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On colorings of maps

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Abstract

A fixed-point free map $f : X \to X$ is said to be colorable with k colors if there exists a closed cover $\mathcal{C}$ of X consisting of k elements such that $C \cap f(C) = \emptyset$ for every C in $\mathcal{C}$. It is shown that every fixed-point free continuous selfmap of a compact space X with $\dim X \leqslant n$ can be colored with $n + 3$ colors. Similar results are obtained for finitely many maps. It is shown that every free $\mathbb{Z}_p$-actionon an n-dimensional compact space X has genus at most $n + 1$.

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1. Introduction

In the paper [2] the terminology "coloring of a map" of a map was introduced. We recall:

Definition 1. Let $f : X \to X$ be a fixed-point free map. We say that f can be colored with k colors or that f is colorable with k colors if there is a closed cover $\mathcal{C} = \{C_1, \ldots, C_k\}$ with k sets such that no C_i contains a pair $\{x, f(x)\}$ or, equivalently, $C_i \cap f(C_i) = \emptyset$ for each $i = 1, \ldots, k$. The elements of $\mathcal{C}$ are called colors and we shall say that $\mathcal{C}$ is a coloring of f.

A theorem of Katětov states that every fixed-point free map $\varphi : D \to D$ on a discrete space can be colored with three colors. The theorem of Liusternik and Schnirel'man is

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the statement that every coloring of the antipodal map $\alpha : S^n \to S^n$ on the n-dimensional sphere S^n needs at least $n+2$ colors.

In [2,4,6] topological versions of this theorem were obtained.

Theorem 2 [2]. (1) *Let X be a paracompact Hausdorff space with* $\dim X \leqslant n$. *Suppose that ι is a fixed-point free involution of X. Then there exists a closed cover $\{C_1, \ldots, C_k\}$ of X with $k \leqslant n+2$ such that no C_i contains a pair $\{x, \iota(x)\}$, $i = 1, \ldots, k$.*

(2) *Let X be a metrizable space with* $\dim X \leqslant n$. *Then every fixed-point free homeomorphism of X onto itself can be colored with $n+3$ colors.*

(3) *Let X be a compact metrizable space with* $\dim X \leqslant n$. *Every fixed-point free continuous map of X to itself can be colored with $n+3$ colors.*

The function dim is the covering dimension. The first goal of this paper is to show that in the second and the third statement of this theorem the metrizability condition can be dropped.

Theorem 3. *Let X be a paracompact space with* $\dim X \leqslant n$. *Then every fixed-point free homeomorphism of X onto itself can be colored with $n+3$ colors.*

Theorem 4. *Let X be a compact space with* $\dim X \leqslant n$. *Every fixed-point free continuous map of X to itself can be colored with $n+3$ colors.*

In Theorem 3 homeomorphism cannot be replaced by continuous map. For basic information we refer to [2,4].

We frequently use the following result.

Lemma 5 [6]. *Every fixed-point free homeomorphism $f : X \to X$ of a finite-dimensional paracompact Hausdorff space onto itself has a fixed-point free Čech–Stone extension $\beta f : \beta X \to \beta X$.*

2. Proofs

Proof of Theorem 3. By Lemma 5 we only have to prove Theorem 3 under the additional assumption that X is compact. (Recall that $\dim X = \dim \beta X$.) Assume $f : X \to X$ is a homeomorphism with X compact. We show that this map is weakly conjugated to a fixed-point free map (even homeomorphism) on a compact metric space of dimension not larger than that of X. This means that we construct a compact metrizable space Y with $\dim Y \leqslant \dim X = n$, a fixed-point free map $f^* : Y \to Y$ and a map $\varphi : X \to Y$ such that $\varphi \circ f = f^* \circ \varphi$. The space Y shall be the Wallman representation of a carefully selected collection of zero-sets of X. If we succeed in this construction we are done with the proof. Indeed, a coloring $\{A_1, \ldots, A_{n+3}\}$ of the constructed map on Y induces the coloring $\{\varphi^{-1}(A_1), \ldots, \varphi^{-1}(A_{n+3})\}$ of the map f.

The idea of this proof as described above is due to K.P. Hart.

Since the proof of Theorem 3 relies on the use of the family of zero-sets $\mathcal{Z}(X)$ we state a lemma from [1] in which the covering dimension is characterized by the zero-sets of the space.

