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Formal paths, iterated integrals and the center problem for ordinary differential equations

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Abstract

We continue the study of the center problem for the ordinary differential equation $v' = \sum_{i=1}^{\infty} a_i(x)v^{i+1}$ started in [A. Brudnyi, An explicit expression for the first return map in the center problem, J. Differential Equations 206 (2004) 306–314; A. Brudnyi, On the center problem for ordinary differential equations, Amer. J. Math. 128 (2006) 419–451; A. Brudnyi, An algebraic model for the center problem, Bull. Sci. Math. 128 (2004) 839–857; A. Brudnyi, On center sets of ODEs determined by moments of their coefficients, Bull. Sci. Math. 130 (2006) 33–48; A. Brudnyi, Vanishing of higher-order moments on Lipschitz curves, Bull. Sci. Math. 132 (3) (2008) 165–181]. In this paper we present the highlights of the algebraic theory of centers.

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1. Introduction

In this paper we describe an algebraic approach to the center problem for the ordinary differential equation

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)v^{i+1}, \quad x \in I_T := [0, T], \tag{1.1}$$

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with coefficients a_i from the Banach space $L^\infty(I_T)$ of bounded measurable complex-valued functions on I_T equipped with the supremum norm. Condition

$$\sup_{x\in I_T,\ i\in\mathbb{N}} \sqrt[i]{|a_i(x)|} < \infty \tag{1.2}$$

guarantees that (1.1) has Lipschitz solutions on I_T for all sufficiently small initial values. By X we denote the complex Fréchet space of sequences $a = (a_1, a_2, \dots)$ satisfying (1.2). We say that Eq. (1.1) determines a *center* if every solution v of (1.1) with a sufficiently small initial value satisfies $v(T) = v(0)$. By $\mathcal{C} \subset X$ we denote the set of centers of (1.1). The center problem is: *given $a \in X$ to determine whether $a \in \mathcal{C}$*. It arises naturally in the framework of the geometric theory of ordinary differential equations created by Poincaré. In particular, there is a relation between the center problem for (1.1) and the classical Poincaré Center–Focus problem for planar polynomial vector fields

$$\frac{dx}{dt} = -y + F(x,y), \qquad \frac{dy}{dt} = x + G(x,y), \tag{1.3}$$

where F and G are polynomials of a given degree without constant and linear terms. This problem asks about conditions on F and G under which all trajectories of (1.3) situated in a small neighborhood of $0 \in \mathbb{R}^2$ are closed. Passing to polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$ in (1.3) and expanding the right-hand side of the resulting equation as a series in r (for F, G with sufficiently small coefficients) we obtain an equation of the form (1.1) whose coefficients are trigonometric polynomials depending polynomially on the coefficients of (1.3). This reduces the Center–Focus problem for (1.3) to the center problem for (1.1) with coefficients depending polynomially on a parameter.

In this paper we continue the study of the center problem for Eq. (1.1) started in [6–10]. One of the basic objects of our approach is a metrizable topological group $G(X)$ defined by the coefficients of Eqs. (1.1) (the, so-called, *group of paths* in $\mathbb{C}^\infty$). Modulo the set of universal centers $\mathcal{U} \subset \mathcal{C}$ of (1.1), described explicitly in [7], the set of centers forms a normal subgroup $\widehat{\mathcal{C}} \subset G(X)$. By $G_f(X)$ and $\widehat{\mathcal{C}}_f$ we denote the groups of *formal paths* and of *formal centers*, respectively, i.e., the completions of $G(X)$ and $\widehat{\mathcal{C}}$ with respect to the metric on $G(X)$. In this paper we study the algebraic properties of $G_f(X)$ and $\widehat{\mathcal{C}}_f$. In particular, we describe Lie algebras of these groups and prove that $G_f(X)$ is the semidirect product of a naturally defined normal subgroup of $\widehat{\mathcal{C}}_f$ and the subgroup $G_f(X^2)$ of formal paths in $G_f(X)$ determined by coefficients of Abel differential equations, i.e., Eqs. (1.1) with $a_k = 0$ for all $k \geqslant 3$. Also, we show that $\widehat{\mathcal{C}}_f$ contains a dense subgroup of centers generated by certain piecewise linear paths in $\mathbb{C}^\infty$.

The paper is organized as follows.

Section 2 is devoted to the study of the group $G_f(X)$ of formal paths in $\mathbb{C}^\infty$.

In Section 2.1 we introduce a natural multiplication on the set X similar to the product of paths in topology. Then we define the group of paths $G(X)$ as the quotient of X by an equivalence relation determined in terms of iterated integrals on X. We equip $G(X)$ with a natural metric d and define the group $G_f(X)$ of formal paths in $\mathbb{C}^\infty$ as the completion of $G(X)$ with respect to d.

The equivalence class $\mathcal{U} \subset X$ corresponding to $1 \in G(X)$ is called the set of universal centers of Eq. (1.1). In Section 2.2 we present some results of [7] on the characterization of elements from $\mathcal{U}$.

Next, in Section 2.3 we will show how to embed $G_f(X)$ in a group G of invertible formal power series in t whose coefficients belong to the associative algebra with unit I of complex non-commutative polynomials in I and free non-commutative variables X_i, $i \in \mathbb{N}$.

Identifying $G_f(X)$ with its image in G we describe in Section 2.4 the Lie algebra $\mathcal{L}_{\mathrm{Lie}}$ of $G_f(X)$ as the subset of Lie elements of the Lie algebra $\mathcal{L}_G$ of G.

In Section 2.5 we prove some structural theorems for $G(X)$ and $G_f(X)$. Namely, we describe the topological lower central series of these groups and their subgroups corresponding to closed paths in $\mathbb{C}^\infty$ in terms of iterated integrals on $G_f(X)$.

Finally, in Section 2.6 we describe some natural subgroups of $G_f(X)$: the groups $G_f(X^n)$ generated by paths in $\mathbb{C}^n$ and $G(X_{\mathbb{F}})$ determined over a field $\mathbb{F} \subset \mathbb{C}$.

Section 3 is devoted to the study of the center problem for Eq. (1.1).

In Section 3.1 we gather some results from [6–8] on the explicit expression for the first return map of (1.1).

Using this we define in Section 3.2 the group $\widehat{\mathcal{C}}_f \subset G_f(X)$ of formal centers of Eq. (1.1).

Then in Section 3.3 we give an explicit description of the Lie algebra of $\widehat{\mathcal{C}}_f$ and show that $\widehat{\mathcal{C}}_f$ is the closure in $G_f(X)$ of the group of centers $\widehat{\mathcal{C}}$ of (1.1). We also prove that $G_f(X)$ is the semidirect product of a naturally defined normal subgroup of $\widehat{\mathcal{C}}_f$ and the subgroup $G_f(X^2)$ of formal paths in $\mathbb{C}^2$ (i.e., determined by coefficients of Abel differential equations). At the end of this section we briefly discuss the center problem over a field $\mathbb{F} \subset \mathbb{C}$.

Finally, in Section 3.4 we introduce the subgroup $PL \subset G_f(X)$ of piecewise linear paths in $\mathbb{C}^\infty$. We give a characterization of centers belonging to this group and show that the set of such centers is dense in $\widehat{\mathcal{C}}_f$.

2. Group of formal paths

2.1. *Definition of the group of paths*

2.1.1. Let us consider X as a semigroup with the operations given for $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ by

$$a * b = (a_1 * b_1, a_2 * b_2, \ldots) \in X \quad \text{and} \quad a^{-1} = \left(a_1^{-1}, a_2^{-1}, \ldots\right) \in X,$$

where for $i \in \mathbb{N}$

$$(a_i * b_i)(x) = \begin{cases} 2b_i(2x) & \text{if } 0 \leqslant x \leqslant T/2, \\ 2a_i(2x - T) & \text{if } T/2 < x \leqslant T \end{cases}$$

and

$$a_i^{-1}(x) = -a_i(T - x), \quad 0 \leqslant x \leqslant T.$$

Let $\mathbb{C}^\infty$ be the vector space of sequences of complex numbers $(c_1, c_2, \ldots)$ equipped with the product topology. For $a = (a_1, a_2, \ldots) \in X$ by $\widetilde{a} = (\widetilde{a}_1, \widetilde{a}_2, \ldots) : I_T \to \mathbb{C}^\infty$, $\widetilde{a}_k(x) := \int_0^x a_k(t)\,dt$ for all $k \in \mathbb{N}$, we denote a path in $\mathbb{C}^\infty$ starting at 0. The one-to-one map $a \mapsto \widetilde{a}$ sends the product $a * b$ to the product of paths $\widetilde{a} \circ \widetilde{b}$, that is, the path obtained by translating $\widetilde{a}$ so that its beginning meets the end of $\widetilde{b}$ and then forming the composite path. Similarly, $\widetilde{a^{-1}}$ is the path obtained by translating $\widetilde{a}$ so that its end meets 0 and then taking it with the opposite orientation.

2.1.2. For $a \in X$ let us consider the *basic iterated integrals*

$$I_{i_1,\ldots,i_k}(a) := \int\cdots\int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_k \leqslant T} a_{i_k}(s_k) \cdots a_{i_1}(s_1)\,ds_k \cdots ds_1 \tag{2.1}$$

(for $k=0$ we assume that this equals 1). By the Ree shuffle formula [18] the linear space generated by all such functions on X is an algebra. The number k in (2.1) is called the *order* of the iterated integral. Also, the basic iterated integrals satisfy the following equations (see, e.g., [15, Propositions 2.9 and 2.12]):

$$I_{i_1,\dots,i_k}(a*b)=I_{i_1,\dots,i_k}(a)+\sum_{j=1}^{k-1}I_{i_1,\dots,i_j}(a)\cdot I_{i_{j+1},\dots,i_k}(b)+I_{i_1,\dots,i_k}(b). \tag{2.2}$$

$$I_{i_1,\dots,i_k}(a^{-1})=(-1)^k I_{i_1,\dots,i_k}(a). \tag{2.3}$$

For $a,b\in X$ we write $a\sim b$ if all basic iterated integrals vanish at $a*b^{-1}$. Eqs. (2.2) and (2.3) imply that $a\sim b$ if and only if $I_{i_1,\dots,i_k}(a)=I_{i_1\dots,i_k}(b)$ for all basic iterated integrals. In particular, $\sim$ is an equivalence relation on X. By $G(X)$ we denote the set of equivalence classes. Then $G(X)$ is a group with the product induced by the product $*$ on X. By $\pi: X\to G(X)$ we denote the map determined by the equivalence relation. By the definition each iterated integral $I.$ is constant on fibres of π and therefore it determines a function $\widehat{I.}$ on $G(X)$ such that $I.=\widehat{I.}\circ\pi$. The functions $\widehat{I.}$ will be referred to as iterated integrals on $G(X)$. These functions separate the points on $G(X)$.

Next, we equip $G(X)$ with the weakest topology τ in which all basic iterated integrals $\widehat{I}_{i_1,\dots,i_k}$ are continuous. Then $(G(X),\tau)$ is a topological group. Moreover, $G(X)$ is metrizable: for $g,h\in G(X)$ the formula

$$d(g,h):=\sum_{n=1}^{\infty}\frac{1}{4^n}\cdot\left(\sum_{i_1+\cdots+i_k=n}\frac{|\widehat{I}_{i_1,\dots,i_k}(g)-\widehat{I}_{i_1,\dots,i_k}(h)|}{1+|\widehat{I}_{i_1,\dots,i_k}(g)-\widehat{I}_{i_1,\dots,i_k}(h)|}\right) \tag{2.4}$$

determines a metric on $G(X)$ compatible with topology τ (see [8, Theorem 2.4]). We mention also that $G(X)$ is contractible, residually torsion free nilpotent (i.e., finite-dimensional unipotent representations of $G(X)$ separate the points on $G(X)$) and is the union of an increasing sequence of compact subsets (see [8]).

By $G_f(X)$ we denote the completion of $G(X)$ with respect to the metric d. Then $G_f(X)$ is a topological group which will be called the *group of formal paths* in $\mathbb{C}^\infty$.

2.2. *Structure of the set of universal centers*

By $\mathcal{U}\subset X$ we denote the set of elements $a\in X$ such that all basic iterated integrals vanish at a. According to (2.2), $\mathcal{U}$ is a sub-semigroup of X. It was shown in [7] that $\mathcal{U}\subset\mathcal{C}$, the set of centers of Eq. (1.1). We call $\mathcal{U}$ the *set of universal centers* of (1.1). In this section we formulate some results on the characterization of elements from $\mathcal{U}$ established in [7].

Let $X^k:=\{a=(a_1,a_2,\dots)\in X\colon\ a_j=0 \text{ for all } j>k\}$. By $p_k: X\to X^k$, $(a_1,a_2,\dots)\mapsto(a_1,\dots,a_k,0,0,\dots)$, we denote the natural projection. Clearly $a\in\mathcal{U}$ if and only if $p_k(a)\in\mathcal{U}$ for all $k\in\mathbb{N}$. Therefore it suffices to characterize elements from the sets $\mathcal{U}^k:=\mathcal{U}\cap X^k$.

For $a=(a_1,\dots,a_k,0,\dots)\in X^k$ consider the Lipschitz curve $A_k: I_T\to\mathbb{C}^k$ determined by the formula

$$A_k(x):=\left(\int_0^x a_1(t)\,dt,\dots,\int_0^x a_k(t)\,dt\right),\quad x\in I_T. \tag{2.5}$$

We set $\Gamma_k:=A_k(I_T)$.

Next, we require

Definition 2.1. The polynomially convex hull $\widehat{K}$ of a compact set $K \subset \mathbb{C}^k$ is the set of points $z \in \mathbb{C}^k$ such that if p is any holomorphic polynomial in k variables

$$\left|p(z)\right| \leqslant \max_{x \in K}\left|p(x)\right|.$$

It is well known (see, e.g., [3]) that $\widehat{K}$ is compact, and if K is connected then $\widehat{K}$ is connected. The following basic result was proved in [7, Theorem 1.10].

Theorem 2.2. *Suppose that $a \in \mathcal{U}^k$. Then for any domain $U \subset \mathbb{C}^k$ containing $\widehat{\Gamma_k}$ the path $A_k : I_T \to U$ is closed and represents the unit element of the fundamental group $\pi_1(U)$ of U.*

Since A_k is Lipschitz, Γ_k is of a finite linear measure. Then according to the result of Alexander [1], $\widehat{\Gamma_k} \setminus \Gamma_k$ is a (possibly empty) pure one-dimensional complex analytic subset of $\mathbb{C}^k \setminus \Gamma_k$. In particular, since the covering dimension of Γ_k is 1, the covering dimension of $\widehat{\Gamma_k}$ is 2. Therefore according to the Freudenthal expansion theorem [14], $\widehat{\Gamma_k}$ can be presented as an inverse limit of a sequence of compact polyhedra $\{Q_{kj}\}_{j \in \mathbb{N}}$ with $\dim Q_{kj} \leqslant 2$. Let $\pi_{kj} : \widehat{\Gamma_k} \to Q_{kj}$ be the continuous projections generated by the inverse limit construction. It is easy to check that Theorem 2.2 is equivalent to the following statement: *if $a \in \mathcal{U}^k$, then for all j the continuous paths $\pi_{kj} \circ A_k : I_T \to Q_{kj}$ are closed and represent unit elements of $\pi_1(Q_{kj})$.*

Let us formulate two corollaries of Theorem 2.2. In the first one we use the notion of a *bordered Riemann surface*. This is a compact connected set which consists of a (possibly singular) one-dimensional complex analytic space with a C^2-boundary.

Corollary 2.3. *Suppose that $a \in X^k$ is such that the corresponding set $\widehat{\Gamma_k}$ belongs to a bordered Riemann surface $S \subset \mathbb{C}^k$. Then $a \in \mathcal{U}^k$ if and only if the path $A_k : I_T \to S$ is closed and represents the unit element of the fundamental group $\pi_1(S)$ of S.*

The proof of Corollary 2.3 repeats literally the arguments of the proofs of [7, Corollary 1.17] and [10, Corollary 3.7]. Using the covering homotopy theorem one obtains the following reformulation of the above result.

Let $\pi : \widetilde{S} \to S$ be the universal covering of S. *Under the hypotheses of Corollary* 2.3, *$a \in \mathcal{U}^k$ if and only if there is a closed Lipschitz path $\widetilde{A}_k : I_T \to \widetilde{S}$ such that $A_k = \pi \circ \widetilde{A}_k$.*

Example 2.4. (1) Suppose that $\Gamma_k \subset \mathbb{C}^k$ is the image of the unit circle under a holomorphic embedding of its neighborhood. Then by the result of Wermer [19], $\widehat{\Gamma_k}$ belongs to a bordered Riemann surface.

