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The Łojasiewicz exponent of a set of weighted homogeneous ideals

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ABSTRACT

We give an expression for the Łojasiewicz exponent of a set of ideals which are pieces of a weighted homogeneous filtration. We also study the application of this formula to the computation of the Łojasiewicz exponent of the gradient of a semi-weighted homogeneous function $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin.

1. Introduction

Let R be a Noetherian ring and let I be an ideal of R. Let ν_I be the order function of R with respect to I, that is, $\nu_I(h) = \sup\{r : h \in I^r\}$, for all $h \in R$, $h \neq 0$, and $\nu(0) = \infty$. Let us consider the function $\overline{\nu}_I : R \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ defined by $\overline{\nu}_I(h) = \lim_{s\to\infty} \frac{\nu_I(h^s)}{s}$, for all $h \in R$. It was proven by Samuel [17] and Rees [14] that this limit exists and Nagata proved in [12] that, when finite, the number $\overline{\nu}_I(h)$ is a rational number. The function $\overline{\nu}$ is called the *asymptotic Samuel function of* I. If J is another ideal of R, then the number $\overline{\nu}_I(J)$ is defined analogously and if $h_1, \ldots, h_r$ is a generating system of J then $\overline{\nu}_I(J) = \min\{\overline{\nu}_I(h_1), \ldots, \overline{\nu}_I(h_r)\}$. Let us denote by $\overline{I}$ the integral closure of I. As a consequence of the theorem of existence of the Rees valuations of an ideal (see for instance [8, p. 192]), it is known that, if J is another ideal and $p, q \in \mathbb{Z}_{\geqslant 1}$, then $J^q \subseteq \overline{I^p}$ if and only if $\overline{\nu}_I(J) \geqslant \frac{p}{q}$.

Let $\mathcal{O}_n$ denote the ring of analytic function germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and let m_n denote its maximal ideal, that will be also denoted by m if no confusion arises. Let I be an ideal of $\mathcal{O}_n$ of finite colength. Lejeune and Teissier proved in [10, p. 832] that $\frac{1}{\overline{\nu}_I(m)}$ is equal to the Łojasiewicz exponent of I (in fact, this result was proven in a more general context, that is, for ideals in a structural ring $\mathcal{O}_X$, where X is a reduced complex analytic space). If $g_1, \ldots, g_r$ is a generating system of I, then the *Łojasiewicz exponent of* I is defined as the infimum of those $\alpha > 0$ for which there exist a constant $C > 0$ and an open neighbourhood U of $0 \in \mathbb{C}^n$ with

$$\|x\|^\alpha \leqslant C \sup_i |g_i(x)|$$

for all $x \in U$. Let us denote this number by $\mathcal{L}_0(I)$ and let $e(I)$ denote the Samuel multiplicity of I. Therefore we have that $\mathcal{L}_0(I) = \inf\{\frac{p}{q} : m^p \subseteq \overline{I^q},\ p, q \in \mathbb{Z}_{>0}\}$ and hence, by the Rees multiplicity theorem (see [8, p. 222]) it follows that $\mathcal{L}_0(I) = \inf\{\frac{p}{q} : e(I^q) = e(I^q + m^p),\ p, q \in \mathbb{Z}_{>0}\}$. This expression of $\mathcal{L}_0(I)$ is one of the motivations that led the first author to introduce the notion of Łojasiewicz exponent of a set of ideals in [4]. This notion is based on the Rees mixed multiplicity of a set of ideals (Definition 2.1).

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Łojasiewicz exponents have important applications in singularity theory. Here we recall one of them. If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is an analytic map germ such that $g^{-1}(0) = \{0\}$ then we denote by $\mathcal{L}_0(g)$ the Łojasiewicz exponent of the ideal generated by the component functions of g. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated singularity at the origin. Then $\nabla f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denotes the gradient map of f, that is, $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. The *Jacobian ideal of* f, that we will denote by $J(f)$, is the ideal generated by the components of ∇f. The *degree of* C^0*-determinacy of* f, denoted by $s_0(f)$, is defined as the smallest integer r such that f is topologically equivalent to $f + g$, for all $g \in \mathcal{O}_n$ with $\nu_{m_n}(g) \geqslant r + 1$. Teissier proved in [19, p. 280] that $s_0(f) = [\mathcal{L}_0(\nabla f)] + 1$, where $[a]$ stands for the integer part of a given $a \in \mathbb{R}$. Despite the fact that this equality connects $\mathcal{L}_0(\nabla f)$ with a fundamental topological aspect of f, the problem of determining whether the Łojasiewicz exponent $\mathcal{L}_0(\nabla f)$ is a topological invariant of f is still an open problem.

