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Parameter identification in stationary groundwater flow problems in drainage basins

B. Malengier

Department of Mathematical Analysis, Research Group NfaM[2], Ghent University, Galglaan 2, B-9000 Ghent, Belgium

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Abstract

We discuss the recovery of some parameters in an elliptic boundary value problem, which models a specific stationary flow problem in drainage basins, by using the measured values of the hydraulic head in discrete points through the physical domain. The underlying *direct* problem is the one considered by Tòth (J. Geophys. Res. 67 (1962) 4375; 67 (1963) 4795), among others. The *inverse* problem is solved by means of the Levenberg–Marquardt method (in: G.A. Watson (Ed.), Numerical Analysis, Lecture Notes in Mathematics, Vol. 630, Springer, Berlin, New York, 1977, pp. 105–116). We also show how the infinite element method allows the identification of the far field value of the hydraulic head.

MSC: 76; 65

Keywords: Groundwater flow; Parameter identification; Drainage basin

1. Introduction

In this paper, we deal with the problem of parameter identification in stationary groundwater flow in small drainage basins in the absence of sources. The *direct* problem, i.e., the boundary value problem with given data, has been presented first in [6] and theoretically analysed in [7]. The basin has vertical impenetrable boundaries because of symmetry considerations. The longitudinal component can be neglected as in most small basins the slopes of the valley flanks greatly exceed the longitudinal slopes of the valley floors. Therefore, a two-dimensional model can be adopted

E-mail address: bm@cage.rug.ac.be (B. Malengier).

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in (x,z) coordinates, with x the horizontal coordinate along the valley flank and z the depth. The basin furthermore has the special property that the water table follows the surface, which is possible when the aquifer has a low conductivity and there is abundant rainfall. Finally, the domain is bounded on top by a sloping sinusoidal curve. Tòth [6,7], studied the problem for the hydraulic head $\phi(x,z)$ in a finite, saturated and isotropic region with a constant hydraulic conductivity. He approximates the problem by reducing the domain to a rectangle with the given top boundary values projected onto the top of the rectangle. In [5], Shivakumar et al. have taken the top boundary condition into account exactly, but assumed the region to be semi-infinite in the z direction. The hydraulic conductivity is taken to be decreasing exponentially with depth, i.e., $K(z) = c\mathrm{e}^{dz}$, which is supported by some experimental data. Their assumption of the region to be semi-infinite gives reliable results for deep basins but not for the standard basin of depth from 600 up to 1000 ft as studied by Tòth.

To solve the *direct* problem, in [3] we first extended the *semi-analytical* method of [5], allowing for both a nonpermeable base (homegeneous Neumann boundary condition) and a given profile of the hydraulic head at the basis (inhomogeneous Dirichlet BC). The procedure can be extended to the case of other curved boundaries on top than the sinusoidal one by means of numerical integration. Moreover, we also implemented a standard Galerkin finite element method. Furthermore, an *infinite element method* was developed to reduce the CPU-time needed when dealing with deep drainage basins.

In the present paper, we first discuss the recovery of unknown parameters entering the boundary value problem considered, viz. the coefficient d, appearing in the hydraulic conductivity, and a parameter in the prescribed profile of the hydraulic head at the basis. To this end the measured values of the hydraulic head at a moderate number of discrete points throughout the physical domain are used. To solve these *inverse problems*, we will use the Levenberg–Marquardt method, which, of course, leans upon solving a small number of *direct* problems.

In this paper, we also show how the infinite element method allows for the identification of the far field value of the hydraulic head.

2. The direct problem

2.1. Mathematical model for a region with sinusoidal top

As in [7,5,3], we consider the following governing differential equation for the hydraulic head $\phi(x,z)$ in the stationary regime in the absence of sources:

$$\nabla \cdot (\mathrm{e}^{dz}\nabla\phi(x,z)) = 0 \quad \text{in } \Omega. \tag{1}$$

The hydraulic conductivity is $K = c\mathrm{e}^{dz}$ (c a positive constant, $d \geqslant 0$). The domain Ω under consideration is given by (see Fig. 1)

$$0 < x < L \quad \text{and} \; -T < z < g(x) \equiv -\left[\frac{ax}{L} + V\sin\left(\frac{2\pi n x}{L}\right)\right], \tag{2}$$

where $L > 0$, $T > 0$, $a \geqslant 0$, and V are constants and n is a fixed positive integer.