(2) The assumptions of Corollary 2.3 are also fulfilled if Γ_k belongs to a one-dimensional complex analytic subset of a domain $U \subset \mathbb{C}^k$ such that $U = \bigcup_j K_j$ with $K_j \subset\subset K_{j+1}$ and $\widehat{K}_j = K_j$ for all $j \in \mathbb{N}$, cf. [7, Corollary 1.17]. In particular, this is valid for a convex U.

To formulate the second corollary of Theorem 2.2 we recall the definition of a Lipschitz triangulable curve, see, e.g., [7].

Definition 2.5. A compact curve $C \subset \mathbb{R}^N$ is called Lipschitz triangulable if

1. $C = \bigcup_{j=1}^{s} C_j$ and for $i \neq j$ the intersection $C_i \cap C_j$ consists of at most one point;
2. There are Lipschitz embeddings $f_j : [0,1] \to \mathbb{R}^N$ such that $f_j([0,1]) = C_j$;

3. The inverse maps $f_j^{-1}: C_j \to \mathbb{R}$ are locally Lipschitz on $C_j \setminus \{f_j(0) \cup f_j(1)\}$.

The following corollary extends one of the main results of Chen [13].

Corollary 2.6. *Suppose that $a \in X^k$ is such that Γ_k is a Lipschitz triangulable curve and $\widehat{\Gamma}_k = \Gamma_k$. Then $a \in \mathcal{U}^k$ if and only if the path $A_k : I_T \to \Gamma_k$ is closed and represents the unit element of $\pi_1(\Gamma_k)$.*

Using the covering homotopy theorem one reformulates this corollary as follows:

Under the hypotheses of Corollary 2.6, *$a \in \mathcal{U}^k$ if and only if there are a Lipschitz triangulable curve $\mathcal{T}$ homeomorphic to a finite tree, a locally bi-Lipschitz map $\pi : \mathcal{T} \to \Gamma_k$ and a closed Lipschitz path $\widetilde{A}_k : I_T \to \mathcal{T}$ such that $A_k = \pi \circ \widetilde{A}_k$.*

Example 2.7. (1) The condition $\widehat{\Gamma}_k = \Gamma_k$ is fulfilled if, e.g., Γ_k belongs to a compact set K in a C^1-manifold M with no complex tangents such that $\widehat{K} = K$ (for the proof see, e.g., [3, Theorem 17.1]). For instance, one can take as such K any compact subset of $M = \mathbb{R}^k$.

(2) Γ_k is a Lipschitz triangulable curve if, e.g., $A_k : I_T \to \mathbb{C}^k$ is non-constant analytic.

Corollaries 2.3 and 2.6 reveal a connection of the center problem for Eq. (1.1) with the so-called *composition condition* whose role and importance was studied in [2,4,5,20] for the special case of Abel differential equations.

2.3. Representation of paths by non-commutative formal power series

2.3.1. Let $\mathbb{C}\langle X_1, X_2, \ldots\rangle$ be the associative algebra with unit I of complex non-commutative polynomials in I and free non-commutative variables $X_1, X_2, \ldots$ (i.e., there are no nontrivial relations between these variables). By $\mathbb{C}\langle X_1, X_2, \ldots\rangle[[t]]$ we denote the associative algebra of formal power series in t with coefficients from $\mathbb{C}\langle X_1, X_2, \ldots\rangle$. Also, by $\mathcal{A} \subset \mathbb{C}\langle X_1, X_2, \ldots\rangle[[t]]$ we denote the subalgebra of series f of the form

$$f = c_0 I + \sum_{n=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} X_{i_1}\cdots X_{i_k}\right)t^n \tag{2.6}$$

with $c_0, c_{i_1,\ldots,i_k} \in \mathbb{C}$ for all $i_1, \ldots, i_k, k \in \mathbb{N}$.

Remark 2.8. Let $S \subset \mathbb{C}\langle X_1, X_2, \ldots\rangle$ be the multiplicative semigroup generated by $I, X_1, X_2, \ldots$. Consider a grading function $w : S \to \mathbb{Z}_+$ determined by the conditions

$$w(I) = 0,\ w(X_i) = i, \quad i \in \mathbb{N}, \quad \text{and} \quad w(x \cdot y) := w(x) + w(y) \quad \text{for all } x, y \in S.$$

This splits S in a disjoint union $S = \bigsqcup_{n=0}^{\infty} S_n$, where $S_n = \{s \in S:\ w(s) = n\}$. Now, each $f \in \mathcal{A}$ is written as

$$f = \sum_{n=0}^{\infty} f_n t^n \quad \text{where } f_n \in V_n := \operatorname{span}_{\mathbb{C}}(S_n),\ n \in \mathbb{Z}_+. \tag{2.7}$$

We equip $\mathcal{A}$ with the weakest topology in which all coefficients $c_{i_1,\ldots,i_k}$ in (2.6) considered as functions in $f \in \mathcal{A}$ are continuous. Since the set of these functions is countable, $\mathcal{A}$ is metrizable, cf. (2.4). Moreover, if d is a metric on $\mathcal{A}$ compatible with the topology, then $(\mathcal{A}, d)$ is a complete metric space. Also, by the definition the multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous in this topology.

Remark 2.9. A sequence $\{f_k = \sum_{n=0}^{\infty} f_{nk} t^n\colon\ f_{nk} \in V_n,\ n \in \mathbb{Z}_+\}_{k\in\mathbb{N}}$ converges to $f = \sum_{n=0}^{\infty} f_n t^n$, $f_n \in V_n$, $n \in \mathbb{Z}_+$, in the topology of $\mathcal{A}$ if and only if each $\{f_{kn}\}_{k\in\mathbb{N}}$ converges to f_n in V_n naturally identified with the hermitian space $\mathbb{C}^{d(n)}$ where $d(n) := \#S_n$.

By $G \subset \mathcal{A}$ we denote the closed subset of elements f of form (2.6) with $c_0 = 1$. Then $(G, \cdot)$ is a topological group. Its Lie algebra $\mathcal{L}_G \subset \mathcal{A}$ consists of elements f of form (2.6) with $c_0 = 0$. (For $f, g \in \mathcal{L}_G$ their product is defined by the formula $[f, g] := f \cdot g - g \cdot f$.) Also, the map $\exp\colon \mathcal{L}_G \to G$, $\exp(f) := e^f = \sum_{n=0}^{\infty} \frac{f^n}{n!}$, is a homeomorphism.

2.3.2. For an element $a = (a_1, a_2, \ldots) \in X$ let us consider the equation

$$F'(x) = \left(\sum_{i=1}^{\infty} a_i(x)\, t^i X_i\right) F(x), \quad x \in I_T. \tag{2.8}$$

This can be solved by Picard iteration to obtain a solution $F_a : I_T \to G$, $F_a(0) = I$, whose coefficients in expansion in $X_1, X_2, \ldots$ and t are Lipschitz functions on I_T. We set

$$E(a) := F_a(T), \quad a \in X. \tag{2.9}$$

By the definition, see Section 2.1.1, we have, cf. [11, Theorem 6.1],

$$E(a * b) = E(a) \cdot E(b), \quad a, b \in X. \tag{2.10}$$

Also, an explicit calculation leads to the formula

$$E(a) = I + \sum_{n=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=n} I_{i_1,\ldots,i_k}(a) X_{i_1} \cdots X_{i_k}\right) t^n. \tag{2.11}$$

The last formula implies that the kernel of the homomorphism $E : X \to G$ is the set of universal centers $\mathcal{U}$. In particular, there is a homomorphism $\widehat{E} : G(X) \to G$ such that $E = \widehat{E} \circ \pi$, that is,

$$\widehat{E}(g) = I + \sum_{n=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=n} \widehat{I}_{i_1,\ldots,i_k}(g) X_{i_1} \cdots X_{i_k}\right) t^n, \quad g \in G(X). \tag{2.12}$$

Formula (2.12) shows that $\widehat{E} : G(X) \to G$ is a continuous embedding. Moreover, one can determine a metric d_1 on $\mathcal{A}$ compatible with topology such that $\widehat{E} : (G(X), d) \to (G, d_1)$ is an isometric embedding, cf. (2.4). Therefore $\widehat{E}$ is naturally extended to a continuous embedding $G_f(X) \to G$ (denoted also by $\widehat{E}$). By the definition, $\widehat{E} : G_f(X) \to G$ is an injective homomorphism of topological groups and $\widehat{E}(G_f(X))$ is the closure of $\widehat{E}(G(X))$ in the topology of G.

In what follows we identify $G(X)$ and $G_f(X)$ with their images under $\widehat{E}$.

2.4. Lie algebra of the group of formal paths

2.4.1. Recall that each element $g \in \mathcal{L}_G$ can be written as $g = \sum_{n=1}^{\infty} g_n t^n$, $g_n \in V_n$, $n \in \mathbb{N}$. We say that such g is a *Lie element* if each g_n belongs to the free Lie algebra generated by $X_1, \ldots, X_n$. In this case each g_n has the form

$$g_n = \sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k}[X_{i_1}, [X_{i_2}, [\cdots, [X_{i_{k-1}}, X_{i_k}]\cdots]]]. \tag{2.13}$$

with all $c_{i_1,\dots,i_k} \in \mathbb{C}$. (Here the term with $i_k = n$ is $c_n X_n$.)

Let $L_n \subset V_n$ be the subspace of elements g_n of form (2.13). It follows from [17, Theorem 3.2] that

$$\dim_{\mathbb{C}} L_n = \frac{1}{n} \sum_{d|n} (2^{n/d} - 1) \cdot \mu(d) \tag{2.14}$$

where the sum is taken over all numbers $d \in \mathbb{N}$ that divide n, and $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function defined as follows. If d has a prime factorization

$$d = p_1^{n_1} p_2^{n_2} \cdots p_q^{n_q}, \quad n_i > 0,$$

then

$$\mu(d) = \begin{cases} 1 & \text{for } d = 1, \\ (-1)^q & \text{if all } n_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By $\mathcal{L}_{\mathrm{Lie}}$ we denote the subset of Lie elements of $\mathcal{L}_G$. Then $\mathcal{L}_{\mathrm{Lie}}$ is a closed (in the topology of $\mathcal{A}$) Lie subalgebra of $\mathcal{L}_G$.

Theorem 2.10. *The exponential map* $\exp : \mathcal{L}_G \to G$ *maps* $\mathcal{L}_{\mathrm{Lie}}$ *homeomorphically onto* $G_f(X)$.

Thus $\mathcal{L}_{\mathrm{Lie}}$ can be considered as the Lie algebra of $G_f(X)$.

2.4.2. Proof of Theorem 2.10

Let $\log : G \to \mathcal{L}_G$, $\log(f) = -\sum_{n=1}^{\infty} \frac{(I-f)^n}{n}$, be the logarithmic map. By the definition it is continuous and inverse to the exponential map exp. From [12, Theorem 4.2] and [18] follow that log maps $G(X)$ into $\mathcal{L}_{\mathrm{Lie}}$. Since $\mathcal{L}_{\mathrm{Lie}}$ is a closed subspace of $\mathcal{L}_G$ and log is continuous, it maps $G_f(X)$ into $\mathcal{L}_{\mathrm{Lie}}$, as well. In particular, $G_f(X) \subset \exp(\mathcal{L}_{\mathrm{Lie}})$. Let us prove the converse implication.

Let $J \subset \mathcal{A}$ be the two-sided ideal of elements f of form (2.6) with $c_0 = 0$. By J^l we denote the lth power of J. Let $q_l : \mathcal{A} \to \mathcal{A}/J^l =: A_l$ be the quotient homomorphism. We set $\overline{X}_s = q_l(X_s \cdot t^s)$, $1 \leqslant s \leqslant l-1$. Then for $f \in \mathcal{A}$ of form (2.6) we have

$$q_l(f) := c_0 I + \sum_{n=1}^{l-1} \left(\sum_{i_1+\dots+i_k=n} c_{i_1,\dots,i_k} \overline{X}_{i_1} \cdots \overline{X}_{i_k} \right). \tag{2.15}$$

(Here I is the unit of A_l.) We naturally identify A_l with the hermitian space $\mathbb{C}^{n(k)}$, $n(k) = \dim_{\mathbb{C}} A_l$, so that q_l is a continuous map. Then $G_l := q_l(G) \subset A_l$ is a complex nilpotent Lie group.

Further, let $X_{\mathrm{rect}} \subset X$ be the sub-semigroup of rectangular paths, i.e., elements $a \in X$ whose first integrals $\tilde{a} : I_T \to \mathbb{C}^\infty$ are paths consisting of segments each going in the direction of some particular coordinate. By $G(X_{\mathrm{rect}}) \subset G(X)$ we denote the subgroup generated by X_{rect}. Identifying $G(X)$ with its image in G by $\widehat{E}$ we obtain from (2.11) that $G(X_{\mathrm{rect}})$ is a subgroup of G generated by elements $e^{c_n X_n t^n}$, $c_n \in \mathbb{C}$, $n \in \mathbb{N}$. (In particular, $G(X_{\mathrm{rect}})$ is isomorphic to the free product of countably many copies of $\mathbb{C}$.)

Proposition 2.11. *The images of* $G(X_{\mathrm{rect}})$, $G(X)$ *and* $G_f(X)$ *in* G_l *coincide and form a complex Lie subgroup of* G_l.

Proof. We set for brevity

$$\overline{Q}_l := q_l\big(G(X_{\mathrm{rect}})\big), \qquad Q_l := q_l\big(G(X)\big), \qquad \widetilde{Q}_l := q_l\big(G_f(X)\big). \tag{2.16}$$

For any $c \in \mathbb{C}$, $a = (a_1, a_2, \ldots) \in X$ by ca we denote the element $(ca_1, c^2a_2, \ldots) \in X$. Suppose that $S \subset X$ is a sub-semigroup such that for each $s \in S$ and any $c \in \mathbb{C}$ elements s^{-1}, cs belong to S. Let $S_l := (q_l \circ \widehat{E} \circ \pi)(S)$ be the image of S in G_l. By the definition S_l is a subgroup of G_l. We will use the following

Lemma 2.12. *S_l is a complex Lie subgroup of G_l.*

Proof. Let $g \in G(X) \subset \mathcal{A}$ be the image of an element $a \in X$. For any $c \in \mathbb{C}$ by $cg \in G(X)$ we denote the image of $ca \in X$. If $g = \sum_{n=0}^{\infty} g_n t^n$, $g_n \in V_n$, see (2.7), then

$$q_l(g) = I + \sum_{n=1}^{l-1} q_n\big(g_n t^n\big) \quad \text{and} \quad q_l(cg) := I + \sum_{n=1}^{l-1} c^n \cdot q_n\big(g_n t^n\big). \tag{2.17}$$

We will naturally identify G_l with $\mathbb{C}^N$ where $N := \dim_{\mathbb{C}} G_l$.

Let $K = \{g_1, \ldots, g_k\} \subset \pi(S) \subset G(X)$ be a finite set. We define a holomorphic polynomial map $F_K : \mathbb{C}^k \to \mathbb{C}^N$ by the formula

$$F_K(z_1, \ldots, z_k) := q_l\big((z_1 g_1) \cdots (z_k g_k)\big). \tag{2.18}$$

Let Z_K be the Zariski closure of $F_K(\mathbb{C}^k)$ in $\mathbb{C}^N$. Then Z_K is an irreducible complex algebraic subvariety of $\mathbb{C}^N$ and $F_K(\mathbb{C}^k)$ contains an open dense subset of Z_K (see, e.g., the book of Mumford [16] for the basic facts of Algebraic Geometry). By the definition we have

$$F_{K_1}\big(\mathbb{C}^{\#K_1}\big) \subset F_{K_2}\big(\mathbb{C}^{\#K_2}\big) \quad \text{and} \quad Z_{K_1} \subset Z_{K_2} \quad \text{for } K_1 \subset K_2. \tag{2.19}$$

Let $Z := \bigcup_K Z_K$ where K runs over all finite subsets of G_l. Since $\dim_{\mathbb{C}} Z_K \leqslant N$ and all Z_K are irreducible, (2.19) implies that there is a finite set $K \subset \pi(S)$ such that $Z_K = Z$. Observe that by the definition the group S_l is dense in $Z = Z_K$. Hence Z is the Zariski closure of S_l. In particular, Z is a complex Lie subgroup of G_l. Also, from the identity $Z = Z_K$ it follows that S_l contains an open dense subset of Z. Since the topologies of the groups S_l and Z are induced from that of G_l, the latter implies that $S_l = Z$ completing the proof of the lemma. □

Continuing the proof of the proposition we choose as the S in Lemma 2.12 semigroups X_{rect} and X. Then we conclude that $\overline{Q}_l \subset Q_l$ are complex Lie subgroups of G_l (in particular, they are closed subsets of G_l). Since Q_l is dense in $\widetilde{Q}_l$, see Section 2.1.2, the latter implies that $Q_l = \widetilde{Q}_l$.

