infty)) = 0$. Thus $0 = x(x(-\infty)) > x(-\infty)$ because of the strict monotonicity of $x$.

*Case* 3: $x' < 0$. If $a > -\infty$ then, by Lemma 2, $x(a) = b$. $x'(a) = x(x(a)) \geqslant a > -\infty$. So $b = x(a) < \infty$. Thus, by Lemma 2, $x(b) = a$. $b = x(x(b)) = x'(b)$ which is $<0$ by Theorem 1. As above $a < -1 < b$.

If $a = -\infty$ then let $t_0 \in ]-\infty, b[$. For $t < t_0$ we have $x'(t) < x'(t_0) < 0$. $x(t) > x(t_0) + (t_0 - t) \cdot (-x'(t_0))$. Thus $x(-\infty) = \infty$ and $b = \infty$. $x(\infty) = x(x(-\infty)) = x'(-\infty) < x'(t_0) < 0$. $x(x(\infty)) = x'(\infty) \leqslant 0 < \infty$. So $x(\infty) > -\infty$. Thus $x'(\infty) = 0$. So $x(x(\infty)) = 0$. ∎

## 4. Existence, Uniqueness, Analyticity and Analytic Dependence

Lemma 3 (Local Existence, Uniqueness, Analytic Dependence). *There is a real-valued analytic function $x$ on an open neighbourhood of $\{(a, a) \mid -1 < a < 1\}$ in $\mathbb{R} \times \mathbb{R}$ such that, for any $a \in ]-1, 1[$, $x(a, \cdot)$ is the unique solution of $(*)$ with the initial condition $x(a, a) = a$.*

*$x$ can be extended to give a holomorphic function $y$ of a neighbourhood of $\{(a, a) \mid a \in \mathbb{C} \wedge |a| < 1\}$ in $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C}$ such that, for any $a \in \mathbb{C}$ with $|a| < 1$, $y(a, \cdot)$ is the unique holomorphic solution of $(*)$ with the initial condition $y(a, a) = a$.*

*Here "unique" means it coincides with every other solution of $(*)$ with the same initial condition on the intersection of their domains of definition.* ∎

*Proof.* Let $\mathbb{A}$ be either the set of reals or the set of complex numbers. Let $A$ be a subset of $\mathbb{A}$ such that $|a| \leqslant 1 - \varepsilon$ for all $a \in A$ for some $\varepsilon \in ]0, \frac{1}{2}[$. For every $a \in \mathbb{A}$ we denote the open ball with center $a$ and radius $\varepsilon$ by $B_\varepsilon(a)$. Let $S := \{(a, t) \mid a \in A \wedge t \in B_\varepsilon(a)\}$ and $C_b(S, \mathbb{A})$ be the set of continuous bounded functions from $S$ into $\mathbb{A}$. We set $B := C_b(S, \mathbb{R})$ if $\mathbb{A} = \mathbb{R}$ and $B := \{x \in C_b(S, \mathbb{C}) \mid x$ is holomorphic in $\mathring{S}$, and $x(a, \cdot)$ is holomorphic on $B_\varepsilon(a)$ for each $a \in A\}$ if $\mathbb{A} = \mathbb{C}$, where $\mathring{S}$ is the union of all open subsets of $S$ in $\mathbb{C} \times \mathbb{C}$. Let $\|\cdot\|$ be the supremum norm on $B$. Then $(B, \|\cdot\|)$ is a Banach space. The set

$$X := \{x \in B \mid x(a, a) = a \text{ for each } a \in A, \text{ and } |x(a, t_2) - x(a, t_1)| \leqslant |t_2 - t_1| \text{ when } (a, t_1), (a, t_2) \in S\}$$

is a closed subset of $B$.