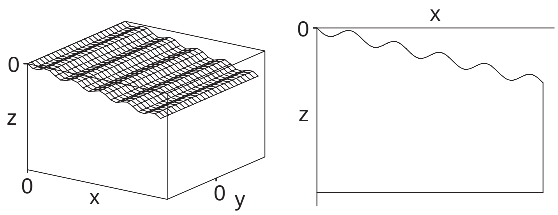

Fig. 1. The domain.

The boundary conditions are given by

$$\frac{\partial \phi}{\partial x}\Big|_{x=0} = \frac{\partial \phi}{\partial x}\,|_{x=L} = 0 \ , \tag{3}$$

$$\phi(x,z) = z \quad \text{on} \quad z = g(x) \quad 0 < x < L \tag{4}$$

and

$$\frac{\partial \phi}{\partial z}\,|_{z=-T} = 0 \tag{5}$$

or

$$\phi(x,z)|_{z=-T} = f(x). \tag{6}$$

Here, as in [5], g is defined by (2). Moreover, f is a piecewise smooth function on $[0,L]$. We recall that in [5], the depth T of the soil layer is taken to be infinity, the boundary condition being that ϕ is bounded for $z \to -\infty$.

2.2. Numerical algorithms and examples

As shown in [3], for the boundary value problem above a semi-analytical approximation can be developed and implemented in a standard mathematical package. The procedure for a sinusoidal top, $z = g(x)$, g given by (2), can be extended to other surfaces by means of numerical integration.

A more straightforward approach is to construct a finite element algorithm (FEM). This method is general and fast, and has no restrictions on the geometry of the domain. For the FEM we consider a regular triangulation of the domain, in which the curved boundary is approximated by piecewise linear functions.

The results of the FEM are found to be in full agreement with those of the semi-analytical procedure. Actually, the equipotential lines obtained with both methods are nearly identical.

However, the FEM does require more CPU-time with increasing depth of the basin. Moreover, it cannot be applied to the BVP in [5] on a semi-infinite region (i.e., $T = -\infty$). In [3] we cope with this difficulty by setting up an infinite element method. The hydraulic head is obtained for a semi-infinite region by using a small number of elements in the mesh, leading to minimal CPU-time.

We give some numerical results. In Fig. 2 we present to the left a result with the same data as in [7], i.e., for a Neumann BC at the base. The equipotential lines are in full agreement with those of [7]. They are obtained over the entire domain, in contrast with [7]. Note that the flow lines of the groundwater flow are perpendicular to these equipotential lines. In the left part of Fig. 2 we have regional flow, that is flow from the highest part towards the lowest part of the region.

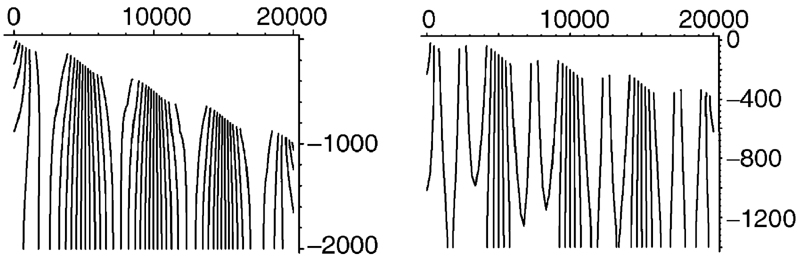

Fig. 2. Equipotential lines (in the xz-cross section) in case of a Neumann BC. Data in left part: $a/L = 0.05$, $V = 50$, $d = 0$, $n = 4$, $L = 20{,}000$, $T = 2000$. Data in right part: $a/L = 0.02$, $V = 50$, $d = 0.00235$, $n = 4$, $L = 20{,}000$, $T = 1400$.