Let $\mathcal{L}_{G_l}$ be the Lie algebra of G_l. By the definition it consists of all elements of form (2.15) with $c_0 = 0$. Let $\exp : \mathcal{L}_{G_l} \to G_l$, $\exp(f) := e^f$, be the corresponding exponential map. Clearly we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L}_G & \xrightarrow{\exp} & G \\ \downarrow{\scriptstyle q_l} & & \downarrow{\scriptstyle q_l} \\ \mathcal{L}_{G_l} & \xrightarrow{\exp} & G_l. \end{array} \tag{2.20}$$

This implies that for each $f \in Q_l$ the element $\log(f) := -\sum_{n=1}^{l-1} \frac{(I-f)^n}{n}$ belongs to the free nilpotent Lie algebra $\mathcal{L}_{\mathrm{Lie}}^l \subset \mathcal{L}_{G_l}$ generated by elements $\overline{X}_1, \ldots, \overline{X}_{l-1}$, i.e., each element g of this algebra has the form

$$g = \sum_{n=1}^{l-1} \left(\sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} [\overline{X}_{i_1}, [\overline{X}_{i_2}, [\cdots, [\overline{X}_{i_{k-1}}, \overline{X}_{i_k}] \cdots]]] \right) \tag{2.21}$$

with all $c_{i_1,\ldots,i_k} \in \mathbb{C}$, $i_1, \ldots, i_k \in \{1, \ldots, l-1\}$. Thus the Lie algebra $\mathcal{L}_{Q_l} := \log(Q_l)$ of Q_l is a subset of $\mathcal{L}_{\mathrm{Lie}}^l$. Next, the Lie algebra $\mathcal{L}_{\overline{Q}_l} := \log(\overline{Q}_l)$ of $\overline{Q}_l$ is a subset of $\mathcal{L}_{\mathrm{Lie}}^l$, as well. Moreover, since $\overline{Q}_l$ is generated by elements $e^{c_s \overline{X}_s}$, $c_s \in \mathbb{C}$, $1 \leqslant s \leqslant l-1$, its Lie algebra is generated by elements $\overline{X}_s$, $1 \leqslant s \leqslant l-1$, and therefore coincides with $\mathcal{L}_{\mathrm{Lie}}^l$. This implies that $\mathcal{L}_{\overline{Q}_l} = \mathcal{L}_{Q_l} = \mathcal{L}_{\mathrm{Lie}}^l$, and $Q_l = \overline{Q}_l$.

The proof of the proposition is complete. □

Let us finish the proof of the theorem. Let $g \in \mathcal{L}_{\mathrm{Lie}}$. Consider the elements $g_l := q_l(g) \in \mathcal{L}_{\mathrm{Lie}}^l$, $l \geqslant 2$. According to Proposition 2.11 there are elements $f_l \in G(X_{\mathrm{rect}})$ such that $q_l(f_l) = e^{g_l}$. Now, for $m > l$ we have

$$q_l(f_m \cdot f_l^{-1}) = q_l(f_m) \cdot \big(q_l(f_l)\big)^{-1} = q_l\big(q_m(f_m)\big) \cdot e^{-g_l} = q_l(e^{g_m}) \cdot e^{-g_l} = e^{g_l} \cdot e^{-g_l} = I.$$

Thus $f_m - f_l \in J^l$. By the definition of the topology of G this implies that the sequence $\{f_l\}_{l \geqslant 2}$ converges in G to an element $f \in G_f(X)$, so that $q_l(f) = e^{g_l}$, $l \geqslant 2$. Taking here the limit as l tends to ∞ we get $f = e^g$. This shows that $\exp(\mathcal{L}_{\mathrm{Lie}}) \subset G_f(X)$ and completes the proof of the theorem. □

Remark 2.13. We also established in the proof that $G(X_{\mathrm{rect}})$ is a dense subgroup of $G_f(X)$.

2.4.3. Shuffles

Definition 2.14. A permutation σ of $\{1, 2, \ldots, r+s\}$ is a shuffle of type (r, s) if

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$$

and

$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s).$$

(The term "shuffle" is used because such permutations arise in riffle shuffling a deck of $r+s$ cards cut into one pile of r cards and a second pile of s cards.)

The following result is a corollary of Theorem 2.10.

Theorem 2.15. *An element*

$$f = I + \sum_{n=1}^{\infty} \left(\sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k} \right) t^n \in G$$

belongs to $G_f(X)$ if and only if its coefficients satisfy the system of Ree shuffle relations:

$$c_{i_1,\ldots,i_r} \cdot c_{i_{r+1},\ldots,i_{r+s}} = \sum_{\sigma} c_{i_{\sigma(1)},\ldots,i_{\sigma(r+s)}}, \quad i_1, \ldots, i_{r+s} \in \mathbb{N}, \tag{2.22}$$

where the sum is taken over the set of shuffles of type (r, s).

Proof. According to the main result of Ree [18], $\log(f) \in \mathcal{L}_{\mathrm{Lie}}$ for $f \in G$ if and only if the coefficients of f satisfy Eqs. (2.22). This and Theorem 2.10 imply the required result. □

2.5. Topological lower central series of some groups of paths

2.5.1. In the section we describe the topological lower central series of groups $G(X)$ and $G_f(X)$.

Let G be a topological group. We set $G_n := [G, G_{n-1}]$ and $G_1 = [G, G]$ (the commutator subgroup of G). For $H \subset G$ by $\overline{H} \subset G$ we denote the closure of H.

Definition 2.16. The sequence

$$G \supset \overline{G}_1 \supset \overline{G}_2 \supset \cdots$$

is called the topological lower central series of G.

Next, consider the family $\{\widehat{I}_{i_1,\dots,i_k}\}$ of all basic integrals on $G(X)$, see Section 2.1.2. By the definition each function of this family admits a continuous extension to $G_f(X)$. We retain the same symbols for the extended functions and call them the *basic iterated integrals* on $G_f(X)$. Observe that if $c_{i_1,\dots,i_k} : \mathcal{A} \to \mathbb{C}$ is the function whose value at $f \in \mathcal{A}$ is the coefficient corresponding to the monomial $X_{i_1} \cdots X_{i_k}$ in the series expansion (2.6) of f, then $\widehat{I}_{i_1,\dots,i_k} = c_{i_1,\dots,i_k} \circ \widehat{E}$.

Theorem 2.17. (1) *An element $g \in G_f(X)$ belongs to $\overline{G_f(X)_n}$ if and only if all basic iterated integrals of order $\leqslant n$ vanish at g.*

(2)

$$\overline{G(X)_n} = \overline{G_f(X)_n} \cap G(X).$$

Let us recall that the order of a basic iterated integral is the number of its indices.

Proof. We use the notation of Section 2.4.2. Since the map $q_l : G(X) \to Q_l$, see (2.16), is surjective, $q_l(G(X)_n) = (Q_l)_n$. Moreover, since Q_l is a complex nilpotent Lie group, $(Q_l)_n$ is a complex nilpotent Lie subgroup of Q_l. In particular, $q_l(\overline{G(X)_n}) = (Q_l)_n$ (because $q_l(G(X)_n)$ is dense in $q_l(\overline{G(X)_n})$). Similarly, $q_l(\overline{G_f(X)_n}) = q_l(G_f(X)_n) = (Q_l)_n$. Further, the Lie algebra $\mathcal{L}^l_{\mathrm{Lie}}$ of Q_l is the free nilpotent Lie subalgebra of $\mathcal{L}_{G_l}$ generated by elements $\overline{X}_1, \dots, \overline{X}_{l-1}$, see (2.21). Then the nth term $(\mathcal{L}^l_{\mathrm{Lie}})_n$ of the lower central series of $\mathcal{L}^l_{\mathrm{Lie}}$ consists of the elements of form (2.21) with all $c_{i_1,\dots,i_k} = 0$ for $k \leqslant n$ (i.e., the number of brackets in each term of this formula must be $\geqslant n$). Since the exponential map exp maps $(\mathcal{L}^l_{\mathrm{Lie}})_n$ surjectively onto $(Q_l)_n$, an explicit computation shows that $(Q_l)_n \subset Q_l$ consists of elements of form (2.15) with $c_0 = 1$ and all $c_{i_1,\dots,i_k} = 0$ for $k \leqslant n$.

Now, suppose that $g \in \overline{G_f(X)_n}$. Identifying $G_f(X)$ with its image under $\widehat{E}$ we have

$$g = I + \sum_{n=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k}(g)\, X_{i_1} \cdots X_{i_k}\right)t^n. \tag{2.23}$$

Since $q_l(g) \in (Q_l)_n$ for any l, formula (2.15) and the above description of $(Q_l)_n$ imply that $c_{i_1,\dots,i_k}(g) = 0$ for all $k \leqslant n$. Equivalently, all basic iterated integrals of order $\leqslant n$ vanish at g.

Conversely, assume that $g \in G_f(X)$ is of form (2.23) with $c_{i_1,\dots,i_k}(g) = 0$ for all $k \leqslant n$. Then from the description of $(Q_l)_n$ and (2.15) we obtain that $q_l(g) \in (Q_l)_n$ for any $l \geqslant 2$.

Since$(Q_l)_n = q_l(G(X)_n)$, there are elements $g_l \in G(X)_n$, $l \geqslant 2$, such that $q_l(g^{-1}g_l) = I$. As in the proof of Theorem 2.10 this implies that $\{g_l\}_{l\geqslant 2}$ converges to g in the topology of $G_f(X)$, that is, $g \in \overline{G_f(X)_n}$. This proves (1).

Further, if $g \in \overline{G_f(X)_n} \cap G(X)$, then as above there is a sequence $\{g_l\}_{l\geqslant 2}$ with $g_l \in G(X)_n$ such that $\lim_{l\to\infty} g_l = g$. Thus $g \in \overline{G(X)_n}$. Since the implication $\overline{G(X)_n} \subset \overline{G_f(X)_n} \cap G(X)$ is obvious, we obtain the proof of (2). □

2.5.2. By $X_* \subset X$ we denote the set of elements $a \in X$ such that $I_s(a) = 0$ for all $s \in \mathbb{N}$, i.e., $a \in X_*$ if and only if its first integral $\widetilde{a} : I_T \to \mathbb{C}^\infty$ is a closed path. We call the image $G(X_*) := \pi(X_*) \subset G(X)$ the *subgroup of closed paths* and its closure $G_f(X_*)$ in $G_f(X)$ the *subgroup of formal closed paths*. According to Theorem 2.17, $G(X_*) = \overline{G(X)_1}$ and $G_f(X_*) = \overline{G_f(X)_1}$. In this section we describe the topological lower central series of $G(X_*)$ and $G_f(X_*)$.

Given $a = (a_1, a_2, \ldots) \in X$ we define

$$\widetilde{a}_i(x) := \int_0^x a_i(s)\,ds, \quad x \in I_T. \tag{2.24}$$

By $\mathcal{P}(X)$ we denote the set of functions on $X \times I_T$ of the form

$$\left(\widetilde{a}_{i_1}(x)\right)^{n_1} \cdots \left(\widetilde{a}_{i_k}(x)\right)^{n_k} \cdot a_{i_{k+1}}(x), \quad i_1, \ldots, i_{k+1} \in \mathbb{N},\ n_1, \ldots, n_k \in \mathbb{Z}_+. \tag{2.25}$$

Definition 2.18. A moment of order k on X is an iterated integral of the form

$$m(a) := \int \cdots \int_{0\leqslant s_1\leqslant\cdots\leqslant s_k\leqslant T} p_k(a; s_k) \cdots p_1(a; s_1)\,ds_k \cdots ds_1 \tag{2.26}$$

where each $p_j \in \mathcal{P}(X)$, $1 \leqslant j \leqslant k$.

Moments of the first order play an important role in the study of the center problem for Abel differential equations (see, e.g., [2,4,5,20]). Also, it was proved in [9, Theorem 2.1] that such moments determine centers of Eqs. (1.1) whose coefficients are either polynomials in $e^{\pm 2\pi i x/T}$ or in x, a result on complexity of the set of centers for these equations. (E.g., this class contains equations obtained from the Poincaré Center–Focus problem, see the Introduction.)

According to the Ree shuffle formula (2.22) each moment m is a linear combination with natural coefficients of some basic iterated integrals. In particular, there is a continuous function $\widehat{m}$ on $G(X)$ from the vector space generated by all basic iterated integrals on $G(X)$ such that $m = \widehat{m} \circ \pi$. Thus, every such $\widehat{m}$ admits a continuous extension (denoted by the same symbol) to $G_f(X)$. The extended function will be called a *moment* on $G_f(X)$. By the definition the *order* of $\widehat{m}$ is the order of the moment m on X representing $\widehat{m}$.

Theorem 2.19. (1) *An element $g \in G_f(X_*)$ belongs to $\overline{G_f(X_*)_n}$ if and only if all moments of order $\leqslant n$ vanish at g.*

(2) $\overline{G(X_*)_n} = \overline{G_f(X_*)_n} \cap G(X_*)$.

Proof. We first prove the particular case of (1) for $g \in G(X_*)$. Namely we will prove

Proposition 2.20. *An element $g \in G(X_*)$ belongs to $\overline{G(X_*)_n}$ if and only if all moments of order $\leqslant n$ vanish at g.*

Proof. This result was stated in [10, Theorem 3.2]. In its proof given in [10] some details were omitted. Here we will give the complete proof of this fact.

First, assume that $g \in G(X_*)_n$. Since g represents a closed path in $\mathbb{C}^\infty$ all moments of order $\leqslant n$ vanish at g (see, e.g., [15] for properties of iterated integrals over closed paths). Since each moment is a continuous function on $G(X_*)$, by continuity we obtain also that for $g \in \overline{G(X_*)_n}$ all moments of order $\leqslant n$ vanish at g. Thus we must prove a converse statement.

So assume that $g \in G(X_*)$ is such that all moments of order $\leqslant n$ vanish at g. We will prove that $g \in \overline{G(X_*)_n}$.

For an element $p = \widetilde{a}_{i_1}^{n_1} \cdots \widetilde{a}_{i_k}^{n_k} \cdot a_{i_{k+1}} \in \mathcal{P}(X)$ the number $i_1 n_1 + \cdots + i_k n_k + i_{k+1}$ will be called the degree of p. Next, for a moment m on X its degree $\deg(m)$ is the maximum of degrees of elements $p_j \in \mathcal{P}(X)$ in its definition, see (2.26). In turn, the degree of the moment $\widehat{m}$ on $G(X)$ representing m is defined as $\deg(\widehat{m}) := \deg(m)$.