Obviously, for $x \in X$ and $a \in A$, $x(a, \cdot)$ maps $B_\varepsilon(a)$ into $B_\varepsilon(a)$. Therefore a map $T: X \to B$ can be defined by

$$Tx(a, t) := a + \int_a^t x(a, x(a, \tau))\, d\tau.$$

If $x \in X$ and $(a, \tau) \in S$, then

$$\begin{aligned}|x(a, x(a, \tau))| &\leqslant |x(a, a)| + |x(a, x(a, \tau)) - x(a, a)| \\ &\leqslant |a| + |\tau - a| < |a| + \varepsilon \leqslant 1.\end{aligned}$$

Therefore, if $x \in X \wedge a \in A \wedge t_1, t_2 \in B_\varepsilon(a)$ then

$$|Tx(a, t_2) - Tx(a, t_1)| = \left| \int_{t_1}^{t_2} x(a, x(a, \tau))\, d\tau \right| \leqslant |t_2 - t_1|,$$

i.e., $T$ maps $X$ into $X$.

We prove that $T$ is contractive. Let $x, y \in X$.

$$\begin{aligned}|(Tx - Ty)(a, t)| &= \left| \int_a^t x(a, x(a, \tau)) - y(a, y(a, \tau))\, d\tau \right| \\ &\leqslant \left| \int_a^t |x(a, x(a, \tau)) - y(a, x(a, \tau))|\, d\tau \right| + \\ &\quad \left| \int_a^t |y(a, x(a, \tau)) - y(a, y(a, \tau))|\, d\tau \right|.\end{aligned}$$

The integrand in the first term on the right hand side of the inequality is $\leqslant \|x - y\|$ and the integrand in the second term is $\leqslant |x(a, \tau) - y(a, \tau)| \leqslant \|x - y\|$. Hence, $|(Tx - Ty)(a, t)| \leqslant 2 \cdot |t - a| \cdot \|x - y\| \leqslant 2\varepsilon \|x - y\|$. So $\|Tx - Ty\| \leqslant 2\varepsilon \|x - y\|$ for all $x, y \in X$, and $2\varepsilon < 1$.

By Banach's fixed point theorem, $T$ has a unique fixed point.

Now let $a \in \,]-1, 1[$. Setting $A := \{a\}$ and $\mathbb{A} = \mathbb{R}$ shows that there is, locally, a unique solution $\underset{a}{x}$ of $(*)$ with the initial condition, because every such solution must lie in $X$ for some $\varepsilon$ and be a fixed point of $T$. For $A := \{a\} \wedge \mathbb{A} = \mathbb{C}$ let $Z := \{x \in X \mid x(b, t)$ is real when $b$ and $t$ are real$\}$. $Z$ is a closed subset of $X$, and $T$ maps $Z$ into $Z$. So the fixed point $z$ of $T$ must lie in $Z$. So $z|_{\mathbb{P}} = \underset{a}{x}$. Thus $\underset{a}{x}$ is analytic. Also, $z$ is the only holomorphic solution.

Now let $a \in \mathbb{C}$ and $|a| < 1$. Let $0 < \delta < 1 - |a|$, $A := B_\delta(a)$, $\mathbb{A} = \mathbb{C}$. Let $y$ be the least common extension of all the fixed points thus generated, starting from any $a$ with $|a| < 1$. ∎

LEMMA 4. *There is a unique solution of* $(*)$ *with the initial condition* $x(1)=1$ *in the interval* $[\frac{2}{3}, 1]$. ∎

*Proof.* Let

$$X := \{x \in C_b([\tfrac{2}{3}, 1], \mathbb{R}) \mid x(1)=1 \text{ and } 0 \leqslant x(t_2)-x(t_1) \leqslant t_2-t_1$$
$$\text{whenever } \tfrac{2}{3} \leqslant t_1 \leqslant t_2 \leqslant 1\}.$$

Any $x \in X$ maps $[\frac{2}{3}, 1]$ into $[\frac{2}{3}, 1]$.