The effective computation of $\mathcal{L}_0(I)$ has proven to be a challenging problem in algebraic geometry that, by virtue of the results of Lejeune and Teissier is directly related with the computation of the integral closure of an ideal. In [5] the authors relate the problem of computing $\mathcal{L}_0(I)$ with the algorithms of resolution of singularities. The approach that we give in this paper is based on techniques of commutative algebra.

We recall that, if $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$, then a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called weighted homogeneous of degree d with respect to w when f is written as a sum of monomials $x_1^{k_1} \cdots x_n^{k_n}$ such that $w_1x_1 + \cdots + w_nx_n = d$. This paper is motivated by the main result of Krasiński et al. in [9], which says that if $f : \mathbb{C}^3 \to \mathbb{C}$ is a weighted homogeneous polynomial of degree d with respect to (w_1, w_2, w_3) with an isolated singularity at the origin, then $\mathcal{L}_0(\nabla f)$ is given by the expression

$$\mathcal{L}_0(\nabla f) = \frac{d - \min\{w_1, w_2, w_3\}}{\min\{w_1, w_2, w_3\}}$$

provided that $d \geqslant 2w_i$, for all $i = 1, 2, 3$. That is, $\mathcal{L}_0(\nabla f)$ depends only on the weights w_i and the degree d in this case. Therefore it is concluded that $\mathcal{L}_0(\nabla f)$ is a topological invariant of f, by virtue of the results of [16,21]. In view of the above equality it is reasonable to conjecture that the analogous result holds in general, that is, if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous polynomial, or even a semi-weighted homogeneous function (see Definition 4.1), with respect to $(w_1, \dots, w_n)$ of degree d with an isolated singularity at the origin, and if $d \geqslant 2w_i$, for all $i = 1, \dots, n$, then

$$\mathcal{L}_0(\nabla f) = \frac{d - \min\{w_1, \dots, w_n\}}{\min\{w_1, \dots, w_n\}}. \tag{1}$$

We point out that inequality ($\leqslant$) always holds in (1) for semi-weighted homogeneous functions (see Corollary 4.11).

In this paper we obtain the equality (1) for semi-weighted homogeneous germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ under a restriction expressed in terms of the supports of the component functions of ∇f (see Corollary 4.11). This result arises as a consequence of a more general result involving the Łojasiewicz exponent of a set of ideals coming from a weighted homogeneous filtration (see Theorem 4.7). Our approach to Łojasiewicz exponents is purely algebraic and comes from the techniques developed in [3,4]. This new point of view of the subject has led us to detect a broad class of semi-weighted homogeneous functions where relation (1) holds.

For the sake of completeness we recall in Section 2 the definition of the Rees mixed multiplicity and basic facts about this notion. In Section 3 we show some results about the notion of Łojasiewicz exponent of a set of ideals that will be applied in Section 4. The main results appear in Section 4.

2. The Rees mixed multiplicity of a set of ideals

Let (R, m) be a Noetherian local ring and let I be an ideal of R. We denote by $e(I)$ the Samuel multiplicity of I. Let $\dim R = n$ and let us fix a set of n ideals $I_1, \dots, I_n$ of R of finite colength. Then we denote by $e(I_1, \dots, I_n)$ the mixed multiplicity of $I_1, \dots, I_n$, as defined by Teissier and Risler in [20] (we refer to [8, Section 17] and [18] for fundamental results about mixed multiplicities of ideals). We recall that, if the ideals $I_1, \dots, I_n$ are equal to a given ideal, say I, then $e(I_1, \dots, I_n) = e(I)$.