We retain the notation of Section 2.4.2. Also, for $Q_l := q_l(G(X)) \subset G_l$ we set $R_l := [Q_l, Q_l]$. Our proof is based on the following

Lemma 2.21. *Suppose that $g \in G(X_*)$ is such that all moments of order $\leqslant n$ and of degree $\leqslant l-1$ vanish at g. Then $q_l(g) \in (R_l)_n$.*

Proof. Let $a = (a_1, a_2, \ldots) \in X$ be such that $\pi(a) = g$. By the definitions of E, see Section 2.3.2, and of q_l, see (2.15), $q_l(g)$ is the monodromy of the equation

$$H'(x) = \left(\sum_{i=1}^{l-1} a_i(x)\,\overline{X}_i\right) H(x), \quad x \in I_T. \tag{2.27}$$

This equation can be solved by Picard iteration to obtain a solution $H_a : I_T \to G_l$, $H_a(0) = I$, whose coefficients in expansion in $\overline{X}_1, \ldots, \overline{X}_{l-1}$ are Lipschitz functions on I_T. Then $q_l(g) := H_a(T)$. We write

$$H_a = H_1 \cdots H_{l-1} \cdot H_l \quad \text{where } H_i := e^{\widetilde{a}_i \overline{X}_i},\ 1 \leqslant i \leqslant l-1, \tag{2.28}$$

see (2.24). Since $g \in G(X_*)$, $H_i(T) = I$ for $1 \leqslant i \leqslant l-1$. This implies that $q_l(g) = H_a(T) = H_l(T)$. From (2.28) follows that H_l satisfies the equation

$$\begin{aligned} &H_l' = \omega \cdot H_l \quad \text{where} \\ &\omega := F^{-1} \cdot \left(\sum_{i=1}^{l-1} a_i\,\overline{X}_i\right) \cdot F - F^{-1} \cdot F', \quad F := H_1 \cdots H_{l-1}. \end{aligned} \tag{2.29}$$

Claim 2.22. *ω is a function on I_T with values in the Lie algebra $\mathcal{L}_{R_l}$ of R_l.*

Indeed, the first term in the definition of ω is the logarithm of

$$F^{-1}(x) \cdot \exp\left(\sum_{i=1}^{l-1} a_i(x)\overline{X}_i\right) \cdot F(x), \quad x \in I_T.$$

By the definition of F for any $x \in I_T$ each term of this product belongs to Q_l (see Section 2.4.2 after formula (2.21)). Thus its logarithm belongs to the Lie algebra $\mathcal{L}_{Q_l}$ of Q_l. Next, the second term in the definition of ω is equal to

$$\left(\sum_{s=2}^{l-1} H_{l-1}^{-1}(x)\cdots H_s^{-1}(x)\cdot \widetilde{a}_{s-1}(x)\,\overline{X}_{s-1}\cdot H_{l-1}(x)\cdots H_s(x)\right)+\widetilde{a}_{l-1}(x)\overline{X}_{l-1}, \quad x \in I_T.$$

By the same reason as above, for any $x \in I_T$ each term of this sum belongs to $\mathcal{L}_{Q_l}$. Thus $\omega(x) \in \mathcal{L}_{Q_l}$ for any $x \in I_T$. Observe also that from (2.29) follows that $\omega(x)$ considered as a polynomial in $\overline{X}_i$, $1 \leqslant i \leqslant l-1$, does not contain linear terms. Then by the definition of R_l, see also the proof of Theorem 2.17, $\omega(x) \in \mathcal{L}_{R_l}$.

This completes the proof of the Claim.

An explicit computation of ω based on the Campbell–Hausdorff formula shows that

$$\omega(x)=\sum_{n=1}^{l-1}\left(\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k}(x)\,[\overline{X}_{i_1},[\overline{X}_{i_2},[\cdots,[\overline{X}_{i_{k-1}},\overline{X}_{i_k}]\cdots]]]\right) \tag{2.30}$$

where each $c_{i_1,\dots,i_k} \in \mathrm{span}_{\mathbb{Q}}(\mathcal{P}(X))$. Moreover, each term of $\mathcal{P}(X)$ in the definition of $c_{i_1,\dots,i_k}$ is of degree $i_1+\cdots+i_k$.

We set

$$S := \bigl\{[\overline{X}_{i_1},[\overline{X}_{i_2},[\cdots,[\overline{X}_{i_{k-1}},\overline{X}_{i_k}]\cdots]]] \neq 0\colon\ i_1+\cdots+i_k=n,\ 1 \leqslant n \leqslant l-1\bigr\},$$

and arrange S into a sequence $\{v_i\}_{1\leqslant i\leqslant N}$, $N := \#S$. Then ω can be written as

$$\omega(x)=\sum_{i=1}^{N} f_i(x)\,v_i, \quad x \in I_T,$$

where $f_i \in \mathrm{span}_{\mathbb{Q}}(\mathcal{P}(X))$ and each term of $\mathcal{P}(X)$ in the definition of f_i is of degree $\leqslant l-1$.

Let $\mathbb{C}[[X_1,\dots,X_N]]$ be the associative algebra with unit I of non-commutative formal power series in free variables $X_1,\dots,X_N$. Then there is a homomorphism $\phi:\mathbb{C}[[X_1,\dots,X_N]] \to A$, where A is the associative complex subalgebra of $\mathcal{A}_l$ (see Section 2.4.2) generated by I and $v_1,\dots,v_n$, determined by $\phi(X_i)=v_i$, $1 \leqslant i \leqslant N$, $\phi(I)=I$.

Next, consider the equation

$$G'(x)=\left(\sum_{i=1}^{N} f_i(x)X_i\right)G(x), \quad x \in I_T.$$

Solving this equation by Picard iteration we get a solution $G: I_T \to \mathbb{C}[[X_1,\dots,X_N]]$, $G(0)=I$, whose coefficients in expansion in $X_1,\dots,X_N$ are Lipschitz functions on I_T. Also, by the definition we have $\phi(G(T))=H_l(T)=q_l(g)$. Observe now that

$$G(T)=I+\sum_{k=1}^{\infty}\left(\sum_{1\leqslant i_1,\dots,i_k\leqslant N} I_{i_1,\dots,i_k}(f)\,X_{i_1}\cdots X_{i_k}\right),$$

cf. (2.11), where $f=(f_1,\dots,f_N,0,\dots) \in X$. By the definition, each basic iterated integral $I_{i_1,\dots,i_k}(f)$ in this formula is a linear combination of moments of order k and of degree $\leqslant l-1$ on X. In particular, by the hypothesis of the lemma, $I_{i_1,\dots,i_k}(f)=0$ for all $k \leqslant n$, i.e., the first sum in the definition of $G(T)$ can be considered for $k \geqslant n+1$ only. Further, by the Ree theorem [18]

$\log(G(T))$ is a Lie element in $\mathbb{C}[[X_1,\dots,X_N]]$. Since the degree of each monomial in $G(T)$ is greater than n,

$$\log\bigl(G(T)\bigr)=\sum_{k=n+1}^{\infty}\biggl(\sum_{1\leqslant i_1,\dots,i_k\leqslant N} g_{i_1,\dots,i_k}[X_{i_1},[X_{i_2},[\,\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]]\biggr).$$

This implies that

$$\begin{aligned}\phi\bigl(\log\bigl(G(T)\bigr)\bigr)&=\log\bigl(\phi\bigl(G(T)\bigr)\bigr)=\log\bigl(q_l(g)\bigr)\\&=\sum_{k=n+1}^{\infty}\biggl(\sum_{1\leqslant i_1,\dots,i_k\leqslant N} g_{i_1,\dots,i_k}[v_{i_1},[v_{i_2},[\,\cdots,[v_{i_{k-1}},v_{i_k}]\cdots]]]\biggr),\end{aligned}$$

that is, $\log(q_l(g))\in(\mathcal{L}_{R_l})_n$. Since the exponential map exp maps $(\mathcal{L}_{R_l})_n$ surjectively onto $(R_l)_n$, $q_l(g)\in(R_l)_n$.

The proof of the lemma is complete. □

Let us finish the proof of the proposition.

For an element $g\in G(X_*)$ such that all moments of order $\leqslant n$ vanish at g by Lemma 2.21 we have $q_l(g)\in(R_l)_n$ for all $l\geqslant 2$. Since q_l maps $G(X_*)$ surjectively onto R_l (see the proof of Theorem 2.10), there are elements $g_l\in G(X_*)_n$, $l\geqslant 2$, such that $q_l(g^{-1}\cdot g_l)=I$ for all l. As in the proof of Theorem 2.17 this implies that $\lim_{l\to\infty}g_l=g$. Thus $g\in\overline{G(X_*)_n}$. This completes the proof of the proposition. □

Using Proposition 2.20 we prove now Theorem 2.19.

Assume that $g\in\overline{G_f(X_*)_n}$. Since $G_f(X_*)$ is the closure in $G_f(X)$ of $G(X_*)$, $\overline{G_f(X_*)_n}$ is the closure in $G_f(X)$ of $G(X_*)_n$. In particular, all moments of order $\leqslant n$ vanish at g (see the beginning of the proof of Proposition 2.20).

Conversely, suppose that $g\in G_f(X_*)$ is such that all moments of order $\leqslant n$ vanish at g. We will prove first that $q_l(g)\in(R_l)_n$, for all $l\geqslant 2$.

By the Ree shuffle formula, given $l\geqslant 2$ each moment $\widehat{m}$ of order $\leqslant n$ and of degree $\leqslant l-1$ on $G_f(X_*)$ can be presented as a linear combination with natural coefficients of basic iterated integrals $\widehat{I}_{i_1,\dots,i_k}$ (on $G_f(X)$) of order $k\leqslant l+n-1$ with $1\leqslant i_1,\dots,i_k\leqslant l-1$. We set $s:=(l+n-1)\cdot(l-1)+1$ and consider $Q_s:=q_s(G(X))=q_s(G_f(X))$ (cf. the proof of Proposition 2.11). By the definitions of $\widehat{E}$ and q_s, see (2.12) and (2.15), we have

$$q_s(g)=I+\sum_{m=1}^{s-1}\biggl(\sum_{i_1+\cdots+i_k=m}\widehat{I}_{i_1,\dots,i_k}(g)\,\overline{X}_{i_1}\cdots\overline{X}_{i_k}\biggr).$$

Since $R_s:=[Q_s,Q_s]=q_s(G(X_*))=q_s(G_f(X_*))$ and $g\in G_f(X_*)$, there is $\widetilde{g}\in G(X_*)$ such that $q_s(g)=q_s(\widetilde{g})$. In particular,

$$\widehat{I}_{i_1,\dots,i_k}(g)=\widehat{I}_{i_1,\dots,i_k}(\widetilde{g})\quad\text{for all } i_1+\cdots+i_k=m,\ 1\leqslant m\leqslant s-1.$$

From this by the above description of moments of order $\leqslant n$ and of degree $\leqslant l-1$ we obtain that for each such a moment $\widehat{m}$ on $G_f(X_*)$, $\widehat{m}(g)=\widehat{m}(\widetilde{g})$. Since $\widehat{m}(g)=0$ by our hypothesis, $\widehat{m}(\widetilde{g})=0$, as well. Hence $\widetilde{g}$ satisfies the conditions of Lemma 2.21. According to this lemma, $q_l(\widetilde{g})\in(R_l)_n$. But by our construction $q_l(g)=q_l(\widetilde{g})$. That is, $q_l(g)\in(R_l)_n$. Using this and arguing as in the proof of Proposition 2.20 we find a sequence $\{g_l\}_{l\geqslant 2}\subset G(X_*)_n$ such that $\lim_{l\to\infty}g_l=g$.

In particular, $g \in \overline{G_f(X_*)_n}$. This completes the proof of part (1) of Theorem 2.19. The second statement of this theorem follows from the first one and from Proposition 2.20. $\square$

In conclusion let us mention that in [10, Section 3.3] a topological characterization of paths representing elements of $\overline{G(X_*)_n}$ is given, similar to that for elements of the set of universal centers $\mathcal{U}$, cf. Section 2.2.

2.6. Subgroups of the group of formal paths

2.6.1. By $X^k \subset X$ we denote the subset of elements $a = (a_1, \ldots, a_k, 0, 0, \ldots) \in X$. Then X^k is a sub-semigroup of X. The first integrals of elements of X^k are paths in $\mathbb{C}^k$. We set $G(X^k) := \pi(X^k) \subset G(X)$ and let $G_f(X^k)$ be the closure of $G(X^k)$ in $G_f(X)$. The group $G_f(X^k)$ will be called the *group of formal paths* in $\mathbb{C}^k$. Let $p_k : X \to X^k$, $p_k(a_1, a_2, \ldots) := (a_1, \ldots, a_k, 0, 0, \ldots)$, be the natural projection. It induces a surjective homomorphism of topological groups $\widehat{p}_k : G_f(X) \to G_f(X^k)$. In particular, $G_f(X)$ is the semidirect product of groups $\operatorname{Ker} \widehat{p}_k$ and $G_f(X^k)$. In turn, the homomorphism $\widehat{p}_k$ determines a continuous Lie algebra homomorphism $\phi_k : \mathcal{L}_{\mathrm{Lie}} \to \mathcal{L}_{\mathrm{Lie}}$ where $\mathcal{L}_{\mathrm{Lie}}$ is the Lie algebra of $G_f(X)$, see Section 2.4.1. It is determined by the conditions

$$\phi_k(X_s) := \begin{cases} X_s & \text{if } 1 \leqslant s \leqslant k, \\ 0 & \text{if } s > k. \end{cases}$$

The image of ϕ_k is a closed Lie subalgebra $\mathcal{L}^k_{\mathrm{Lie}}$ of $\mathcal{L}_{\mathrm{Lie}}$ consisting of Lie elements in variables $X_1, \ldots, X_k$ and t. Identifying $G_f(X)$ with its image under map $\widehat{E}$, see Section 2.3.2, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{\mathrm{Lie}} & \xrightarrow{\exp} & G_f(X) \\ {\scriptstyle \phi_k}\downarrow & & \downarrow{\scriptstyle \widehat{p}_k} \\ \mathcal{L}^k_{\mathrm{Lie}} & \xrightarrow{\exp} & G_f(X^k). \end{array} \tag{2.31}$$

Thus $\mathcal{L}^k_{\mathrm{Lie}}$ can be regarded as the Lie algebra of $G_f(X^k)$.

Also, analogs of Theorems 2.17 and 2.19 are valid for topological lower central series of $G_f(X^k)$ and $G_f(X^k_*)$ (the *group of formal closed paths in* $\mathbb{C}^k$) where in these results we consider basic iterated integrals and moments on $G_f(X^k)$, respectively.

2.6.2. Let $\mathbb{F} \subset \mathbb{C}$ be a field. By $X_{\mathbb{F}} \subset X$ we denote the subset of elements $a \in X$ such that $I(a) \in \mathbb{F}$ for all basic iterated integrals on X. Formulas (2.2) and (2.3) imply that $X_{\mathbb{F}}$ is a sub-semigroup of X. By $G(X_{\mathbb{F}}) := \pi(X_{\mathbb{F}})$ we denote the subgroup of $G_f(X)$ generated by $X_{\mathbb{F}}$. The homomorphism $\widehat{E}$, see (2.12), embeds $G(X_{\mathbb{F}})$ into the subalgebra $\mathcal{A}_{\mathbb{F}}$ of $\mathcal{A}$, see Section 2.3.1, of formal power series whose coefficients in expansion in $I, X_1, X_2, \ldots$ and t belong to $\mathbb{F}$. We will identify $G(X_{\mathbb{F}})$ with its image under $\widehat{E}$.

Next, by $J_{\mathbb{F}} \subset \mathcal{A}_{\mathbb{F}}$ we denote the two-sided ideal of elements f whose series expansions do not contain terms with I. By $J^k_{\mathbb{F}}$ we denote the kth power of $J_{\mathbb{F}}$. Let us introduce the $J_{\mathbb{F}}$-adic topology on $\mathcal{A}_{\mathbb{F}}$, i.e., a sequence $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{A}_{\mathbb{F}}$ converges in this topology to $f \in \mathcal{A}_{\mathbb{F}}$ if and only if for any $l \in \mathbb{N}$ there is a natural number N_l such that for all $n \geqslant N_l$ the images of f_n and f in the quotient algebra $\mathcal{A}_{\mathbb{F}}/J^l_{\mathbb{F}}$ coincide. Observe that $\mathcal{A}_{\mathbb{F}}$ is complete in this topology. By $G_f(X_{\mathbb{F}}) \subset \mathcal{A}_{\mathbb{F}}$ we denote the completion of $G(X_{\mathbb{F}})$ in the $J_{\mathbb{F}}$-adic topology. We call it the *group of formal paths over* $\mathbb{F}$.

Let $[X_{\mathbb{F}}]_{\text{rect}}$ be a sub-semigroup of the semigroup of rectangular paths X_{rect} generated by elements $a_i = (a_{1i}, a_{2i}, \ldots)$ where $a_{ki} = 0$ for $k \neq i$ and $a_{ii} = c_i/T$, $c_i \in \mathbb{F}$. Then $G([X_{\mathbb{F}}]_{\text{rect}})$ is the subgroup of $G(X_{\text{rect}})$ generated by elements $e^{c_i X_i t^i}$, $c_i \in \mathbb{F}$, $i \in \mathbb{N}$. Based on the results of Sections 2.4.1 and 2.4.2 one obtains that $G([X_{\mathbb{F}}]_{\text{rect}})$ is dense in $G_f(X_{\mathbb{F}})$. Moreover, the Lie algebra $\mathcal{L}_{\text{Lie}(\mathbb{F})} \subset \mathcal{L}_{\text{Lie}}$ of $G_f(X_{\mathbb{F}})$ consists of all Lie elements with coefficients from $\mathbb{F}$.