We can, similarly as in Lemma 3, define a $T: X \to X$ by

$$Tx(t) := 1 + \int_1^t x(x(\tau))\, d\tau.$$

Again, $T$ is contractive in $(X, \|\cdot\|)$ ($\|\cdot\| =$ sup-norm). As $(X, \|\cdot\|)$ is complete, there is a unique fixed point of $T$. This fixed point is the unique solution. ∎

LEMMA 5 (Existence, Uniqueness, Analytic Dependence). *There is an open neighbourhood* $N$ *of* $\{(a,a) \mid 0 \leqslant a < 1\}$ *in* $\mathbb{R} \times \mathbb{R}$ *and an analytic function* $x: N \to \mathbb{R}$ *such that, for any* $a \in [0, 1[$, $x(a, \cdot)$ *is the unique* inextendable *solution of* $(*)$ *with the initial condition* $x(a,a)=a$. ∎

*Proof.* For any function $x$ as in Lemma 3 let $\hat{x}$ be the unique maximal solution of the ordinary differential equation

$$\hat{x}(a, \cdot)'(t) = x(a, \hat{x}(a, t))$$

with the initial condition

$$\hat{x}(a, a) = a$$

for $a \in ]-1, 1[$. Here $\hat{x}(a, \cdot)'(t)$ or $\partial_2 \hat{x}(a, t)$ denotes the derivative of $\hat{x}$ with respect to the second argument at $(a, t)$. Then $\hat{x}$ is analytic again and is as in Lemma 3. $\hat{x}$ is an extension of $x$. Now let, for some $x_0$ as in Lemma 3,

$$x_{n+1} := \hat{x}_n \qquad \text{for all} \quad n \in \mathbb{N}_0.$$

Let $x$ be the least common extension of all $x_n$. Then $\hat{x} = x$.

For $a \in ]0, 1[$, the domain of definition of $x(a, \cdot)$ is some interval $]r, s[$. Since $x(a, \cdot)'(a) = a > 0$, it follows from Theorem 2 that $x(a, \cdot)'$ is positive everywhere and $s < \infty$. By a similar argument as in the proof of Lemma 2 we obtain from $\hat{x} = x$:

$$x(a, r) = r \qquad \text{if} \quad r > -\infty \qquad \text{and} \qquad x(a, s) = s.$$

But $x(a, a) = a$ and $r < a < s$. So the strict monotonicity of $x(a, \cdot)'$ implies $r = -\infty$ and $s = x(a, \cdot)'(s) > 1$. Thus $x(a, \cdot)$ is inextendable. In the case $a = 0$ the inextendability of $x(a, \cdot)$ follows immediately from the definition of $x$. ∎

THEOREM 3 (Existence, Uniqueness, Analytic Dependence). *Every strictly monotonically increasing inextendable solution of* (∗) *is of type* (2)(ii) (*in Theorem* 2) *with* $b \geqslant 1$. *It is analytic with a singularity at* $b$ *if* $b > 1$.[2]

*There is a set* $U \subset \mathbb{R} \times \mathbb{R}$ *with*

$$\{(t_0, x_0) \mid t_0, x_0 \geqslant 0 \text{ and } x_0 \leqslant t_0\} \subset U$$

*and*

$$\{(t_0, t_0) \mid 0 < t_0 < 1\} \subset \mathring{U}$$

*such that there is a unique inextendable solution* $x_u$ *of* (∗) *through every* $u \in U$ (*i.e., with the initial condition* $x_u(t_0) = x_0$ *where* $u = (t_0, x_0)$). *If* $u = (t_0, x_0)$ *then* $x_u \equiv 0$ *for* $x_0 = 0$ *and* $x_u$ *is of type* (2)(ii) *if* $x_0 > 0$.

*Let* $x$ *be the function* $(u, t) \mapsto x_u(t)$, *and let* $S$ *be its domain of definition. Then* $x$ *is continuous. The set* $\{(u, t) \mid u \in \mathring{U} \text{ and } (u, t) \in S\}$ *is open, i.e., it is* $\mathring{S}$. *The restriction* $x|_{\mathring{S}}$ *of* $x$ *to* $\mathring{S}$ *is analytic.*

*If we define, for any* $u \in U$, $b_u \in \mathbb{R}_\infty$ *as the supremum of the domain of definition of* $x_u$, *then* $b_u$ *depends continuously on* $u$ (*see Definition* 2 *for the topology of* $\mathbb{R}_\infty$). ∎

*Proof.* We know from Lemma 5 that every type (2)(i) solution is extendable, and that the same is true for any type (2)(ii) solution with $b < 1$. So, by Theorem 2, every strictly monotonically increasing inextendable solution is of type (2)(ii) with $b \geqslant 1$. If $x$ is a type (2)(ii) solution with $b > 1$ then there is an $s \in \left]0, 1\right[$ with $x(s) = s$. By Lemma 5, $x$ is analytic.