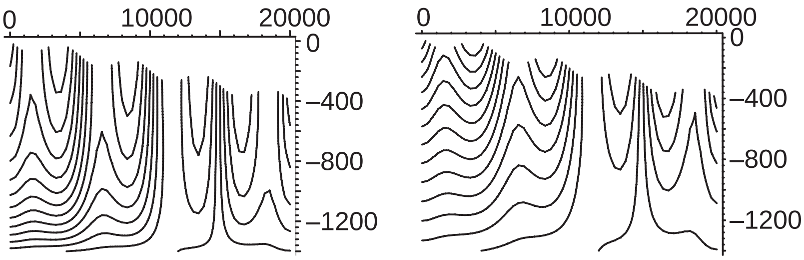

Fig. 3. Equipotential lines (in the xz-cross section) in case of a Dirichlet BC at the base for $a/L = 0.02$, $V = 50$, $n = 4$, $L = 20{,}000$, $T = 1400$, $u = -50/L$, $v = -250$. Left: $d = 0.00235$. Right: $d = 0.0$.

We may also present results in a shallow basin with decreasing conductivity and an impermeable base. In the right part of Fig. 2 we show the equipotential lines for $d = 0.00235$.

Next, we consider a flow problem with a Dirichlet BC at the base. We take the function f appearing in (6) to be

$$f(x) = ux + v, \quad u \text{ and } v \text{ constant.} \tag{7}$$

This BC can be interpreted as corresponding to an underlying, highly conductive aquifer. The function f then represents the Dupuit–Forcheimer flow (see [2]) in this aquifer. The resulting equipotential lines are depicted in Fig. 3 for 2 specific choices of the data.

When $T \gg a/2$, the numerical results for the equipotential lines are found to be in good agreement with those from [5], as it should.

3. Parameter identification

3.1. The far field hydraulic head

In the infinite element algorithm (IFEM), see e.g. [1] or [8], the identification of the hydraulic head at great depth comes out naturally. Comparison with the FEM shows that both results are in close agreement when $T \gg a/2$.

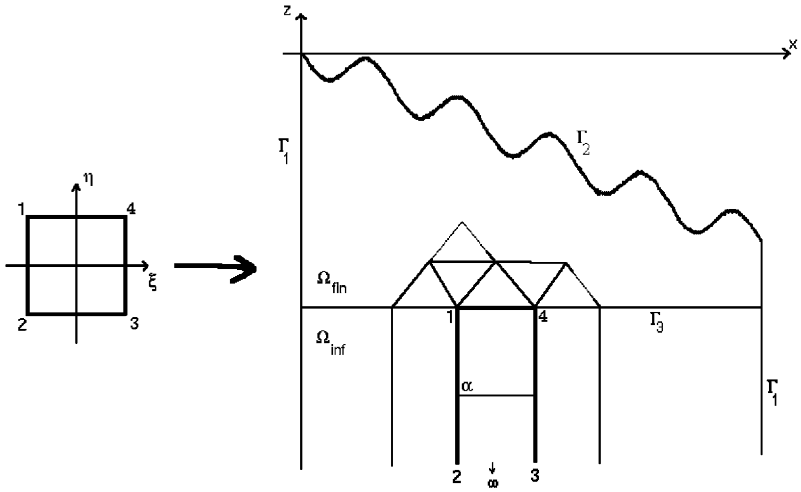

Fig. 4. The mapping of the infinite rectangular elements.

The IFEM is developed with a constant but unknown far field value of the hydraulic head for $z \to -\infty$, which replaces the boundedness assumption in [5].