Lemma 6. *Let X be a compact Hausdorff space. The following statements are equivalent.*

(1) $\dim X \leqslant n$.

(2) *There exists a base $\mathcal{B}$ of zero-sets such that for every finite collection $\mathcal{F} \subset \mathcal{B}$ with $\bigcap \mathcal{F} = \emptyset$ there exists a finite collection $\mathcal{G} \subset \mathcal{B}$ such that:*

- (a) *for every element $G \in \mathcal{G}$ there exists an $F \in \mathcal{F}$ with $F \subset G$,*
- (b) $\bigcap \mathcal{G} = \emptyset$,
- (c) *every subcollection of G with more than $n+1$ elements is a covering of X.*

We are looking for a countable collection $\mathcal{T} \subset \mathcal{Z}(X)$ with the following properties.

List 7. *Properties of $\mathcal{T}$.*

(1) $\{\emptyset, X\} \subset \mathcal{T}$.

(2) *For $S, T \in \mathcal{T}$ we have $\{S \cap T, S \cup T\} \subset \mathcal{T}$.*

(3) *For $T \in \mathcal{T}$ we have $\{f^{-1}(T), f(T)\} \subset \mathcal{T}$.*

(4) *The lattice $\mathcal{T}$ is normal, i.e., for every disjoint $S, T \in \mathcal{T}$ there exist G and H in $\mathcal{T}$ such that $G \cap S = \emptyset = T \cap H$ and $G \cup H = X$. (This property is also known as the screening-property.)*

(5) *For every $T \in \mathcal{T}$ there exist $T_n \in \mathcal{T}$ such that $X \backslash T = \bigcup\{T_n : n \in \mathbb{N}\}$.*

(6) *For every finite collection $\mathcal{F} \subset \mathcal{T}$ with $\bigcap \mathcal{F} = \emptyset$ there exists a finite collection $\mathcal{G} \subset \mathcal{T}$ with:*

- (a) $\bigcap \mathcal{G} = \emptyset$,
- (b) *every subcollection of $\mathcal{G}$ with more than $n+1$ elements is a covering of X,*
- (c) *for every element $G \in \mathcal{G}$ there exists an F in $\mathcal{F}$ with $F \subset G$.*

(7) *There is a finite subcollection $\{C_1, \ldots, C_k\}$ of $\mathcal{T}$ satisfying $f(C_i) \cap C_i = \emptyset$, for all $1 \leqslant i \leqslant k$ and $C_1 \cup \cdots \cup C_k = X$.*

The following lemma is the heart of the construction.

Lemma 8. *Given a countable collection $\mathcal{S} \subset \mathcal{Z}(X)$ there exists a countable collection $\mathcal{T} \subset \mathcal{Z}(X)$ with $\mathcal{T} \supset \mathcal{S}$ and satisfying the properties $1, \ldots, 6$ from List 7.*

Proof. The proof is trivial, since all the conditions in List 7 are satisfied in $\mathcal{Z}(X)$.

As X is compact and the map f has no fixed points there exists a finite covering $\{C_1, \ldots, C_k\}$ of zero-sets such that $f(C_i) \cap C_i = \emptyset$ for every i. Now apply Lemma 8 to the collection $\mathcal{S} = \{\emptyset, C_1, \ldots, C_k, X\}$ and we obtain the required countable collection $\mathcal{T}$ satisfying the 7 properties from List 7.

The space Y will be the Wallman representation of the collection $\mathcal{T}$ (as described by Wallman in [10]). So Y is the collection of all $\mathcal{T}$-ultrafilters with as a closed base the family $\{\mathcal{S}^*\colon\ \mathcal{S} \in \mathcal{T}\}$, where $S^* = \{\mathcal{F} \in Y\colon\ S \in \mathcal{F}\}$. (Note that the fifth property implies that $S = T$ if and only if $S^* = T^*$.) Lemma 6 together with the fact that $\mathcal{T}$ is countable implies the following.

Lemma 9. *The constructed space Y is compact metrizable and* $\dim Y \leqslant \dim X$.