Now let $x$ be the function of Lemma 5. For $a \in \left[0, 1\right[$ we have $x(a, a) = a$ and $\partial_2 x(a, a) = x(a, x(a, a)) = a$. So, by Taylor's theorem, $x(a, t) = a + a \cdot (t - a) + \frac{1}{2}\partial_2^2 x(a, \tau) \cdot (t - a)^2$ where $\tau$ lies between $a$ and $t$. Thus $\partial_1 x(a, a) = 1 - a > 0$. So, there is an open neighbourhood $V$ of $\{(a, a) \mid 0 < a < 1\}$ such that $\partial_1 x$ is positive everywhere in $V$. We can assume that $V$ is a subset of the neighbourhood $N$ in Lemma 5, and we can assume $t_0, x_0 \geqslant 0$ for all $(t_0, x_0) \in V$. Let $U := \{(t, x(a, t)) \mid (a, t) \in V\} \cup \{(t_0, x_0) \mid t_0, x_0 \geqslant 0 \text{ and } x_0 \leqslant t_0\}$.

Since $\{(t, x(a, t)) \mid (a, t) \in V\}$ is open we have $\{(t_0, x_0) \mid t_0, x_0 \geqslant 0 \text{ and } x_0 \leqslant t_0\} \subset U$ and $\{(t_0, x_0) \mid 0 < t_0 < 1\} \subset \mathring{U}$.

[2] If $b = 1$ then it is not analytic at least at 1.

For any $a \in [0, 1]$ let

$$A_a := \{(t_0, x_0) \in U \mid ((a, t_0) \in V \text{ or } a < t_0 < b_{(a,a)}) \text{ and } x_0 > x(a, t_0)\}.$$

For $u \in U$ let $a[u] := \inf\{a \in [0, 1] \mid u \notin A_a\}$ and $\underset{u}{x} := x(a[u], \cdot)$. By the continuity of $x$ the solution $\underset{u}{x}$ obeys the initial conditions $\underset{u}{x}(t_0) = x_0$. The uniqueness of the solution follows from $\partial_1 x(a, t) > 0$ which is true for $u := (t, x(a, t)) \in \{(t, x(a, t)) \mid (a, t) \in V\}$ by the construction of $V$ and can easily be seen to be true also for $u \in \{(t_0, x_0) \mid t_0, x_0 > 0 \text{ and } x_0 < t_0\}$ and therefore for all $u \in \mathring{U}$. So there is a unique solution $x_u$ through every $u \in U$, and $\{(u, t) \mid u \in \mathring{U} \text{ and } (u, t) \in S\}$ is open and $(u, t) \mapsto x_u(t)\colon \mathring{S} \to \mathbb{R}$ is analytic. A simple estimate shows that $a \mapsto b_{(a,a)}$ is strictly monotonically decreasing and continuous.

It remains to show that type (2)(ii) solutions are not analytic at $b$ if $b \geqslant 1$. Assume $x$ is such a solution, with no singularity at $b$. Let

$$x_n := \underbrace{x \circ x \circ \cdots \circ x}_{n\,\text{times}} \qquad \text{for} \quad n \in \mathbb{N}.$$

Then $x'_n = x_{n+1} \cdot x_n \cdot \cdots \cdot x_3 \cdot x_2$.