In the same way one can formulate analogs of Theorems 2.17 and 2.19 for topological lower central series of $G(X_{\mathbb{F}})$ and $G_f(X_{\mathbb{F}})$ in terms of basic iterated integrals and moments on $G_f(X_{\mathbb{F}})$, respectively. (We leave the details to the reader.)

2.6.3. In the sequel we will use the following result.

Let $R \subset \mathcal{L}_{\text{Lie}}$ be a subset. By $A_R \subset \mathcal{L}_{\text{Lie}}$ we denote the minimal closed Lie subalgebra containing R. Consider the subgroup $H_R \subset G_f(X)(\subset G)$ generated by elements e^{cr} for all possible $r \in R$ and $c \in \mathbb{C}$. By $\overline{H}_R$ we denote the closure of H_R in $G_f(X)$.

Proposition 2.23. *We have*

$$\log(\overline{H}_R) = A_R.$$

Moreover, $\overline{H}_R$ is a normal subgroup of $G_f(X)$ if and only if A_R is a normal Lie subalgebra of $\mathcal{L}_{\text{Lie}}$.

Proof. We retain the notations of Section 2.4.2.

Consider the image $(H_R)_l := q_l(H_R) \subset Q_l := q_l(G_f(X))$. Then, $(H_R)_l$ is a complex Lie subgroup of Q_l. It can be shown similarly to the statement of Lemma 2.12 where instead of the map F_K given by (2.18) we determine now a new map $F_K : \mathbb{C}^k \to \mathbb{C}^N$, $N := \dim_{\mathbb{C}} Q_l$, with $K = \{r_1, \ldots, r_k\} \subset R$ by the formula

$$F_K(z_1, \ldots, z_k) := q_l(e^{z_1 r_1} \cdots e^{z_k r_k}).$$

Then, as in the proof of the lemma, F_K is a holomorphic polynomial map. Applying now the arguments of Lemma 2.12 to the family of such maps F_K, we finally get the required: $(H_R)_l$ is a complex Lie subgroup of Q_l. In particular, we also have $(\overline{H}_R)_l := q_l(\overline{H}_R) = (H_R)_l$.

From the above statement we obtain that $\log((H_R)_l) \subset \mathcal{L}^l_{\text{Lie}}$ is the Lie algebra of $(H_R)_l$ (cf. (2.21)). Since $\log((H_R)_l)$ contains $q_l(R)$ and $(H_R)_l$ is generated by elements $e^{cq_l(r)}$ for all possible $r \in R$ and $c \in \mathbb{C}$, the Campbell–Hausdorff formula implies that

$$\log\big((\overline{H}_R)_l\big) = \log\big((H_R)_l\big) = q_l(A_R). \tag{2.32}$$

Assume now that $h \in \log(\overline{H}_R)$. According to (2.32) there is a sequence $\{h_l\}_{l \geqslant 2} \subset A_R$ such that $q_l(h - h_l) = 0$ for all l. This implies that $\lim_{l\to\infty} h_l = h$, that is, $\log(\overline{H}_R) \subset A_R$. The inclusion $A_R \subset \log(\overline{H}_R)$ is obtained similarly using the fact that $\log(\overline{H}_R)$ is a closed subset of $\mathcal{L}_{\text{Lie}}$.

Now, if $\overline{H}_R$ is a normal subgroup of $G_f(X)$, then $(\overline{H}_R)_l$ is a normal Lie subgroup of Q_l. This and (2.32) imply that $q_l(A_R)$ is a normal Lie subalgebra of $\mathcal{L}^l_{\text{Lie}}$ (the standard fact of the theory of finite-dimensional complex Lie groups). Thus, if $a \in \mathcal{L}_{\text{Lie}}$, $h \in A_R$, then $q_l([a,h]) = [q_l(a), q_l(h)] \in q_l(A_R)$ for all l. As above, the latter implies that $\lim_{l\to\infty} g_l = [a,h]$ for some $\{g_l\}_{l\geqslant 2} \subset A_R$, i.e., $[a,h] \in A_R$. Hence, A_R is a normal subalgebra of $\mathcal{L}^l_{\text{Lie}}$. The converse statement can be obtained in the same way (we leave the details to the reader). □

Remark 2.24. The above result can be proved by means of the Campbell–Hausdorff formula only. This method of the proof works also to establish a similar result for $R \subset \mathcal{L}_{\mathrm{Lie}(\mathbb{F})}$ and $H_R \subset G_f(X_{\mathbb{F}})$ generated by elements e^{cr} for all possible $r \in R$ and $c \in \mathbb{F}$.

3. Center problem for ODEs

3.1. An explicit expression for the first return map

3.1.1. Let $\mathbb{C}[[z]]$ be the algebra of formal complex power series in z. By $D, L : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ we denote the differentiation and the left translation operators defined on $f(z) = \sum_{k=0}^{\infty} c_k z^k$ by

$$(Df)(z) := \sum_{k=0}^{\infty}(k+1)c_{k+1}z^k, \qquad (Lf)(z) := \sum_{k=0}^{\infty} c_{k+1}z^k. \tag{3.1}$$

Let $\mathcal{A}(D, L)$ be the associative algebra with unit I of complex polynomials in I, D and L. By $\mathcal{A}(D, L)[[t]]$ we denote the associative algebra of formal power series in t with coefficients from $\mathcal{A}(D, L)$. Also, by $G_0(D, L)[[t]]$ we denote the group of invertible elements of $\mathcal{A}(D, L)[[t]]$ consisting of elements whose expansions in t begin with I.

Further, consider Eq. (1.1) corresponding to an $a = (a_1, a_2, \ldots) \in X$:

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)v^{i+1}, \quad x \in I_T. \tag{3.2}$$

Using the method of linearization of (3.2) from [6] we associate to this equation the following system of ODEs:

$$H'(x) = \left(\sum_{i=1}^{\infty} a_i(x)DL^{i-1}t^i\right)H(x), \quad x \in I_T. \tag{3.3}$$

Solving (3.3) by Picard iteration we obtain a solution $H_a : I_T \to G_0(D, L)[[t]]$, $H_a(0) = I$, whose coefficients in the series expansion in D, L and t are Lipschitz functions on I_T. It was established in [7, Theorem 1.1] that (3.2) determines a center (i.e., $a \in \mathcal{C}$) if and only if $H_a(T) = I$. This implies the following result (see [10, Proposition 2.1]).

Theorem 3.1.

$$a \in \mathcal{C} \iff \sum_{i_1+\cdots+i_k=i} p_{i_1,\ldots,i_k} I_{i_1,\ldots,i_k}(a) \equiv 0 \quad \text{for all } i \in \mathbb{N}, \tag{3.4}$$

where $p_{i_1,\ldots,i_k}$ is a polynomial of degree k defined by

$$p_{i_1,\ldots,i_k}(t) = (t - i_1 + 1)(t - i_1 - i_2 + 1)(t - i_1 - i_2 - i_3 + 1)\cdots(t - i + 1), \quad t \in \mathbb{C}. \tag{3.5}$$

3.1.2. Let $G[[r]]$ be the set of formal complex power series $f(r) = r + \sum_{i=1}^{\infty} d_i r^{i+1}$. Let $d_i : G[[r]] \to \mathbb{C}$ be such that $d_i(f)$ is the $(i+1)$st coefficient in the series expansion of f. We equip $G[[r]]$ with the weakest topology in which all d_i are continuous functions and consider the multiplication $\circ$ on $G[[r]]$ defined by the composition of series. Then $G[[r]]$ is a separable topological group. Moreover, it is contractible and residually torsion free nilpotent. By $G_c[[r]] \subset$

$G[[r]]$ we denote the subgroup of power series locally convergent near 0 equipped with the induced topology. Next, we define the map $P: X \to G[[r]]$ by the formula

$$P(a) := r + \sum_{i=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k}(i)\cdot I_{i_1,\dots,i_k}(a)\right)r^{i+1}, \tag{3.6}$$

see (3.5). It follows from [8] that $P(a*b) = P(a)\circ P(b)$ and $P(X) = G_c[[r]]$. Moreover, let $v(x;r;a)$, $x\in I_T$, be the Lipschitz solution of Eq. (3.2) with initial value $v(0;r;a) = r$. Clearly for every $x\in I_T$ we have $v(x;r;a)\in G_c[[r]]$. It was proved in [6] that $P(a) = v(T;\cdot;a)$ (i.e., $P(a)$ is the *first return map* of (3.2)). In particular, we have

$$a\in\mathcal{C} \iff \sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k}(i)\cdot I_{i_1,\dots,i_k}(a)\equiv 0 \quad \text{for all } i\in\mathbb{N}. \tag{3.7}$$

Also, (3.6) implies that there is a continuous homomorphism $\widehat{P}: G(X)\to G[[r]]$ such that $P = \widehat{P}\circ\pi$ (where $\pi: X\to G(X)$ is the quotient map). We extend it by continuity to $G_f(X)$ retaining the same symbol for the extension.

3.2. Group of formal centers

3.2.1. Let $\mathbb{C}\langle X_1, X_2\rangle$ be the associative algebra with unit I of complex polynomials in I and free non-commutative variables X_1, X_2. Consider a homomorphism $\phi: \mathbb{C}\langle X_1, X_2\rangle \to \mathcal{A}(D,L)$ defined by conditions: $\phi(X_1) := D$, $\phi(X_2) := L$. Then $\operatorname{Ker}\phi \subset \mathbb{C}\langle X_1, X_2\rangle$ is a two-sided ideal generated by the element $X_1X_2 - X_2X_1 + X_2^2$, see [7, Proposition 2.2]. In particular, see [7, Lemma 2.4], each $p\in\mathcal{A}(D,L)$ is uniquely presented as

$$p(D,L,I) = a_0 I + \sum_{k=1}^{n} F_k(D,L) \quad \text{where } F_k(D,L) = \sum_{i=0}^{k} a_{ik-i,k} D^i L^{k-i} \tag{3.8}$$

with all $a_0, a_{ij,k}\in\mathbb{C}$.

We say that such p has *degree n* if the polynomial $p(x,y,1)$ in commutative variables x, y has degree n. By $P_n\subset\mathcal{A}(D,L)$ we denote the complex vector space of polynomials of degree $\leqslant n$. We naturally identify P_n with the hermitian space $\mathbb{C}^{k(n)}$ where $k(n) = \dim_{\mathbb{C}} P_n$.

Let $\mathcal{A}_*\subset\mathcal{A}(D,L)[[t]]$ be the subalgebra of series

$$f = \sum_{n=0}^{\infty} f_n t^n \quad \text{with } f_n\in P_n,\ n\in\mathbb{Z}_+. \tag{3.9}$$

We equip $\mathcal{A}_*$ with the weakest topology in which all coefficients f_n in expansion (3.9) considered as functions in f are continuous maps of $\mathcal{A}_*$ into $\mathbb{C}^{k(n)}$, $n\in\mathbb{Z}_+$. Since the set of such maps is countable, $\mathcal{A}_*$ is metrizable, cf. (2.4). Moreover, if d is a metric on $\mathcal{A}_*$ compatible with topology, then $(\mathcal{A}_*, d)$ is a complete metric space (i.e., a sequence $\{f_k = \sum_{n=0}^{\infty} f_{nk}t^n\}_{k\in\mathbb{N}}\subset\mathcal{A}_*$ such that each $\{f_{nk}\}_{k\in\mathbb{N}}\subset P_n$ is a Cauchy sequence converges to $f = \sum_{n=0}^{\infty} f_n t^n\in\mathcal{A}_*$ where $f_n = \lim_{k\to\infty} f_{nk}$, $n\in\mathbb{Z}_+$).

By $G_*\subset G_0(D,L)[[t]]$ we denote the subgroup of elements $f\in\mathcal{A}_*$ with $f_0 = I$ equipped with the induced topology. Then (G_*, d) is a complete metric space.

Next, consider an algebra homomorphism $\Psi: \mathcal{A}\to\mathcal{A}_*$ determined by conditions

$$\Psi(X_i) := DL^{i-1}, \quad i\in\mathbb{N}. \tag{3.10}$$

(Recall that $\mathcal{A} \subset \mathbb{C}\langle X_1, X_2, \ldots\rangle[[t]]$ is defined by (2.6).) By the definition for

$$f = c_0 I + \sum_{n=1}^{\infty}\Bigl(\sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} X_{i_1}\cdots X_{i_k}\Bigr)t^n \in \mathcal{A}$$

we have

$$\Psi(f) := c_0 I + \sum_{n=1}^{\infty}\Bigl(\sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} DL^{i_1-1}\cdots DL^{i_k-1}\Bigr)t^n \in \mathcal{A}_*.$$

Expressing each $DL^{i_1-1}\cdots DL^{i_k-1}$ in the form (3.8) (using identity $DL - LD = -L^2$), we conclude that Ψ is a continuous homomorphism of topological algebras. Moreover, $\Psi|_G : G \to G_*$ is a continuous homomorphism of topological groups. (Recall that G is the subset of elements of $\mathcal{A}$ whose expansions in t begin with I.)

Observe that Ψ transfers Eq. (2.8) (determining $E : X \to G$, see (2.9)) to Eq. (3.3). In particular, we have

$$\Psi\bigl(E(a)\bigr) = H_a, \quad a \in X.$$

The last identity gives rise to the formula

$$\Psi\bigl(\widehat{E}(g)\bigr) = I + \sum_{n=1}^{\infty}\Bigl(\sum_{i_1+\cdots+i_k=n} \widehat{I}_{i_1,\ldots,i_k}(g) DL^{i_1-1}\cdots DL^{i_k-1}\Bigr)t^n, \quad g \in G_f(X). \tag{3.11}$$

(Here, as before, we regard the basic iterated integrals $\widehat{I}_{\cdot}$ as continuous functions on $G_f(X)$ extending them by continuity from $G(X)$.)

3.2.2. Let us observe that the Lie algebra $\mathcal{L}_{G_*}$ of G_* consists of elements of $\mathcal{A}_*$ of form (3.9) with $f_0 = 0$. As usual, for $f, g \in \mathcal{L}_{G_*}$ their product is defined by the formula $[f, g] := f \cdot g - g \cdot f$. Also, the map $\exp : \mathcal{L}_{G_*} \to G_*$, $\exp(f) := e^f$, is a homeomorphism.

The group homomorphism $\Psi : G \to G_*$ determines a continuous homomorphism of the corresponding Lie algebras such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{L}_G \arrow[r, "\exp"] \arrow[d, "\Psi"'] & G \arrow[d, "\Psi"] \\ \mathcal{L}_{G_*} \arrow[r, "\exp"] & G_*. \end{tikzcd} \tag{3.12}$$

By $\mathcal{L}_S \subset \mathcal{L}_{G_*}$ we denote the image under Ψ of the Lie algebra $\mathcal{L}_{\mathrm{Lie}}$ of the group $\widehat{E}(G_f(X))(\cong G_f(X))$, see Section 2.4.1.

In our calculations we will use the following result.

Lemma 3.2.

$$[DL^i, DL^j] = (i-j)DL^{i+j+1}.$$

Proof. It suffices to check the identity for elements $z^n \in \mathbb{C}[[z]]$ with $n \geqslant i + j + 2$. Then we have

$$(DL^i DL^j)(z^n) = (n-j-i-1)(n-j)z^{n-j-i-2},$$
$$(DL^j DL^i)(z^n) = (n-i-j-1)(n-i)z^{n-i-j-2}, \quad \text{and}$$
$$(i-j)(DL^{i+j+1})(z^n) = (i-j)(n-i-j-1)z^{n-i-j-2}.$$

These identities imply the required result. □

Now, for an element

$$g=\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n}c_{i_1,\dots,i_k}[X_{i_1},[X_{i_2},[\,\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]]\biggr)t^n\in\mathcal{L}_{\mathrm{Lie}}$$

we have

$$\Psi(g):=\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n}c_{i_1,\dots,i_k}[DL^{i_1-1},[DL^{i_2-1},[\,\cdots,[DL^{i_{k-1}-1},DL^{i_k-1}]\cdots]]]\biggr)t^n.$$

Simplifying the right-hand side by Lemma 3.2 we finally obtain

$$\Psi(g)=\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n}c_{i_1,\dots,i_k}\cdot\gamma_{i_1,\dots,i_k}DL^{n-1}\biggr)t^n \tag{3.13}$$

where $\gamma_n=1$ and

$$\gamma_{i_1,\dots,i_k}:=(-1)^{k-1}(i_k-i_{k-1})(i_{k-1}+i_k-i_{k-2})\cdots(i_2+\cdots+i_k-i_1)\quad\text{for } k\geqslant 2.$$

Proposition 3.3. *The following is true*:

$$\mathcal{L}_S=\Biggl\{g\in\mathcal{L}_{G_*}\colon\ g=\sum_{n=1}^{\infty}g_n\,DL^{n-1}t^n,\ g_n\in\mathbb{C},\ n\in\mathbb{N}\Biggr\}. \tag{3.14}$$

Proof. By V we denote the vector space on the right-hand side of (3.14). According to Lemma 3.2, V is a closed Lie subalgebra of $\mathcal{L}_{G_*}$. Moreover, by (3.13) $\mathcal{L}_S\subset V$.