Let Ω be the semi-infinite region shown in Fig. 4. The boundary parts are denoted by Γ_1 and Γ_2 with $\Gamma_1 \cap \Gamma_2 = \emptyset$. For simplicity we assume that Γ_1 are vertical lines extending to $-\infty$. The boundary conditions are

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma_1,$$

$$\phi = g \quad \text{on} \quad \Gamma_2, \phi \to c_{\mathrm{ff}} \quad \text{as} \quad z \to -\infty, c_{\mathrm{ff}} \,\text{constant},$$

where g is a given sufficiently smooth function defined on Γ_2, where Γ_2 is assumed to be Lipschitz continuous. The far field value of the hydraulic head, c_{ff}, is unknown.

We construct a mesh for $\bar{\Omega}_h$. Here, $\bar{\Omega}_h$ is a semi-infinite polygonal domain that approximates Ω, with boundary $\bar{\Gamma}_1 \cup \bar{\Gamma}_{2h}$, where the polygonal line Γ_{2h} piecewise linearly interpolates the curved boundary Γ_2 of Ω. The domain Ω_h is splitted into a bounded part Ω_{fin} and an unbounded part Ω_{inf} by means of the horizontal line Γ_3, $z = -T$. For Ω_{fin} we consider a regular triangulation τ_h, while on Ω_{inf} we consider a mesh ρ_h of semi-infinite rectangles matching perfectly with the elements of τ_h, see Fig. 4. For the IFEM we take globally continuous, elementwise polynomials of degree 1 on Ω_{fin}. On Ω_{inf} we use globally continuous functions, which are defined in a generic semi-infinite rectangular element K by the images of the bilinear polynomials on the master square $\hat{K}$ under a suitable mapping of $\hat{K}$ onto K. The details of the construction of a proper infinite element space and its cardinal basis can be found in [3]. In the numerical method we use the piecewise linear Lagrange-interpolant g_h of g on $\bar{\Gamma}_{2h}$.

The results of the IFEM are found to be in full agreement with those from [5]. If we set for example $L = 8000$, $a/L = 0.1$, $V = 80$, $d = 0.00235$, $T = 3000$ and $\alpha = -12,000$, and divide the domain in 3025 triangles and 40 semi-infinite rectangles, we obtain a far field value $c_{\mathrm{ff}} = -411.971$, which corresponds with the far field value that we obtained with the semi-analytical method from [5], namely $c_{\mathrm{ff}} = -411.868$.

3.2. The inverse method

In the direct problem above, all parameters where given: the constant d in (1) or the coefficients u and v of the function $f(x)$ in (7) or the coefficients of a higher order form of $f(x)$. In the inverse problem some or all of these data are not given a priori. We denote by $p=(p_1,\dots,p_n)$ the parameter vector of the unknown parameters which we want to recover from the measured hydraulic head ϕ_i^* at N discrete points (x_i,z_i). Denote by ϕ_p the hydraulic head obtained from the direct problem with parameter vector p. Furthermore, σ_i is the relative or absolute standard deviation of the measurement in (x_i,z_i). Then, in the inverse problem we look for a parameter vector p such that the *penalty functional*

$$\Phi(p)=\sum_{i=1}^{N}\left(\frac{\phi_p(x_i,z_i)-\phi_i^*}{\sigma_i}\right)^2 \tag{8}$$

reaches a minimum.

We solve this inverse problem by using the Levenberg–Marquardt method, which is known to be fairly general and reliable. It includes the Newton–Gauss method as a special case, see [4]. For this method we need the gradient $\nabla_p\Phi$, which reduces to calculating $\nabla_p\phi_p(x_i,z_i)$. This gradient can be obtained numerically by solving the direct problem several times. If we start from an initial parameter set p^k, the improved vector p^{k+1} is given by

$$(J_k^{\mathrm{T}}J_k+\lambda I)(p^{k+1}-p^k)=J_k^{\mathrm{T}}F_k. \tag{9}$$

Here, I is the unity matrix of order n, J_k^{T} is the transposed matrix of the matrix J_k defined by

$$(J_k)_{i,j}=\partial_{p_j}\phi_{p^k}(x_i,z_i).$$

Furthermore, F_k is the column matrix defined by

$$(F_k)_i=\frac{\phi_{p^k}(x_i,z_i)-\phi_i^*}{\sigma_i}$$

and λ is a multiplicator.