Next, define the map $\varphi : X \to Y$ by

$$\varphi(x) = \{S \in \mathcal{T}\colon\ x \in S\}.$$

Clearly $\varphi(x) \in Y$ and the compactness of X implies that φ is surjective. Moreover, the following identities are easy to verify.

$$\varphi(S) = S^*, \quad \varphi^{-1}(S^*) = S.$$

Finally, we define the continuous function $f^* : Y \to Y$ by:

$$f^*(\mathcal{F}) = \{f(F)\colon\ F \in \mathcal{F}\}.$$

Obviously $f^*(\mathcal{F}) \in Y$. The following equalities are easily seen to be true.

$$(f^*)^{-1}(S^*) = \left(f^{-1}(S)\right)^*, \quad f^*(S^*) = \left(f(S)\right)^*.$$

In particular, f^* is continuous. (In fact it is not difficult to see that the map f^* is a homeomorphism.) We check that the map f^* is fixed-point free. Indeed, $Y = C_1^* \cup \cdots \cup C_k^*$ and as $f(C_i) \cap C_i = \emptyset$ it follows that $(f(C_i))^* \cap C_i^* = \emptyset$ and so $f^*(C_i^*) \cap C_i^* = \emptyset$.

Finally we observe that the maps φ and f^* defined as above satisfy:

$$\varphi \circ f = f^* \circ \varphi.$$

This ends the proof of Theorem 3.

Proof of Theorem 4. This proof can be copied from [2]. Here it was verified that if the fixed-point free homeomorphisms of every compact n-dimensional (metric) space X can be colored with $n + 3$ colors then the same is true for all fixed-point free continuous maps.

3. A topological version of the De Bruijn–Erdős theorem

There exists a generalization of the Katětov theorem due to de Bruijn and Erdős.

Theorem 10 [5]. *Let D be a set and $\varphi_1, \ldots, \varphi_p : D \to D$ be fixed-point free maps. Then $D = A_1 \cup \cdots \cup A_{2p+1}$ with $A_i \cap \varphi_j(A_i) = \emptyset$ for all $i \leqslant 2p + 1$ and $j \leqslant p$.*

In [4] a topological version of this theorem was obtained for the class of compact zero-dimensional spaces. Here we present one for the class of finite dimensional spaces.

Definition 11. Let $f_j : X \to X$ $(j = 1, \ldots, \ell)$ be fixed-point free maps. We say that these maps can be colored with k colors if there is a closed cover $\mathcal{C} = \{C_1, \ldots, C_k\}$

with k sets such that $f_j(C_i) \cap C_i = \emptyset$, for all $i \leqslant k$ and $j \leqslant \ell$. The elements of $\mathcal{C}$ are called colors and we shall say that $\mathcal{C}$ is a coloring of $f_1, \ldots, f_\ell$.

We have a good generalization of Theorem 10 and Theorem 3 if we restrict ourself to *homeomorphisms* on the class of finite dimensional paracompact spaces.

Theorem 12. *Let X be a paracompact space with $\dim X \leqslant n$. Let $f_i : X \to X$ ($i = 1, \ldots, \ell$) be fixed-point free homeomorphisms of X and let $g_j : X \to X$ ($j = 1, \ldots, k$) be fixed-point free involutions of X. Then $\{f_1, \ldots, f_\ell, g_1, \ldots, g_k\}$ can be colored with $n + 2\ell + k + 1$ colors.*

But for the larger class of continuous maps we can only prove a similar statement under the condition that the maps commute.

Theorem 13. *Let X be a compact space with $\dim X \leqslant n$. Let $f_i : X \to X$ ($i = 1, \ldots, \ell$) be fixed-point free continuous maps and let $g_j : X \to X$ ($j = 1, \ldots, k$) be fixed-point free involutions of X such that any two functions from $\{f_1, \ldots, f_\ell, g_1, \ldots, g_k\}$ commute. Then $\{f_1, \ldots, f_\ell, g_1, \ldots, g_k\}$ can be colored with $n + 2\ell + k + 1$ colors.*

We are quite convinced that the assumption on commutativity can be dropped, but we do not see yet how to do it.

In the final remark of this paper we show that the upperbound $n + 2\ell + k + 1$ is the best upperbound for all n, ℓ and $k \in \{0, 1\}$. (For other k we do not have examples.) In particular, the number $n + 3$ in Theorem 3 is the best upperbound, for all n.