The converse implication is obvious: for an element $g=\sum_{n=1}^{\infty}g_nDL^{n-1}t^n\in V$ consider $\widetilde{g}:=\sum_{n=1}^{\infty}g_nX_nt^n\in\mathcal{L}_{\mathrm{Lie}}$. Then $\Psi(\widetilde{g})=g$, i.e., $g\in\mathcal{L}_S$. □

From Proposition 3.3 and diagram (3.12) we immediately obtain:

$S:=(\Psi\circ\widehat{E})\bigl(G_f(X)\bigr)$ *is a closed subgroup of* G_* *with the Lie algebra* $\mathcal{L}_S$.

3.2.3. According to (3.11) the normal subgroup $\mathrm{Ker}(\Psi\circ\widehat{E})\subset G_f(X)$ consists of elements g such that

$$\sum_{i_1+\cdots+i_k=n}\widehat{I}_{i_1,\dots,i_k}(g)DL^{i_1-1}\cdots DL^{i_k-1}=0\quad\text{for all } n\in\mathbb{N}.$$

Repeating literally the arguments of the proof of [10, Proposition 2.1] we obtain

$$g\in\mathrm{Ker}(\Psi\circ\widehat{E})\quad\Longleftrightarrow\quad\sum_{i_1+\cdots+i_k=i}p_{i_1,\dots,i_k}\widehat{I}_{i_1,\dots,i_k}(g)\equiv 0\quad\text{for all } i\in\mathbb{N}, \tag{3.15}$$

where $p_{i_1,\dots,i_k}$ is the polynomial defined in (3.5).

Next, recall that the homomorphism $\widehat{P}:G_f(X)\to G[[r]]$ is determined by

$$\widehat{P}(g):=r+\sum_{i=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=i}p_{i_1,\dots,i_k}(i)\cdot\widehat{I}_{i_1,\dots,i_k}(g)\biggr)r^{i+1}, \tag{3.16}$$

see Section 3.1.2. Then from (3.15) we obtain that

$$\mathrm{Ker}(\Psi \circ \widehat{E}) \subset \mathrm{Ker}\,\widehat{P}.$$

This implies that there is a homomorphism $\Phi : S \to G[[r]]$ such that

$$\widehat{P} = \Phi \circ \Psi \circ \widehat{E}. \tag{3.17}$$

Proposition 3.4. $\Phi : S \to G[[r]]$ *is an isomorphism of topological groups.*

Proof. Suppose that

$$s := \exp\left(\sum_{n=1}^{\infty} s_n DL^{n-1} t^n\right) \in S. \tag{3.18}$$

Then, from (3.16), (3.17) we obtain

$$\Phi(s) = r + \sum_{i=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=i} \frac{p_{i_1,\dots,i_k}(i)\cdot s_{i_1}\cdots s_{i_k}\cdot T^k}{k!}\right) r^{i+1}. \tag{3.19}$$

Since the map $\log : S \to \mathcal{L}_S$, $\log(f) := -\sum_{i=1}^{\infty} \frac{(I-f)^i}{i}$, is a homeomorphism, formula (3.19) and the definitions of topologies on S and $G[[r]]$ imply that Φ is a continuous homomorphism.

Further, the expression in the brackets of (3.19) can be written in the form $s_i T + p_i(s_1 T, \dots, s_{i-1} T)$ where p_i is a polynomial of degree i with rational coefficients on $\mathbb{R}^{i-1}$. In particular, for any sequence $\{d_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ one can solve consequently the equations

$$s_i T + p_i(s_1 T, \dots, s_{i-1} T) = d_i, \quad i \in \mathbb{N},$$

to get an element s of form (3.18) such that $\Phi(s) = r + \sum_{i=1}^{\infty} d_i r^{i+1}$. Moreover, each s_i in the definition of s is a polynomial in variables $d_1, \dots, d_i$. This implies that Φ has a continuous inverse $\Phi^{-1} : G[[r]] \to S$ and completes the proof of the proposition. □

Remark 3.5. The Lie algebra $\mathcal{L}_S$ is isomorphic to the algebra $W_1(1)$, the nilpotent part of the Witt algebra of formal vector fields on $\mathbb{R}$, which is known to be the Lie algebra of $G[[r]]$. Recall that $W_1(1)$ has the natural basis $e_i := r^{i+1}\frac{d}{dr}$, $i \in \mathbb{N}$. Then the isomorphism $w : \mathcal{L}_S \to W_1(1)$ is given by $w(DL^{i-1}) = -e_i$, $i \in \mathbb{N}$. Identifying $\mathcal{L}_S$ with $W_1(1)$ by w we can regard the map $\Phi \circ \exp$ as an exponential map $W_1(1) \to G[[r]]$.

According to (3.17) and Proposition 3.4 we have

$$\mathrm{Ker}(\Psi \circ \widehat{E}) = \mathrm{Ker}\,\widehat{P}.$$

This group is denoted by $\widehat{\mathcal{C}}_f$ and called the *group of formal centers* of Eq. (1.1). By the definition $\widehat{\mathcal{C}}_f$ is a closed normal subgroup of $G_f(X)$. Moreover, $\widehat{\mathcal{C}}_f$ contains the subgroup $\widehat{\mathcal{C}} := \pi(\mathcal{C}) \subset G(X)$, the *group of centers* of Eq. (1.1).

3.3. Properties of the group of formal centers

3.3.1. In what follows we identify $G_f(X)$ and $G(X)$ with their images under $\widehat{E}$.

Formulas (3.15), (3.16) imply, cf. Section 3.1:

$$g \in \widehat{\mathcal{C}}_f \iff \sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k}\widehat{I}_{i_1,\dots,i_k}(g) \equiv 0 \quad \text{for all } i \in \mathbb{N}$$
$$\iff \sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k}(i)\cdot\widehat{I}_{i_1,\dots,i_k}(g) = 0 \quad \text{for all } i \in \mathbb{N}, \tag{3.20}$$

where $p_{i_1,\dots,i_k}(t) = (t-i_1+1)(t-i_1-i_2+1)\cdots(t-i+1)$.

In turn, (3.12) and (3.13) imply that the Lie algebra $\mathcal{L}_{\widehat{\mathcal{C}}_f} \subset \mathcal{L}_{\mathrm{Lie}}$ of $\widehat{\mathcal{C}}_f$ consists of elements

$$\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k}[X_{i_1},[X_{i_2},[\,\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]]\biggr)t^n$$

such that

$$\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k}\cdot\gamma_{i_1,\dots,i_k} = 0 \quad \text{for all } n \in \mathbb{N} \quad \text{where } \gamma_n = 1 \quad \text{and}$$
$$\gamma_{i_1,\dots,i_k} := (-1)^{k-1}(i_k-i_{k-1})(i_{k-1}+i_k-i_{k-2})\cdots(i_2+\cdots+i_k-i_1) \quad \text{for } k \geqslant 2. \tag{3.21}$$

(In particular, the map $\exp : \mathcal{L}_{\widehat{\mathcal{C}}_f} \to \widehat{\mathcal{C}}_f$ is a homeomorphism.)

Further, by the definition we have

$$\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_f \cap G(X). \tag{3.22}$$

Proposition 3.6. *$\widehat{\mathcal{C}}$ is a dense subgroup of $\widehat{\mathcal{C}}_f$.*

Proof. According to [8, Theorem 2.10] there exists a continuous embedding $T : G_c[[r]] \to G(X)$ such that $\widehat{P}\circ T = \mathrm{id}$. Moreover, $\widetilde{T} : G_c[[r]] \times \widehat{\mathcal{C}} \to G(X)$, $\widetilde{T}(s,g) := T(s)\cdot g$ is a homeomorphism. We can extend T by continuity to a map $T_f : G[[r]] \to G_f(X)$ using formula (4.1) for the definition of T from [8]. Since $\widehat{P}\circ T = \mathrm{id}$, similarly we have $\widehat{P}\circ T_f = \mathrm{id}$. In particular, T_f is an embedding.

Let $\mathrm{cl}(\widehat{\mathcal{C}})$ be the closure of $\widehat{\mathcal{C}}$ in $\widehat{\mathcal{C}}_f$. We can extend the map $\widetilde{T}$ by continuity to a map $\widetilde{T}_f : G[[r]] \times \mathrm{cl}(\widehat{\mathcal{C}}) \to G_f(X)$. Then, $\widetilde{T}_f(s,g) := T_f(s)\cdot g$. Next, since $\widetilde{T}$ is a homeomorphism and $G(X)$ is dense in $G_f(X)$, the map $\widetilde{T}_f$ is a homeomorphism, as well (by the above definition of $\widetilde{T}$ the inverse $\widetilde{T}_f^{-1}$ of $\widetilde{T}_f$ is the extension by continuity of $\widetilde{T}^{-1}$ which can be expressed explicitly). In particular, for $g \in \widehat{\mathcal{C}}_f$ we have $g = T_f(s)\cdot h$ for some $s \in G[[r]]$, $h \in \mathrm{cl}(\widehat{\mathcal{C}})$. This implies

$$1 = \widehat{P}(g) = \widehat{P}\bigl(T_f(s)\cdot h\bigr) = \widehat{P}\bigl(T_f(s)\bigr)\cdot\widetilde{P}(h) = s.$$

Hence, $g = h \in \mathrm{cl}(\widehat{\mathcal{C}})$. $\square$

3.3.2. By $L \subset \mathcal{L}_{\mathrm{Lie}}$ we denote the closed subspace of elements $g = \sum_{n=1}^{\infty} g_n X_n t^n$, $g_n \in \mathbb{C}$, $n \in \mathbb{N}$. Then there is a continuous linear isomorphism $A : \mathcal{L}_S \to L$ determined by $A(DL^{n-1}t^n) := X_n t^n$, $n \in \mathbb{N}$. Thus, $\Psi \circ A = \mathrm{id}$. The map $\Pi := A\circ\Psi : \mathcal{L}_{\mathrm{Lie}} \to L$ is a continuous linear projection onto L. Moreover, $\mathrm{id}-\Pi : \mathcal{L}_{\mathrm{Lie}} \to \mathcal{L}_{\widehat{\mathcal{C}}_f}$ is a continuous linear projection onto $\mathcal{L}_{\widehat{\mathcal{C}}_f}$. Hence $\Pi \oplus (\mathrm{id}-\Pi) : \mathcal{L}_{\mathrm{Lie}} \to L \oplus \mathcal{L}_{\widehat{\mathcal{C}}_f}$ is an isomorphism. Also, every element $g \in \mathcal{L}_{\widehat{\mathcal{C}}_f}$ is presented in the form

$$g = \sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n,\ k\geqslant 2} c_{i_1,\dots,i_k}v_{i_1,\dots,i_k}\biggr)t^n \quad \text{where}$$
$$v_{i_1,\dots,i_k} := [X_{i_1},[X_{i_2},[\,\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]] - \gamma_{i_1,\dots,i_k}X_n. \tag{3.23}$$

Recall that elements $\{v_{i_1,\dots,i_k}\colon i_1+\cdots+i_k=n,\ k\geqslant 2\}$ are not linearly independent, cf. (2.14). The number of linearly independent elements in this set is

$$\frac{1}{n}\left(\sum_{d|n}(2^{n/d}-1)\cdot\mu(d)\right)-1.$$

Proposition 3.7. *$\widehat{\mathcal{C}}_f$ is the closure in $G_f(X)$ of the group H generated by elements $\exp(c_{i_1,\dots,i_k}v_{i_1,\dots,i_k}t^{i_1+\cdots+i_k})$ for all possible $v_{i_1,\dots,i_k}$ and $c_{i_1,\dots,i_k}\in\mathbb{C}$.*

Proof. According to Proposition 2.23, $\log(\overline{H})$ is the minimal closed Lie subalgebra of $\mathcal{L}_{\mathrm{Lie}}$ containing all possible elements $v_{i_1,\dots,i_k}t^{i_1+\cdots+i_k}$. By (3.23) this subalgebra coincides with $\mathcal{L}_{\widehat{\mathcal{C}}_f}$. Thus, $\overline{H}=\widehat{\mathcal{C}}_f$. □

Question 1. Is it true that every nontrivial element $\exp(c_{i_1,\dots,i_k}v_{i_1,\dots,i_k}t^{i_1+\cdots+i_k})$ belongs to $G_f(X)\setminus G(X)$, i.e., it cannot be presented by an element from X?

3.3.3. Let us consider a continuous homomorphism $\pi_2:\mathcal{L}_{\mathrm{Lie}}\to\mathcal{L}_{\mathrm{Lie}}$ determined by the conditions

$$\pi_2(X_i):=\begin{cases} X_i & \text{for } i=1,2,\\ \frac{1}{i-2}\cdot[\pi_2(X_{i-1}),X_1] & \text{for } i\geqslant 3.\end{cases} \tag{3.24}$$

From this definition we get

$$\pi_2(X_i)=\frac{(-1)^i}{(i-2)!}\cdot\underbrace{[X_1,[X_1,[\cdots,[X_1,X_2]\cdots]]]}_{(i-2)\text{-brackets}}\quad\text{for } i\geqslant 3. \tag{3.25}$$

Therefore π_2 maps $\mathcal{L}_{\mathrm{Lie}}$ surjectively onto $\mathcal{L}^2_{\mathrm{Lie}}$, the Lie algebra of the group $G_f(X^2)$ of formal paths in $\mathbb{C}^2$, see Section 2.6.1.

Proposition 3.8. *The kernel* $\mathrm{Ker}(\pi_2)$ *is a subalgebra of $\mathcal{L}_{\widehat{\mathcal{C}}_f}$. It is the minimal normal closed Lie subalgebra $\mathcal{N}$ of $\mathcal{L}_{\mathrm{Lie}}$ containing elements $\{s_it^i\}_{i\geqslant 3}$ where*

$$s_i:=X_i-\frac{1}{i-2}\cdot[X_{i-1},X_1],\quad i\geqslant 3.$$

Proof. By the definition each $s_it^i\in\mathrm{Ker}(\pi_2)$. Since π_2 is continuous, $\mathcal{N}\subset\mathrm{Ker}(\pi_2)$. Next, consider an element

$$g=\sum_{n=1}^{\infty}\left(\sum_{i_1+\cdots+i_k=n}c_{i_1,\dots,i_k}[X_{i_1},[X_{i_2},[\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]]\right)t^n=\sum_{n=1}^{\infty}g_nt^n\in\mathcal{L}_{\mathrm{Lie}}.$$

Using (3.25) one can represent each g_n as f_n+h_n where $f_nt^n\in\mathcal{N}\cap(L_n\cdot t^n)$ and $h_nt^n\in\mathcal{L}^2_{\mathrm{Lie}}\cap(L_n\cdot t^n)$ (here L_n is the complex vector space generated by all possible g_n, see (2.14)). Then we have

$$g=f+h\quad\text{where } f:=\sum_{n=1}^{\infty}f_nt^n\in\mathcal{N}\text{ and } h:=\sum_{n=1}^{\infty}h_nt^n\in\mathcal{L}^2_{\mathrm{Lie}}.$$

In particular, $\pi_2(g)=h$. Thus $g\in\mathrm{Ker}(\pi_2)$ if and only if $h=0$, i.e., $g\in\mathcal{N}$.

Further, each $s_i t^i \in \mathcal{L}_{\widehat{\mathcal{C}}_f}$, see (3.21). Hence $\mathrm{Ker}(\pi_2) = \mathcal{N} \subset \mathcal{L}_{\widehat{\mathcal{C}}_f}$. $\square$

In the proof we have established also the natural decomposition

$$\mathcal{L}_{\mathrm{Lie}} = \mathcal{N} \oplus \mathcal{L}^2_{\mathrm{Lie}}. \tag{3.26}$$

Taking the exponential map in (3.26) we obtain the following result.