Equation $(*)$ implies $x^{(n)} = P_n(x_{n+1}, x_n, \ldots, x_3, x_2)$ where $P_n$ is some polynomial with nonnegative coefficients, and the coefficient of

$$x_{n+1} \cdot x_n \cdot x_{n-1}^2 \cdot x_{n-2}^3 \cdot x_{n-3}^4 \cdot \cdots \cdot x_3^{n-2} \cdot x_2^{n-1}$$

is 1. But $x_k(b) = b$ for all $k$, and $b > 0$. So

$$x^{(n)}(b) \geqslant b \cdot b \cdot b^2 \cdot b^3 \cdot b^4 \cdot \cdots \cdot b^{n-2} \cdot b^{n-1} = b^{n(n-1)/2+1}.$$

If $b > 1$, this grows too fast to be the $n$th derivative of an analytic function. Thus $x$ has a singularity at $b$.

For $a \in [0, 1]$ let $\underset{a}{x}$ be the solution through $(a, a)$, and $b_a := b_{(a,a)}$. Then this implies that the radius of convergence of $\underset{a}{x}$ at $a < 1$ is $\leqslant b_a - a$. But

$$\lim_{\substack{a \to 1 \\ a < 1}} (b_a - a) = 0.$$

As all derivatives of $\underset{a}{x}$ at $a$ are $\geqslant 0$ and increasing in $a$, this implies that $\underset{1}{x}$ is not analytic at 1. ∎

We summarize our results in a

Main Theorem (A Summary). *Every solution of $(*)$ is a restriction of an inextendable solution $x\colon ]a, b[ \to \mathbb{R}$ of $(*)$ which is defined on some open* *interval* $]a, b[$ *where* $a, b \in \underset{\infty}{\mathbb{R}}$. *If* $x\colon ]a, b[ \to \mathbb{R}$ *is an inextendable solution of* $(*)$ *then one and only one of the following statements holds*:

(1) $x$ *vanishes everywhere* $\wedge\, a = -\infty \wedge b = \infty$

(2) $x$ *is strictly monotonically increasing* $\wedge$

$$\wedge\, a = -\infty \wedge 1 \leqslant b < \infty \wedge -\infty < x(-\infty) < 0 \wedge x(b) = b \wedge x(x(-\infty)) = 0$$

(3) $x$ *is strictly monotonically decreasing* $\wedge$

(i) $-\infty < a < -1 < b < 0 \wedge x(a) = b \wedge x(b) = a$

(ii) $a = -\infty \wedge b = \infty \wedge x(-\infty) = \infty \wedge -\infty < x(\infty) < 0 \wedge x(x(\infty)) = 0.$

*If* $x$ *is strictly monotonically increasing and* $b > 1$, *then* $x$ *is analytic with a singularity at* $b$.

*For every* $u = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}$ *with* $t_0, x_0 \geqslant 0 \wedge x_0 \leqslant t_0$ *there is a unique inextendable solution* $x_u$ *of* $(*)$ *with* $x_u(t_0) = x_0$. *The function* $(u, t) \mapsto x_u(t)$ *is analytic at a point* $(u, t) = ((t_0, x_0), t)$ *if* $t$ *is in the domain of definition of* $x_u$ *and* $t_0, x_0 > 0$ *and* $(x_0 < t_0 \vee (x_0 = t_0 \wedge 0 < t_0 < 1))$. *The supremum* $b_u$ *of the domain of definition of* $x_u$ *depends continuously on* $u$. ∎

*Remark.* A strictly monotonically increasing solution of $(*)$ is the restriction of a strictly monotonically increasing inextendable solution of $(*)$ to an interval $]a, b[$ with $a \leqslant r \leqslant b$ where $r$ is defined by $x(r) = r$ and $r \leqslant 1$. So the main theorem gives a complete survey of the monotonically increasing solutions. Only the question of the analyticity of one single inextendable solution $x$, namely, that solution for which $x(1) = 1$ (and its restrictions), is still open.

The strictly monotonically decreasing solutions are advanced in one part of their domain of definition and retarded in the other and if $t$ is in one of those parts then the "retarded or advanced time" $x(t)$ is in the other. I did not consider this case here because I do not see applications of such solutions.

## References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