4. Proofs

Preliminaries for the proof of Theorem 12

By Lemma 5 we only have to prove Theorem 12 under the additional assumption that X is compact. (Note that the βg_j remain fixed-point free involutions.) Using the same construction as in Section 2 we see that we can even assume that the space X is compact metric. Indeed, we replace condition 3 in the List 7 of the Wallman lattice by the condition:

(3) For every $T \in \mathcal{T}$ and i, j we have $\{f_i^{-1}[T], f_i[T], g_j[T]\} \subset \mathcal{T}$. And so we construct a compact metrizable space Y with $\dim Y \leqslant \dim X = n$ and fixed-point free maps $f_i^* : Y \to Y$ and involutions $g_j^* : Y \to Y$ and a map $\varphi : X \to Y$ such that $\varphi \circ f_i = f_i^* \circ \varphi$ and $\varphi \circ g_j = g_j^* \circ \varphi$. So we can follow the method from [2].

Let X be a compact metric space with $\dim X \leqslant n$ and let $f_i : X \to X$ ($i = 1, \ldots, \ell$) be fixed-point free homeomorphisms of X and let $g_j : X \to X$ ($j = 1, \ldots, k$) be fixed-point free involutions of X. The following two propositions are well-known and proofs can be found in [1].

Proposition 14. *Suppose that X is a metrizable space. Let Z be a subspace with $\dim Z \leqslant n$. Then for every pair of disjoint closed subsets F and G there is a partition S between F and G with* $\dim(S \cap Z) \leqslant n-1$.

Proposition 15. *Let $\{Y_k\colon k \in \mathbb{N}\}$ be a collection of F_σ-subsets of a metrizable space X. Suppose that* $\dim Y_k = n_k \geqslant 0$ *for k in $\mathbb{N}$. Then there exists an F_σ-set Z of X such that* $\dim Z = 0$ *and* $\dim(Y_k \backslash Z) = n_k - 1$ *for all k in $\mathbb{N}$.*

The following two lemmas are the key, and the proofs are direct translations from [2]. The details are left to the reader.

Lemma 16. *Suppose X is a metrizable space with* $\dim X \leqslant n$ *and let $h_1, \dots, h_p$ be homeomorphisms of X onto itself without fixed points. Let φ, with or without subscript, denote one of the maps h_i and/or* id. *Suppose that $\mathcal{S} = \{S_i\colon i \in \mathbb{N}\}$ is a family of closed subsets of X such that*

$$\dim\left(\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_p}(S_{i_p})\right) \leqslant n - p$$

whenever $i_1 < \cdots < i_p$ and $1 \leqslant p \leqslant n+1$. Then for each pair of disjoint closed subsets F and G of X there is a partition S between F and G such that

$$\dim\left(\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap \varphi(S)\right) \leqslant n - k$$

whenever $i_1 < \cdots < i_{k-1}$ and $1 \leqslant k \leqslant n+1$.

Lemma 17. *Let X be a metrizable space with* $\dim X \leqslant n$ *and let φ, with or without subscript, denote a map from a finite collection of homeomorphisms from X to X. Let $\mathcal{K} = \{K_i\colon 1 \leqslant i \leqslant m\}$ be a finite closed cover of X and $\mathcal{U} = \{U_i\colon 1 \leqslant i \leqslant m\}$ be an open swelling of $\mathcal{K}$. Then there exists a closed swelling $\mathcal{L} = \{L_i\colon 1 \leqslant i \leqslant m\}$ of $\mathcal{K}$ such that $K_i \subset L_i \subset U_i$ and*

$$\varphi_{i_1}[\partial L_{i_1}] \cap \cdots \cap \varphi_{i_{n+1}}[\partial L_{i_{n+1}}] = \emptyset$$

whenever $1 \leqslant i_1 < \cdots < i_{n+1} \leqslant m$.

Proof of Theorem 12. Let $\mathcal{K} = \{K_1, \dots, K_m\}$ be a finite coloring of $\{f_1, \dots, f_\ell, g_1, \dots, g_k\}$. Let h_i denote one of the maps from this collection of maps. There exists an open swelling $\mathcal{U} = \{U_i\colon 1 \leqslant i \leqslant m\}$ such that $h_j(U_p) \cap U_p =$ for all j and all $1 \leqslant p \leqslant m$.