Proposition 3.9. *The group $G_f(X)$ is the semidirect product of the normal subgroup* $\exp(\mathcal{N}) \subset \widehat{\mathcal{C}}_f$ *and $G_f(X^2)$. Moreover,* $\exp(\mathcal{N})$ *is the minimal normal closed subgroup of $G_f(X)$ containing elements* $\exp(c_i s_i t^i)$ *for all possible s_i and $c_i \in \mathbb{C}$.*

Proof. From the Campbell–Hausdorff formula for elements of $G_f(X)$ using the fact that $\mathcal{N} \subset \mathcal{L}_{\mathrm{Lie}}$ is a closed ideal we obtain for $a \in \mathcal{N}$, $b \in \mathcal{L}^2_{\mathrm{Lie}}$:

$$e^{a+b} = e^a e^b e^{c_1} = e^a (e^b e^{c_1} e^{-b}) e^b = (e^a e^{c_2}) e^b = e^{c_3} e^b$$

for some $c_1, c_2, c_3 \in \mathcal{N}$. This and (3.26) give the first statement of the proposition.

The second statement is the direct consequence of Proposition 2.23 applied to elements of $R := \{\log(e^a e^{s_i t^i} e^{-a}) = e^a s_i t^i e^{-a} \in \mathcal{N}\colon a \in \mathcal{L}_{\mathrm{Lie}},\ i \in \mathbb{N}\}$. $\square$

Remark 3.10. By the definition of s_i each formal path $e^{c_i s_i t^i}$, $c_i \in \mathbb{C}$, belongs to the *subgroup of formal paths in the subspace $W_i \subset \mathbb{C}^\infty$* where

$$W_i := \big\{(z_1, z_2, \dots) \in \mathbb{C}^\infty\colon z_k = 0 \text{ for all } k \notin \{1, i-1, i\}\big\} \cong \mathbb{C}^3.$$

In particular, there exist elements $\gamma_l \in X$ with first integrals $\widetilde{\gamma}_l : I_T \to W_i$, $l \in \mathbb{N}$, such that the sequence $\{\pi(\gamma_l)\}_{l \in \mathbb{N}} \subset G(X)$ converges to $e^{c_i s_i t^i}$. (Recall that we identify $G_f(X)$ with $\widehat{E}(G_f(X))$.)

Let

$$\Psi_2 := \Psi|_{G_f(X^2)} : G_f(X^2) \to S\big(\cong G[[r]]\big) \quad \text{and} \quad \widehat{P}_2 := \widehat{P}|_{G_f(X^2)} : G_f(X^2) \to G[[r]].$$

Then we have, cf. (3.17),

$$\widehat{P}_2 = \Phi \circ \Psi_2. \tag{3.27}$$

Next, we extend the homomorphism π_2 determined by (3.24) to a continuous endomorphism of the associative algebra $\mathcal{A}$, see (2.6). We retain the same symbol for the extension. Then π_2 maps $G_f(X)$ surjectively onto $G_f(X^2)$. Moreover, by Proposition 3.9, $\mathrm{Ker}(\pi_2|_{G_f(X)}) = \exp(\mathcal{N})$.

Proposition 3.11. *The following identity is valid*:

$$\Psi(g) = (\Psi_2 \circ \pi_2)(g) \quad \textit{for all } g \in G_f(X).$$

Proof. Since for $h \in \mathcal{L}_{\mathrm{Lie}}$ we have

$$\Psi(e^h) = e^{\Psi(h)} \quad \text{and} \quad (\Psi_2 \circ \pi_2)(e^h) = e^{(\Psi_2 \circ \pi_2)(h)},$$

it suffices to check the identity of the proposition for elements of $\mathcal{L}_{\mathrm{Lie}}$. Moreover, it suffices to check it for elements X_i, $i \in \mathbb{N}$ (because $\{X_i t^i\}_{i \in \mathbb{N}}$ are generators of $\mathcal{L}_{\mathrm{Lie}}$). In this case we have by the definitions of Ψ and π_2 and by Lemma 3.2

$$\Psi(X_i) = DL^{i-1} \quad \text{and} \quad (\Psi_2 \circ \pi_2)(X_i) = DL^{i-1} \quad \text{for } i = 1, 2,$$
$$(\Psi_2 \circ \pi_2)(X_i) = \frac{(-1)^i}{(i-2)!} \cdot \underbrace{[D, [D, [\cdots, [D, DL] \cdots]]]}_{(i-2)\text{-brackets}} = DL^{i-1} \quad \text{for } i \geqslant 3.$$

This completes the proof of the proposition. □

From this proposition, (3.17) and (3.27) we obtain that

$$\widehat{P} = \widehat{P}_2 \circ \pi_2. \tag{3.28}$$

In particular, the homomorphism $\widehat{P}_2 : G_f(X^2) \to G[[r]]$ corresponding to the first return maps of "generalized" Abel equations is surjective. Moreover, Proposition 3.9 implies that $\mathcal{C}_f$ is the semidirect product of $\exp(\mathcal{N})$ and $\widehat{\mathcal{C}}_f^2 := \widehat{\mathcal{C}}_f \cap G_f(X^2)$ (the *group of formal centers of Abel differential equations*).

3.3.4. It has been shown above that there is a reduction of the center problem for $G_f(X)$ to that for $G_f(X^2)$. In this part we prove some results on the structure of the group $\widehat{\mathcal{C}}_f^2$.

First, observe that $\widehat{\mathcal{C}}_f^2 \subset G_f(X^2)$ is determined by systems of equations of form (3.20) in which all $i_l \in \{1, 2\}$, $l \in \mathbb{N}$. In turn, the Lie algebra $\mathcal{L}_{\widehat{\mathcal{C}}_f^2}$ of $\widehat{\mathcal{C}}_f^2$ is determined by the system of equations of form (3.21) in which also all $i_l \in \{1, 2\}$, $l \in \mathbb{N}$.

By $L_2 \subset \mathcal{L}_{\text{Lie}}^2$ we denote the closed subspace of elements $g = \sum_{n=1}^{\infty} g_n r_n t^n$, $g_n \in \mathbb{C}$, $n \in \mathbb{N}$, where $r_1 = X_1$, $r_2 = X_2$ and

$$r_n := \frac{(-1)^n}{(n-2)!} \cdot \underbrace{[X_1, [X_1, [\cdots, [X_1, X_2] \cdots]]]}_{(n-2)\text{-brackets}} \quad \text{for } n \geqslant 3. \tag{3.29}$$

By the definition, $\Psi_2(r_n) = DL^{n-1}$. Then there is a continuous linear isomorphism $A_2 : \mathcal{L}_S \to L_2$ determined by $A_2(DL^{n-1}t^n) := r_n t^n$, $n \in \mathbb{N}$, such that $\Psi_2 \circ A_2 = \text{id}$. The map $\Pi_2 := A_2 \circ \Psi_2 : \mathcal{L}_{\text{Lie}}^2 \to L_2$ is a continuous linear projection onto L_2. Moreover, $\text{id} - \Pi_2 : \mathcal{L}_{\text{Lie}}^2 \to \mathcal{L}_{\widehat{\mathcal{C}}_f^2}$ is a continuous linear projection onto $\mathcal{L}_{\widehat{\mathcal{C}}_f^2}$. Hence,

$$\Pi_2 \oplus (\text{id} - \Pi_2) : \mathcal{L}_{\text{Lie}}^2 \to L_2 \oplus \mathcal{L}_{\widehat{\mathcal{C}}_f^2} \text{ is an isomorphism.} \tag{3.30}$$

This implies that every element $g \in \mathcal{L}_{\widehat{\mathcal{C}}_f^2}$ is presented in the form

$$g = \sum_{n=1}^{\infty} \left(\sum_{i_1 + \cdots + i_k = n,\ n \geqslant 5} c_{i_1, \ldots, i_k} l_{i_1, \ldots, i_k} \right) t^n \quad \text{where all } i_s \in \{1, 2\},\ s \in \mathbb{N}, \quad \text{and}$$
$$l_{i_1, \ldots, i_k} := [X_{i_1}, [X_{i_2}, [\cdots, [X_{i_{k-1}}, X_{i_k}] \cdots]]] - \gamma_{i_1, \ldots, i_k} r_n. \tag{3.31}$$

The elements $\{l_{i_1, \ldots, i_k} : i_1 + \cdots + i_k = n,\ n \geqslant 5,\ i_s \in \{1, 2\}\}$ are not linearly independent. It follows from [17, Theorem 3.1] that the number of linearly independent elements in this set is

$$\frac{1}{n} \left(\sum_{d|n} \left(\lambda_1^{n/d} + \lambda_2^{n/d} \right) \cdot \mu(d) \right) - 1 \tag{3.32}$$

where $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, μ is the Möbius function, see Section 2.4.1, and the sum ranges over all integers which divide n.

Further, similarly to Proposition 3.7 from (3.30) and (3.31) we get

Proposition 3.12. (1) *There is a continuous map* $T_f^2 : G[[r]] \to G_f(X^2)$ *such that* $\widehat{P}_2 \circ T_f^2 = \mathrm{id}$. *Moreover, the map* $\widetilde{T}_f^2 : G[[r]] \times \widehat{\mathcal{C}}_f^2 \to G_f(X^2)$ *defined by* $\widetilde{T}_f^2(s,g) = T_f^2(s) \cdot g$ *is a homeomorphism.*

(2) *The group* $\widehat{\mathcal{C}}_f^2$ *is the closure in* $G_f(X^2)$ *of the group* H_2 *generated by elements* $\exp(c_{i_1,\dots,i_k} l_{i_1,\dots,i_k} t^{i_1+\dots+i_k})$ *for all possible* $l_{i_1,\dots,i_k}$ *and* $c_{i_1,\dots,i_k} \in \mathbb{C}$ *with* $i_1,\dots,i_k \in \{1,2\}$.

Proof. (1) We define the map T_f^2 by the formula

$$T_f^2 := \exp \circ A_2 \circ \log \circ \Phi^{-1}. \tag{3.33}$$

Then $T_f^2 : G[[r]] \to G_f(X^2)$ is continuous as the composite of continuous maps. Also, from (3.27) by the properties of A_2 we get

$$\begin{aligned}\widehat{P}_2 \circ T_f^2 &= \Phi \circ \Psi_2 \circ \exp \circ A_2 \circ \log \circ \Phi^{-1} = \Phi \circ \exp \circ \Psi_2 \circ A_2 \circ \log \circ \Phi^{-1} \\ &= \Phi \circ \exp \circ \mathrm{id} \circ \log \circ \Phi^{-1} = \Phi \circ \Phi^{-1} = \mathrm{id}.\end{aligned}$$

Now, for the map $\widetilde{T}_f^2 : G[[r]] \times \widehat{\mathcal{C}}_f^2 \to G_f(X^2)$, $\widetilde{T}_f^2(s,g) = T_f^2(s) \cdot g$, we define the map $Q : G_f(X^2) \to G[[r]] \times \widehat{\mathcal{C}}_f^2$ by the formula

$$Q(h) := \left(\widehat{P}_2(h), \left[(T_f^2 \circ \widehat{P}_2)(h)\right]^{-1} \cdot h\right). \tag{3.34}$$

The second term here belongs to $\widehat{\mathcal{C}}_f^2$ because

$$\widehat{P}_2\left[\left(T_f^2\left(\widehat{P}_2(h)\right)\right)^{-1} \cdot h\right] = \left[\left(\widehat{P}_2 \circ T_f^2 \circ \widehat{P}_2\right)(h)\right]^{-1} \cdot \widehat{P}_2(h) = \left[\widehat{P}_2(h)\right]^{-1} \cdot \widehat{P}_2(h) = 1.$$

Clearly, both $\widetilde{T}_f^2$ and Q are continuous maps. Moreover,

$$\begin{aligned}\left(Q \circ \widetilde{T}_f^2\right)(s,g) &= Q\left(T_f^2(s) \cdot g\right) = \left(\widehat{P}_2\left(T_f^2(s) \cdot g\right), \left[(T_f^2 \circ \widehat{P}_2)\left(T_f^2(s) \cdot g\right)\right]^{-1} \cdot T_f^2(s) \cdot g\right) \\ &= \left(s, \left[T_f^2(s)\right]^{-1} \cdot T_f^2(s) \cdot g\right) = (s,g).\end{aligned}$$

Thus Q is the inverse map to $\widetilde{T}_f^2$, i.e., $\widetilde{T}_f^2$ is a homeomorphism.

(2) This statement is proved similarly to the proof of Proposition 3.7. □

Question 2. Is it true that there is a continuous map $T^2 : G_c[[r]] \to G(X^2)$ such that $\widehat{P}_2 \circ T^2 = \mathrm{id}$?

The affirmative answer to this question will show that each locally convergent series from $G_c[[r]]$ can be obtained as the first return map of an Abel differential equation. Moreover, as in the proof of Proposition 3.6 we will get that the group of centers $\widehat{\mathcal{C}}^2 := \widehat{\mathcal{C}}_f^2 \cap G(X^2)$ of Abel differential equations is dense in the group of formal centers $\widehat{\mathcal{C}}_f^2$. Also, $G(X^2)$ will be homeomorphic to $G_c[[r]] \times \widehat{\mathcal{C}}^2$.

3.3.5. In this section we briefly discuss the center problem over a field $\mathbb{F} \subset \mathbb{C}$.

Let $G_{\mathbb{F}}[[r]] \subset G[[r]]$ be the subgroup of formal power series with coefficients from $\mathbb{F}$. Let $I_{\mathbb{F}}^k$ be the normal subgroup of $G_{\mathbb{F}}[[r]]$ consisting of elements f of the form $f(z) := z + d_{k+1}z^{k+1} + d_{k+2}z^{k+2} + \cdots$. We equip $G_{\mathbb{F}}[[r]]$ with $\{I_{\mathbb{F}}^k\}_{k\in\mathbb{N}}$-adic topology, i.e., a sequence $\{f_i\}_{i\in\mathbb{N}} \subset G_{\mathbb{F}}[[r]]$ converges to $f \in G_{\mathbb{F}}[[r]]$ if and only if for any $k \in \mathbb{N}$ there is a number $N_k \in \mathbb{N}$ such that for all

$n \geqslant N_k$ the images of f_n and f in the quotient group $G_{\mathbb{F}}[[r]]/I_{\mathbb{F}}^k$ coincide. By $G_{c,\mathbb{F}}[[r]] \subset G_{\mathbb{F}}[[r]]$ we denote the subgroup of locally convergent power series in $G_{\mathbb{F}}[[r]]$ equipped with the induced topology.

Further, consider the groups $G(X_{\mathbb{F}}) \subset G_f(X_{\mathbb{F}})$ of paths and formal paths over $\mathbb{F}$, see Section 2.6.2. According to (3.16) the homomorphism $\widehat{P}$ (corresponding to the first return maps of Eqs. (1.1)) maps $G(X_{\mathbb{F}})$ and $G_f(X_{\mathbb{F}})$ into $G_{c,\mathbb{F}}[[r]]$ and $G_{\mathbb{F}}[[r]]$, respectively. Also, by the definition of topologies on $G_f(X_{\mathbb{F}})$ and $G_{\mathbb{F}}[[r]]$, $\widehat{P}: G_f(X_{\mathbb{F}}) \to G_{\mathbb{F}}[[r]]$ is a continuous homomorphism of topological groups. We set $\widehat{\mathcal{C}}_{\mathbb{F}} := G(X_{\mathbb{F}}) \cap \widehat{\mathcal{C}}$ and $(\widehat{\mathcal{C}}_{\mathbb{F}})_f := G_f(X_{\mathbb{F}}) \cap \widehat{\mathcal{C}}_f$. These groups are referred to as the *groups of centers and formal centers over* $\mathbb{F}$. Then similarly to the results of the previous sections one can prove the following statements.