To minimize the penalty functional we start from an initial parameter set p_0 and an initial value λ_0 for the multiplicator (we take $\lambda_0=\mathrm{Tr}(J_k^{\mathrm{T}}J_k)/n$). As long as the penalty functional decreases with consecutive steps, we decrease λ. This enlarges the stepsize. If the penalty functional did not decrease from step k to $k+1$, we calculate a new p^{k+1} for a larger value of λ. We stop the method when $\lambda>\lambda_{\max}$. We recall that the Newton–Gauss method follows from the Levenberg–Marquardt method by setting $\lambda=0$. The inverse method works equally well for the FEM or IFEM. In the following, only experiments with the FEM are presented.

3.2.1. Numerical experiment: determination of d

Let us consider the extraction of the parameter vector $p=(d)$ from measured data. We generate the measurements from the solution of the direct problem with $a/L=0.02$, $V=50$, $n=4$, $L=20,000$, $T=1400$, and a known d, which are then perturbated with artificial noise (normally distributed with a relative or absolute standard deviation). This will prove the stability of the solution of the ill-posed inverse problem.

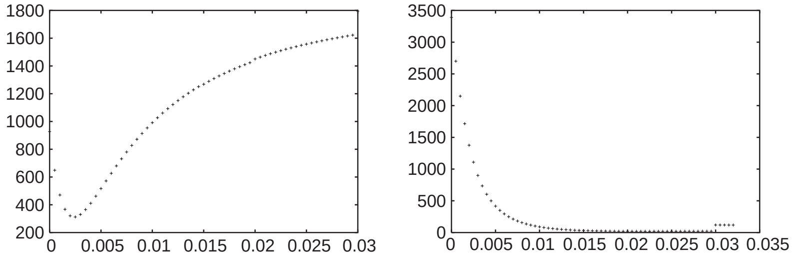

Fig. 5. Penalty functional for $a/L = 0.02$, $V = 50$, $n = 4$, $L = 20,000$, $T = 1400$. Left: initial $d = 0.00235$, recovered $d = 0.00238$. Right: initial $d = 0.0235$, recovered $d = 0.0279$.

We take 15 measurements over the entire length of the basin. In Fig. 5 the penalty functional is plotted against the d-parameter. In the left figure we started from the direct problem with $d = 0.00235$ and an absolute deviation of 5. We retrieved $d = 0.00238$ as extracted value. In the right figure we started from $d = 0.0235$ and a relative deviation of 5%. We retrieved $d = 0.0279$. This is an acceptable result, since for larger d-values, a relatively big deviation does not influence much the solution. This can be interfered from the penalty functional which is very flat around the minimum. For small d-values this is not the case, and we recover the unknown parameter almost exactly. In the absence of noise, the parameters are recovered exactly. From the shapes of the penalty functionals in Fig. 5 it is clear that other inverse methods, like Newton–Gauss, will also converge fastly.

3.2.2. Numerical experiment: areal recharge determination

The FEM can of course be applied to general domains, without the specific sinusoidal top considered in [7] among others. We set up the following experiment. Consider two parallel rivers (at $x = 0$ and L), with an elevation in between. If the slope of the elevation flanks greatly exceeds the longitudinal slopes of the river floors and the longitudinal slope of the elevation top, this longitudinal component can be neglected. Therefore, also in this case a two-dimensional scheme can be adopted, with x the horizontal coordinate along the elevation flanks and z the depth. We may consider the rivers to be no-flow boundaries, similarly as in [6,7]. As before, the groundwater level follows the surface. Under the considered domain, we have an aquifer. This aquifer will undergo an areal recharge due to the overlying basin. Based on the Dupuit–Forcheimer model, we take the bottom boundary condition (6) to be

$$f(x) = -\frac{N}{2}x(x - L) + \frac{\phi_{x=L} - \phi_{x=0}}{L}x + \phi_{x=0} \tag{10}$$

with $\phi_{x=0}$ and $\phi_{x=L}$ the hydraulic head of the aquifer under the rivers, and with N being the areal recharge (dimension L/T), see [2].