Let φ with or without subscript denote f_i, f_i^{-1} or g_j. (Note that $g_j = g_j^{-1}$.) By Lemma 17 we find a closed swelling $\mathcal{L} = \{L_i\colon 1 \leqslant i \leqslant m\}$ of $\mathcal{K}$ such that $K_i \subset L_i \subset U_i$, for all $1 \leqslant i \leqslant m$ and

$$\varphi_{i_1}[\partial L_{i_1}] \cap \cdots \cap \varphi_{i_{n+1}}[\partial L_{i_{n+1}}] = \emptyset.$$

Now put $\mathcal{C} = \{C_1, \dots, C_m\}$ with $C_i = \operatorname{cl}(L_i \backslash \{L_1 \cup \cdots \cup L_{i-1}\})$. Suppose $\zeta > n + 2\ell + k + 1$. We want to remove color C_ζ thus reducing the number of colors. Let $y \in C_\zeta$ and let $\{C_{i_j}\colon j \leqslant t\}$, where $t \leqslant \zeta - 1$, be the collection of colors containing at least one element of the form $f_j(y)$, $f_j^{-1}(y)$ or $g_i(y)$. By using the appropriate functions φ we find

$$y \in \varphi_{i_1}[C_{i_1}] \cap \cdots \cap \varphi_{i_m}[C_{i_t}].$$

Suppose φ_{i_j} is f_j for $t_{j,1}$ indices, φ_{i_j} is f_j^{-1} for $t_{j,2}$ and φ_{i_j} is h_j for $t_{j,3}$ indices. Note that $\sum_{j,b} t_{j,b} = t$.

The question is for how many indices is $y \in \varphi[\partial L_{i_j}]$. By the construction of $\mathcal{C}$ we find y for at least $t_{j,b} - 1$ indices in $\varphi(\partial L_{i_j})$.

This gives $\sum_{j,b} t_{j,b} = t - 2\ell - k \leqslant n$. Which implies y is contained in at most $n+2\ell+k$ sets, hence within the first $n+2\ell+k+1$ sets there is a set not containing any of the previous mentioned forms. We can move y to this set using the same technique as described in [2, Theorem 4].

Proof of Theorem 13. First we prove the theorem in case all the maps are surjective. For notational convenience the maps are called $f_1, \ldots, f_{\ell+k}$. Consider the following inverse system.

$$\cdots \xrightarrow{f_2} X \xrightarrow{f_1} X \xrightarrow{f_{\ell+k}} X \xrightarrow{f_{\ell+k-1}} X \cdots \xrightarrow{f_2} X \xrightarrow{f_1} X.$$

The inverse limit of the sequence with these bonding maps is called Y. Note that Y is a compact space with $\dim Y \leqslant n$. As the maps commute every continuous map f_i can be seen as a map from the inverse sequence to itself and therefore can be lifted to a unique fixed-point free homeomorphism $F_i : Y \to Y$. (Moreover, if f_i is an involution, so is F_i.) As the dimension of Y is at most n the maps $\{F_1, \ldots, F_{\ell+k}\}$ can be colored with $n + 2\ell + k + 1$ colors, say

$$\mathcal{C} = \{C_i\colon\ 1 \leqslant i \leqslant n + 2\ell + k + 1\}.$$

The projection of Y on the jth coordinate space is denoted π_j. Since f is surjective, so is π_j. The projection $\pi_j(\mathcal{C})$ is a closed cover of X. (Here we use that the maps are surjective.)

For fixed index i, the preimages $\pi_j^{-1}(\pi_j(C_i))$ form a descending sequence with intersection $\bigcap\{\pi_j^{-1}(\pi_j(C_i)) \mid j \in \mathbb{N}\} = C_i$. The map F_r satisfies for sufficiently large j_r,

$$\pi_{j_r}^{-1}\big(\pi_{j_r}(C_i)\big) \cap F_r\big(\pi_{j_r}^{-1}\big(\pi_{j_r}(C_i)\big)\big) = \emptyset.$$