(1) $g \in (\widehat{\mathcal{C}}_{\mathbb{F}})_f$ if and only if the element $g \in G_f(X_{\mathbb{F}})$ satisfies Eqs. (3.20).
(2) The Lie algebra $\mathcal{L}_{(\widehat{\mathcal{C}}_{\mathbb{F}})_f} \subset \mathcal{L}_{\mathrm{Lie}(\mathbb{F})}$ of $(\widehat{\mathcal{C}}_{\mathbb{F}})_f$ consists of elements

$$\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k}[X_{i_1},[X_{i_2},[\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]]\biggr)t^n$$

with all $c_{i_1,\dots,i_k} \in \mathbb{F}$ satisfying Eqs. (3.21). In particular the exponential map $\exp : \mathcal{L}_{(\widehat{\mathcal{C}}_{\mathbb{F}})_f} \to (\widehat{\mathcal{C}}_{\mathbb{F}})_f$ is a homeomorphism.
(3) $\widehat{\mathcal{C}}_{\mathbb{F}}$ is a dense subgroup of $(\widehat{\mathcal{C}}_{\mathbb{F}})_f$.

The last statement is proved similarly to Proposition 3.6 using Theorem 2.12 of [8]. This result asserts that there is a continuous embedding $T_{\mathbb{F}}: G_{c,\mathbb{F}}[[r]] \to G(X_{\mathbb{F}})$ such that $\widehat{P} \circ T_{\mathbb{F}} = \mathrm{id}$. In particular, from here we obtain that $G(X_{\mathbb{F}})$ is homeomorphic to $\widehat{\mathcal{C}}_{\mathbb{F}} \times G_{c,\mathbb{F}}[[r]]$ and $G_f(X_{\mathbb{F}})$ is homeomorphic to $(\widehat{\mathcal{C}}_{\mathbb{F}})_f \times G_{\mathbb{F}}[[r]]$.

Further, one can prove a version of Proposition 3.7 (see Remark 2.23):

(4) $(\widehat{\mathcal{C}}_{\mathbb{F}})_f$ is the closure in $G_f(X_{\mathbb{F}})$ of the group $H_{\mathbb{F}}$ generated by elements $\exp(c_{i_1,\dots,i_k} v_{i_1,\dots,i_k} t^{i_1+\cdots+i_k})$ for all possible $v_{i_1,\dots,i_k}$ determined by (3.23) and $c_{i_1,\dots,i_k} \in \mathbb{F}$.

For the group $\mathcal{N} := \mathrm{Ker}(\pi_2)$, see Section 3.3.3, we set $\mathcal{N}_{\mathbb{F}} := \mathcal{N} \cap \mathcal{L}_{\mathrm{Lie}(\mathbb{F})}$. Then repeating word-for-word the proof of Proposition 3.8 we obtain that

(5) $\mathcal{N}_{\mathbb{F}}$ is the minimal normal closed Lie subalgebra of $\mathcal{L}_{\mathrm{Lie}(\mathbb{F})}$ containing elements $s_i t^i = (X_i - \frac{1}{i-2} \cdot [X_{i-1}, X_1])t^i$, $i \geqslant 3$. Moreover, $\mathcal{N}_{\mathbb{F}} \subset \mathcal{L}_{(\widehat{\mathcal{C}}_{\mathbb{F}})_f}$ and

$$\mathcal{L}_{\mathrm{Lie}(\mathbb{F})} = \mathcal{N}_{\mathbb{F}} \oplus \mathcal{L}^2_{\mathrm{Lie}(\mathbb{F})}$$

where $\mathcal{L}^2_{\mathrm{Lie}(\mathbb{F})}$ is the Lie algebra of $G_f(X^2_{\mathbb{F}}) := G_f(X_{\mathbb{F}}) \cap G_f(X^2)$.

From here as in the proof of Proposition 3.9 we obtain

(6) The group $G_f(X_{\mathbb{F}})$ is the semidirect product of the normal subgroup $\exp(\mathcal{N}_{\mathbb{F}}) \subset (\widehat{\mathcal{C}}_{\mathbb{F}})_f$ and $G_f(X^2_{\mathbb{F}})$. Moreover, $\exp(\mathcal{N}_{\mathbb{F}})$ is the minimal closed subgroup of $G_f(X_{\mathbb{F}})$ containing elements $\exp(c_i s_i t^i)$ for all possible s_i and $c_i \in \mathbb{F}$.

Finally, using the methods of the proof of Proposition 3.12 one can prove similar statements for the group $(\widehat{\mathcal{C}}^2_{\mathbb{F}})_f := (\widehat{\mathcal{C}}_{\mathbb{F}})_f \cap G_f(X^2_{\mathbb{F}})$:

(7) There is a continuous map $(T^2_{\mathbb{F}})_f : G_{\mathbb{F}}[[r]] \to G_f(X^2_{\mathbb{F}})$ such that $\widehat{P}_2 \circ (T^2_{\mathbb{F}})_f = \mathrm{id}$. The map $(\widetilde{T}^2_{\mathbb{F}})_f : G_{\mathbb{F}}[[r]] \times (\widehat{\mathcal{C}}^2_{\mathbb{F}})_f \to G_f(X^2_{\mathbb{F}})$ defined by $(\widetilde{T}^2_{\mathbb{F}})_f(s,g) = (T^2_{\mathbb{F}})_f(s) \cdot g$ is a homeomorphism.

(8) $(\widehat{\mathcal{C}}^2_{\mathbb{F}})_f$ is the closure in $G_f(X^2_{\mathbb{F}})$ of the group $(H_{\mathbb{F}})_2$ generated by elements $\exp(c_{i_1,\dots,i_k} l_{i_1,\dots,i_k} t^{i_1+\cdots+i_k})$ for all possible $l_{i_1,\dots,i_k}$ defined by (3.31) and $c_{i_1,\dots,i_k} \in \mathbb{F}$, $i_1,\dots,i_k \in \{1,2\}$.

The main point of the proofs of (4)–(6) and (8) is that all elements $v_{i_1,\dots,i_k} t^{i_1+\cdots+i_k}$, $s_i t^i$ and $l_{i_1,\dots,i_k} t^{i_1+\cdots+i_k}$ belong to $\mathcal{L}_{\mathrm{Lie}(\mathbb{Q})}$ (and therefore to $\mathcal{L}_{\mathrm{Lie}(\mathbb{F})}$ for any field $\mathbb{F} \subset \mathbb{C}$).

3.4. *Group of piecewise linear paths*

Consider elements $g \in G_f(X)$ of the form

$$g = e^h \quad \text{where } h = \sum_{i=1}^{\infty} c_i X_i t^i,\ c_i \in \mathbb{C},\ i \in \mathbb{N}. \tag{3.35}$$

By $PL \subset G_f(X)$ we denote the group generated by all such g. It will be called the *group of piecewise linear paths in* $\mathbb{C}^\infty$.

Remark 3.13. We can naturally extend the semigroup X considering the set $\widetilde{X} := (L^\infty(I_T))^{\mathbb{N}}$ of all possible sequences $a = (a_1, a_2, \dots)$, $a_i \in L^\infty(I_T)$, $i \in \mathbb{N}$, with the multiplication defined in Section 2.1.1. We consider each $L^\infty(I_T)$ with the weak* topology defined by $L^1(I_T)$ (recall that $L^\infty(T) = (L^1(I_T))^*$) and equip $\widetilde{X}$ with the corresponding product topology. Then X is a dense subset of $\widetilde{X}$. Moreover, according to [8, Lemma 3.2] the quotient map $\pi : X \to G_f(X)$ is continuous and so is extended to a continuous map $\widetilde{X} \to G_f(X)$ (denoted also by π). Identifying $G_f(X)$ with its image under $\widehat{E}$, see (2.21), we obtain that PL is the image under π of the sub-semigroup $\widetilde{X}_{PL}$ of $\widetilde{X}$ generated by elements $c = (c_1, c_2, \dots)$, $c_i \in \mathbb{C}$, $i \in \mathbb{N}$. In turn, first integrals of elements of this sub-semigroup are piecewise linear paths in $\mathbb{C}^\infty$. This motivates the above definition.

The group PL contains the subgroup of rectangular paths $G(X_{\mathrm{rect}})$, see Section 2.4.2. In particular, PL is a dense subgroup of $G_f(X)$, see also Proposition 2.23. One can also show (using, e.g., Theorem 2.2) that PL is isomorphic to the free $\mathbb{R}$-product of groups $\mathbb{C}$ (i.e., the set of generators of this product has the cardinality of the continuum).

By $\widehat{\mathcal{C}}_{PL} := \widehat{\mathcal{C}}_f \cap PL$ we denote the *group of formal centers in PL*. Then $\widehat{\mathcal{C}}_{PL}$ is the image in $G_f(X)$ of the semigroup $\mathcal{C}_{PL} \subset \widetilde{X}_{PL}$ consisting of all elements $a = (a_1, a_2, \dots) \in \widetilde{X}_{PL}$ such that monodromies of the equations

$$H'(x) = \left(\sum_{i=1}^{\infty} a_i(x) D L^{i-1} t^i \right) H(x), \quad x \in I_T,$$

are trivial.

Further, recall that there exist a continuous embedding $A : \mathcal{L}_S \to L$, $L := \{c \in \mathcal{L}_{\mathrm{Lie}}:\ c = \sum_{i=1}^{\infty} c_i X_i t^i,\ c_i \in \mathbb{C},\ i \in \mathbb{N}\}$, defined by $A(DL^{n-1}t^n) := X_n t^n$, so that $\Psi \circ A = \mathrm{id}$, see Section 3.3.2. For elements $a, b \in L$ we set

$$s(a,b) := (A \circ \Psi \circ \log)(e^a \cdot e^b) \in L. \tag{3.36}$$

Assume that

$$a=\sum_{i=1}^{\infty} a_i X_i t^i, \quad b=\sum_{i=1}^{\infty} b_i X_i t^i, \quad a_i, b_i \in \mathbb{C},\ i \in \mathbb{N}.$$

Using the Campbell–Hausdorff formula we have

$$\log(e^a \cdot e^b)=\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n} s_{i_1,\dots,i_k}(a,b)[X_{i_1},[X_{i_2},[\,\cdots,[X_{i_{k-1}},X_{i_k}]\cdots]]]\biggr)t^n$$

where each $s_{i_1,\dots,i_k}$ is a universal polynomial with rational coefficients in variables $a_{i_1},\dots,a_{i_k}$, $b_{i_1},\dots,b_{i_k}$ such that $s_{i_1,\dots,i_k}(a_{i_1}^{i_1},\dots,a_{i_k}^{i_k},b_{i_1}^{i_1},\dots,b_{i_k}^{i_k})$ is a homogeneous polynomial of degree $i_1+\cdots+i_k$. Then from (3.23), (3.21) we obtain

$$s(a,b)=\sum_{n=1}^{\infty}\biggl(\sum_{i_1+\cdots+i_k=n}\gamma_{i_1,\dots,i_k}\cdot s_{i_1,\dots,i_k}(a,b)\biggr)X_n t^n. \tag{3.37}$$

In general, the complex vector space spanned by a,b and $s(a,b)$ is 3-dimensional.

Next, formula (3.17) implies that $\widehat{P}(e^{s(a,b)})=\widehat{P}(e^a)\circ\widehat{P}(e^b)$. In particular, $e^a\cdot e^b\cdot e^{-s(a,b)}\in\widehat{\mathcal{C}}_{PL}$.

Let us define a continuous embedding $T_{PL}: G[[r]]\to PL$ by the formula

$$T_{PL}:=\exp\circ A\circ\log\circ\Phi^{-1},$$

cf. (3.33). Then, $\widehat{P}\circ T_{PL}=\mathrm{id}$.

Proposition 3.14. (1) *The map $\widetilde{T}_{PL}: G[[r]]\times\widehat{\mathcal{C}}_{PL}\to PL$ defined by $\widetilde{T}_{PL}(s,g)=T_{PL}(s)\cdot g$ is a homeomorphism.*

(2) *$\widehat{\mathcal{C}}_{PL}$ is generated by elements $e^a\cdot e^b\cdot e^{-s(a,b)}$ for all possible $a,b\in L$.*

(3) *$\widehat{\mathcal{C}}_{PL}$ is a dense subgroup of $\widehat{\mathcal{C}}_f$.*

Proof. (1) The proof repeats literally the proof of Proposition 3.12(1).

(2) It follows easily from the definition of T_{PL} and $\widehat{P}$, see (3.17), that

$$T_{PL}\bigl(\widehat{P}(e^a\cdot e^b)\bigr)=e^{s(a,b)}, \quad a,b\in L. \tag{3.38}$$

Suppose that $g=g_1\cdots g_n\in\widehat{\mathcal{C}}_{PL}$ where $g_i=e^{a_i}$, $a_i\in L$, $1\leqslant i\leqslant n$. We set

$$f_i:=\widehat{P}(g_1\cdots g_i), \quad 1\leqslant i\leqslant n,$$

where by the definition $f_n=1$. We also set for brevity

$$c(h_1,h_2):=h_1\cdot h_2\cdot e^{-s(\log(h_1),\log(h_2))}, \quad h_1,h_2\in PL.$$

Then using (3.38) we obtain

$$g=c\bigl(T_{PL}(f_1),g_2\bigr)\cdot c\bigl(T_{PL}(f_2),g_3\bigr)\cdots c\bigl(T_{PL}(f_{n-1}),g_n\bigr)\cdot T_{PL}(f_n).$$

Since $T_{PL}(f_n)=1$, this implies the required statement.

(3) The proof repeats word-for-word the second part of the proof of Proposition 3.6 and is based on the fact that PL is dense in $G_f(X)$. □

Question 3. Are there nontrivial elements in the group $\widehat{\mathcal{C}}^n_{PL}:=\widehat{\mathcal{C}}^n_f\cap PL$ of piecewise linear centers in $\mathbb{C}^n$? Here $\widehat{\mathcal{C}}^n_f:=\widehat{\mathcal{C}}_f\cap G_f(X^n)$.

We will return to this question in a forthcoming paper.

References

[1] H. Alexander, Polynomial approximation and hulls in sets of finite linear measure in $\mathbb{C}^n$, Amer. J. Math. 93 (1971) 65–74.
[2] M.A.M. Alwash, N.G. Lloyd, Non-autonomous equations related to polynomial two-dimensional systems, Proc. Roy. Soc. Edinburgh Sect. A 105 (1987) 129–152.
[3] H. Alexander, J. Wermer, Several Complex Variables and Banach Algebras, Springer-Verlag, New York, 1998.
[4] M. Briskin, J.-P. Francoise, Y. Yomdin, Center condition II: Parametric and model center problems, Israel J. Math. 118 (2000) 61–82.
[5] M. Briskin, J.-P. Francoise, Y. Yomdin, Center condition III: Parametric and model center problems, Israel J. Math. 118 (2000) 83–108.
[6] A. Brudnyi, An explicit expression for the first return map in the center problem, J. Differential Equations 206 (2004) 306–314.
[7] A. Brudnyi, On the center problem for ordinary differential equations, Amer. J. Math. 128 (2006) 419–451.
[8] A. Brudnyi, An algebraic model for the center problem, Bull. Sci. Math. 128 (2004) 839–857.
[9] A. Brudnyi, On center sets of ODEs determined by moments of their coefficients, Bull. Sci. Math. 130 (2006) 33–48.
[10] A. Brudnyi, Vanishing of higher-order moments on Lipschitz curves, Bull. Sci. Math. 132 (3) (2008) 165–181.
[11] K.-T. Chen, Iterated integrals and exponential homomorphisms, Proc. London Math. Soc. 4 (3) (1954) 502–512.
[12] K.-T. Chen, Integration of paths, geometric invariants and a generalized Baker–Hausdorff formula, Ann. of Math. 65 (1) (1957) 163–178.
[13] K.-T. Chen, Integration of paths—a faithful representation of paths by noncommutative formal power series, Trans. AMS 89 (1958) 8–36.
[14] H. Freudenthal, Über dimensionserhöhende stetige Abbildungen, S.-B. Preuss. Akad. Wiss. H. 5 (1932) 34–38.
[15] R.M. Hain, The geometry of the mixed Hodge structure on the fundamental group, Proc. Symp. Pure Math. 46 (1987) 247–282.
[16] D. Mumford, Algebraic Geometry I. Complex Projective Varieties, Springer-Verlag, New York, 1976.
[17] H. Munthe-Kaas, B. Owren, Computations in a free Lie algebra, Phil. Trans. Royal Soc. A 357 (1999) 957–981.
[18] R. Ree, Lie elements and an algebra associated with shuffles, Ann. of Math. (2) 68 (1958) 550–561.
[19] J. Wermer, The hull of a curve in $\mathbb{C}^n$, Ann. of Math. (2) 68 (1958) 550–561.
[20] Y. Yomdin, Center problem for the Abel equation, compositions of functions and moment conditions, Moscow Math. J. 3 (3) (2003) 1167–1195.