For the experiment, we take the elevation as depicted in Fig. 6.

Consider the recovery of the parameter vector $p = (d, N)$ from measured data, taken not more than 50 m under the surface. We generate the measurements from the solution of the direct problem with $\phi_{x=0} = -290$ m, $\phi_{x=L} = -300$ m, $L = 5500$ m, $d = 0.008$ and $N = 0.05/L$. This means that the areal recharge over the length L totalizes 0.05 m/s.

We use 15 measurements with an absolute deviation of 2 m. In Fig. 7 the penalty functional is plotted versus the d and N-parameter. From the Levenberg–Marquardt method we retrieve that

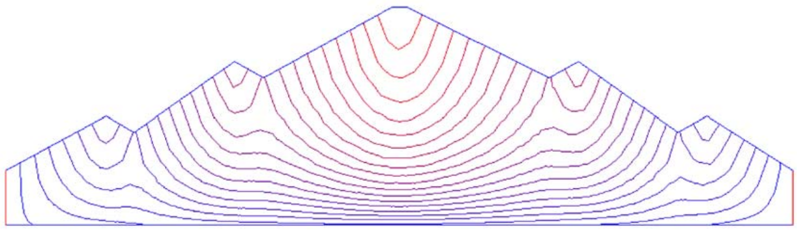

Fig. 6. Topology elevation between two rivers. Top of elevation is at $z = 0$, $L = 5500$ m, underlying aquifer at $z = -400$ m. Equipotential lines for $\phi_{x=0} = -290$ m, $\phi_{x=L} = -300$ m, $N = 0.05/L$ and $d = 0.008$.

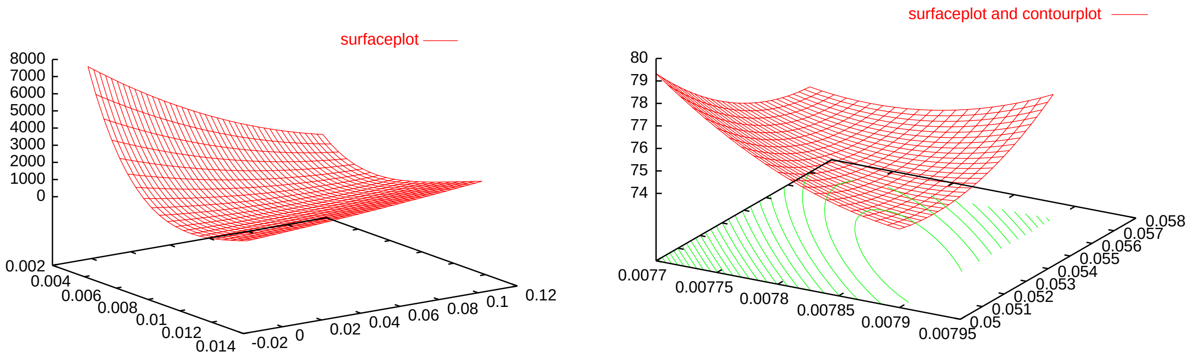

Fig. 7. Penalty functional for the inverse problem from Fig. 6. Right a detail is given around the minimum reached for $d = 0.00786$, $N = 0.0528$, with a contourplot projection.

$d = 0.00786$ and $N = 0.0528$. From Fig. 7 it is clear that the Newton–Gauss method will also give a good result. We may expect a priori that first the d-parameter is recovered, and next the areal recharge, as may be seen from the left part of Fig. 7. This is confirmed by the experiment.

4. Conclusion

The infinite element method allows for an identification of the far field value of the hydraulic head. Other missing parameters can be recovered from measured values of the hydraulic head by using the Levenberg–Marquardt method for the inverse problem based on an accurate direct solver.

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