Sinse $f_r \circ \pi_j = \pi_j \circ F_r$, it follows that $\pi_{j_r}(C_i) \cap f_r(\pi_j(C_i)) = \emptyset$. So, the projection $\pi_j(\mathcal{C})$ is a coloring of all the maps f_r, for sufficiently large j. Now we consider the case in which the maps f_i are arbitrary continuous maps. Consider the sequence of closed subspaces

$$X_1 = f_1(X),\ X_2 = f_2(X_1), \ldots, X_{\ell+k} = f_{\ell+k}(X_{\ell+k+1}),$$

$$X_{\ell+k+1} = f_1(X_{\ell+k}),\ X_{\ell+k+2} = f_2(X_{\ell+k}) \ldots$$

As the maps commute the sequence $\{X_n\colon\ n \geqslant 1\}$ is decreasing. Put $K = \bigcap X_n$. Then $f_i(K) \subset K$ and the maps $f_i : K \to K$ are surjective. By the first result, the maps f_i

on K can be colored by $n+2\ell+k+1$ colors: $\mathcal{C} = \{C_i \mid 1 \leqslant i \leqslant n+2\ell+k+1\}$. In X there is an open collection

$$\mathcal{U} = \{U_i \mid 1 \leqslant i \leqslant n+2\ell+k+1\}$$

such that $C_i \subset U_i$ and $f_j(U_i) \cap U_i = \emptyset$, for all $j = 1, \ldots, \ell+k$ and $i = 1, \ldots, n+2\ell+k+1$. The union U of $\mathcal{U}$ is a neighborhood of K. For m sufficiently large, X_m is contained in U. Let us assume for convenience that $X_m = f_1(X_{m-1})$. Then for sure $\{f_1^{-1}(U_1), \ldots, f_1^{-1}(U_{n+2\ell+k+1})\}$ is an (open) coloring of f_1 on X_{m-1}. But as the maps commute it will also be an open coloring for the other maps:

$$\begin{aligned} f_j^{-1}\big(f_1^{-1}(U_1)\big) \cap f_1^{-1}(U_1) &= f_1^{-1}\big(f_j^{-1}(U_1)\big) \cap f_1^{-1}(U_1) \\ &= f_1^{-1}\big(f_j^{-1}(U_1) \cap U_1\big) = \emptyset. \end{aligned}$$

Next we take preimages under $f_{\ell+k}$ to obtain a covering of X_{m-2}, etc. and in this way an open cover of X is obtained in m steps. A closed shrinking of this cover is the required coloring of the maps f_i by $n+2\ell+k+1$ colors.

5. An application

First we collect some definitions.

Definition 18. Let X be a compact space. A continuous map $f: X \to X$ is called a $\mathbb{Z}_p$-action on X if

(1) $\forall x \in X$: $f^p(x) = x$.

If, moreover:

(2) $\forall x \in X$: the set $\{x, f(x), \ldots, f^{p-1}(x)\}$ has cardinality p then the $\mathbb{Z}_p$-action is called free.

Definition 19. Let X be a compact space and let $f: X \to X$ be a free $\mathbb{Z}_p$-action on X.

(1) A subset $C \subset X$ is called a set of the first type if there exists a closed set $A \subset X$ such that

(a) $C = \bigcup\{f^j(A)\colon j = 0, 1, \ldots, p-1\}$,

(b) $A \cap f^j(A) = \emptyset$, for $j \in 1, 2, \ldots, p-1\}$.

(2) The number $k \in \mathbb{N}$ that is minimal with respect to the property that X is the union of k-many closed sets of first type is called the genus of (X, f), and is denoted by $g(X, f)$. The name "genus of f" is also used and when it is clear from the context what the map is then the name "genus of X" is used.

The theorem of Liusternik and Schnirel'man can be reformulated as follows: the genus of the antipodal map on the n-dimensional sphere $\mathcal{S}^n$ is equal to $n+1$. It is not difficult to see that the first statement in Theorem 2 is equivalent to the following statement.

Proposition 20. *Let X be a paracompact Hausdorff space with* $\dim X \leqslant n$ *and suppose that ι is a fixed-point free involution of X. Then the genus of ι is at most $n+1$.*

There exists a space K_p^n with a $\mathbb{Z}_p$-action $\varphi : K_p^n \to K_p^n$ such that $g(K_p^n, \varphi) = n+1$. We mention that the space K_p^n has dimension n and for odd n: $K_p^n = S^n$, the n-sphere with the standard $\mathbb{Z}_p$-action. For more information see [9].

This space is universal with respect to the spaces of genus at most $n+1$, because of the following property.