Let us suppose that the residue field $k = R/m$ is infinite. Let $a_{i1}, \dots, a_{is_i}$ be a generating system of I_i, where $s_i \geqslant 1$, for $i = 1, \dots, n$. Let $s = s_1 + \cdots + s_n$. We say that a property holds for *sufficiently general* elements of $I_1 \oplus \cdots \oplus I_n$ if there exists a non-empty Zariski-open set U in k^s verifying that the said property holds for all elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$ such that $g_i = \sum_j u_{ij}a_{ij}$, $i = 1, \dots, n$ and the image of $(u_{11}, \dots, u_{1s_1}, \dots, u_{n1}, \dots, u_{ns_n})$ in k^s lies in U.

By virtue of a result of Rees (see [15] or [8, p. 335]), if the ideals $I_1, \dots, I_n$ have finite colength and R/m is infinite, then the mixed multiplicity of $I_1, \dots, I_n$ is obtained as $e(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, for a sufficiently general element $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.

Let us denote by $\mathcal{O}_n$ the ring of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map germ such that $g^{-1}(0) = \{0\}$ and let $g_1, \dots, g_n$ denote the component functions of g. We recall that $e(I) = \dim_{\mathbb{C}} \mathcal{O}_n/I$, where I is the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. It turns out that this number is equal to the geometric multiplicity of g (see [11, p. 258] or [13]).

Now we show the definition of a number associated to a family of ideals that generalizes the notion of mixed multiplicity. This number is fundamental in the results of this paper.

We denote by $\mathbb{Z}_+$ the set of non-negative integers. Let $a \in \mathbb{Z}$, we denote by $\mathbb{Z}_{\geqslant a}$ the set of integers $z \geqslant a$.

Definition 2.1 ([3]). Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \ldots, I_n$ be ideals of R. Then we define the *Rees mixed multiplicity of* $I_1, \ldots, I_n$ *as*

$$\sigma(I_1, \ldots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + m^r, \ldots, I_n + m^r), \tag{2}$$

when the number on the right hand side is finite. If the set of integers $\{e(I_1 + m^r, \ldots, I_n + m^r) : r \in \mathbb{Z}_+\}$ is non-bounded then we set $\sigma(I_1, \ldots, I_n) = \infty$.

We remark that if I_i is an ideal of finite colength, for all $i = 1, \ldots, n$, then $\sigma(I_1, \ldots, I_n) = e(I_1, \ldots, I_n)$. The next proposition characterizes the finiteness of $\sigma(I_1, \ldots, I_n)$.

Proposition 2.2 ([3, p. 393]). *Let $I_1, \ldots, I_n$ be ideals of a Noetherian local ring (R, m) such that the residue field $k = R/m$ is infinite. Then $\sigma(I_1, \ldots, I_n) < \infty$ if and only if there exist elements $g_i \in I_i$, for $i = 1, \ldots, n$, such that $\langle g_1, \ldots, g_n \rangle$ has finite colength. In this case, we have that $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$ for sufficiently general elements $(g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

Remark 2.3. It is worth pointing out that, if $I_1, \ldots, I_n$ is a set of ideals of R such that $\sigma(I_1, \ldots, I_n) < \infty$, then $I_1 + \cdots + I_n$ is an ideal of finite colength. Obviously the converse is not true.

The following result will be useful in subsequent sections.

Lemma 2.4 ([4, p. 392]). *Let (R, m) be a Noetherian local ring of dimension $n \geqslant 1$. Let $J_1, \ldots, J_n$ be ideals of R such that $\sigma(J_1, \ldots, J_n) < \infty$. Let $I_1, \ldots, I_n$ be ideals of R such that $J_i \subseteq I_i$, for all $i = 1, \ldots, n$. Then $\sigma(I_1, \ldots, I_n) < \infty$ and*

$$\sigma(J_1, \ldots, J_n) \geqslant \sigma(I_1, \ldots, I_n).$$

Now we recall some basic definitions. Let us fix a coordinate system $x_1, \ldots, x_n$ in $\mathbb{C}^n$. If $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, we will denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k. If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ denotes the Taylor expansion of h around the origin, then the *support of* h is the set $\mathrm{supp}(h) = \{k \in \mathbb{Z}^n_+ : a_k \neq 0\}$. If $h \neq 0$, the *Newton polyhedron of* h, denoted by $\Gamma_+(h)$, is the convex hull of the set $\{k + v : k \in \mathrm{supp}(h), v \in \mathbb{R}^n_+\}$. If $h = 0$, then we set $\Gamma_+(h) = \emptyset$. If I is an ideal of $\mathcal{O}_n$ and $g_1, \ldots, g_s$ is a generating system of I, then we define the *Newton polyhedron of* I as the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_r)$. It is easy to check that the definition of $\Gamma_+(I)$ does not depend on the chosen generating system of I. We say that I is a *monomial ideal* of $\mathcal{O}_n$ when I admits a generating system formed by monomials.