Lemma 21 [9]. *Let X be a compact space and $f : X \to X$ a free $\mathbb{Z}_p$-action with $g(X, f) \leqslant n+1$. Then f is weakly conjugated (or equivariant) to the map $\varphi : K_p^n \to K_p^n$, i.e., there exists a map $h : X \to K_p^n$ such that $\varphi \circ h = h \circ f$.*

Finally we mention a result by Krasnosel'ski, that generalizes the Liusternik–Schnirel'man theorem.

Theorem 22 (Krasnosel'ski [7]). *The genus of every free $\mathbb{Z}_p$-action $f : S^n \to S^n$ on the n-dimensional sphere S^n is at least $n+1$.*

The application of our results is the following theorem.

Theorem 23. *Let X be a paracompact n-dimensional space and let $f : X \to X$ be a free $\mathbb{Z}_p$-action on X. Then $g(X, f) \leqslant n+1$.*

Proof. *Case* 1: *p is odd, say $p = 2k+1$.*

Claim: if $A \subset X$ and $f^j(A) \cap A = \emptyset$, for $j = 1, \ldots, k$ then $f^j(A) \cap A = \emptyset$ for $j = 1, \ldots, 2k$. This is easy to verify: $A \cap f^i(A) = \emptyset$ implies $f^{p-i}(A) \cap A = \emptyset$, as $f^p = \mathrm{id}$.

Consider the fixed-points free homeomorphismes $\mathcal{F} = \{f, f^2, \ldots, f^k\}$. According to Theorem 12 there exists a closed coloring

$$\mathcal{A} = \{A_1, A_2, \ldots, A_{n+2k}, A_{n+2k+1} = A_{n+p}\}$$

of $\mathcal{F}$.

Note that according to the claim, for all i, the sets $f^j(A_i)$ $(j = 1, \ldots, p)$ are pairwise disjoint. Therefore, for all i, the sets

$$C_i = \bigcup \{f^j(A_i) \colon j = 1, \ldots, p\}$$

are of the first type. If suffices to show that $X = C_1 \cup \cdots \cup C_{n+1}$. If not, then some $x \in X$ is not in $C_1 \cup \cdots \cup C_{n+1}$ and therefore $\{x, f(x), \ldots, f^{p-1}(x)\} \cap C_1 \cup \cdots \cup C_{n+1} = \emptyset$.

The set $\{x, f(x), \ldots, f^{p-1}(x)\}$ is contained in $A_{n+2} \cup \cdots \cup A_{n+p}$ and therefore at least one of these A_i contains at least two points of the set $\{x, f(x), \ldots, f^{p-1}(x)\}$. This is a contradiction, as the sets $f^j(A_i)$ $(j = 1, \ldots, p)$ are pairwise disjoint.

Case 2: *p is even, say $p = 2k$.*

Claim: if $A \subset X$ and $f^j(A) \cap A = \emptyset$, for $j = 1, \ldots, k$, then $f^j(A) \cap A = \emptyset$ for $j = 1, \ldots, 2k-1$.

Consider the fixed-points free homeomorphismes $\mathcal{F} = \{f, f^2, \ldots, f^k\}$. Note that f^k is an involution. According to Theorem 12 there exists a closed coloring $\mathcal{A} = \{A_1, \ldots, A_{n+2(k-1)+1+1} = A_{n+p}\}$ of $\mathcal{F}$.

Note that according to the previous claim, for all i, the sets $f^j(A_i)$ $(j = 1, \ldots, p)$ are pairwise disjoint. Next continue as in the previous case.

Corollary 24. *The genus of every free $\mathbb{Z}_p$-action $f: S^n \to S^n$ on the n-dimensional sphere S^n is equal to $n + 1$.*

We also conclude that the spaces K_p^n are not only universal with respect to genus, but also with respect to dimension.

Corollary 25. *Let X be a compact space with $\dim X \leqslant n$ and let $f: X \to X$ be a free $\mathbb{Z}_p$-action. Then f is weakly conjugated to the map $\varphi: K_p^n \to K_p^n$.*

Remark. The proof of Theorem 23 shows that the spaces K_p^n can be used to see that the upperbound $n + 2\ell + k + 1$ in Theorem 12 is the best upperbound for $k = 0$ and $k = 1$. For other k we do not have examples. The spaces K_3^{2n} of even dimension can be used to see that the $\mathbb{Z}_3$-action on this space needs $2n + 3$ colors. Therefore, the number $n + 3$ in Theorem 3 is the best upperbound, also for even n. (In [2] this was only observed for odd n.)

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