Definition 2.5. Let $I_1, \ldots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \ldots, I_n) < \infty$. Then we denote by $\mathcal{S}(I_1, \ldots, I_n)$ the family of those maps $g = (g_1, \ldots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $g^{-1}(0) = \{0\}$, $g_i \in I_i$, for all $i = 1, \ldots, n$, and $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$, where $e(g_1, \ldots, g_n)$ stands for the multiplicity of the ideal of $\mathcal{O}_n$ generated by $g_1, \ldots, g_n$. The elements of $\mathcal{S}(I_1, \ldots, I_n)$ are characterized in [3, Theorem 3.10].

We denote by $\mathcal{S}_0(I_1, \ldots, I_n)$ the set formed by the maps $g = (g_1, \ldots, g_n) \in \mathcal{S}(I_1, \ldots, I_n)$ such that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \ldots, n$.

3. The Łojasiewicz exponent of a set of ideals

In this section we introduce some results concerning the notion of Łojasiewicz exponent of a set of ideals in a Noetherian ring. These results will be applied in the next section.

Let $I_1, \ldots, I_n$ be ideals of a local ring (R, m) such that $\sigma(I_1, \ldots, I_n) < \infty$. Then we define

$$r(I_1, \ldots, I_n) = \min \left\{ r \in \mathbb{Z}_+ : \sigma(I_1, \ldots, I_n) = e(I_1 + m^r, \ldots, I_n + m^r) \right\}. \tag{3}$$

Theorem 3.1 ([4, p. 398]). *Let $I_1, \ldots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \ldots, I_n)$ is finite. If $g \in \mathcal{S}_0(I_1, \ldots, I_n)$, then $\mathcal{L}_0(g)$ depends only on $I_1, \ldots, I_n$ and it is given by*

$$\mathcal{L}_0(g) = \min_{s \geqslant 1} \frac{r(I_1^s, \ldots, I_n^s)}{s}. \tag{4}$$

By the proof of the above theorem it is concluded that the infimum of the sequence $\{\frac{r(I_1^s, \ldots, I_n^s)}{s}\}_{s \geqslant 1}$ is actually a minimum. Theorem 3.1 motivates the following definition.

Definition 3.2. Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \ldots, I_n$ be ideals of R. Let us suppose that $\sigma(I_1, \ldots, I_n) < \infty$. We define the *Łojasiewicz exponent of* $I_1, \ldots, I_n$ as

$$\mathcal{L}_0(I_1, \ldots, I_n) = \inf_{s \geqslant 1} \frac{r(I_1^s, \ldots, I_n^s)}{s}.$$

As we will see in Lemma 3.3, we have that $r(I_1^s, \dots, I_n^s) \leqslant sr(I_1, \dots, I_n)$, for all $s \in \mathbb{Z}_{\geqslant 1}$. Therefore $\mathcal{L}_0(I_1, \dots, I_n) \leqslant r(I_1, \dots, I_n)$.

We can extend Definition 2.1 by replacing the maximal ideal m by an arbitrary ideal of finite colength, but the resulting number is the same. That is, under the hypothesis of Definition 2.1, let us denote by J an ideal of R of finite colength and let us suppose that $\sigma(I_1, \dots, I_n) < \infty$. Then we define

$$\sigma_J(I_1, \dots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + J^r, \dots, I_n + J^r).$$

An easy computation reveals that $\sigma_J(I_1, \dots, I_n) = \sigma(I_1, \dots, I_n)$. We also define

$$r_J(I_1, \dots, I_n) = \min\left\{r \in \mathbb{Z}_+ : \sigma(I_1, \dots, I_n) = e(I_1 + J^r, \dots, I_n + J^r)\right\}. \tag{5}$$

Let I be an ideal of R of finite colength. Then we denote by $r_J(I)$ the number $r_J(I, \dots, I)$, where I is repeated n times. We deduce from the Rees multiplicity theorem that, if R is quasi-unmixed, then $r_J(I) = \min\{r \geqslant 1 : J^r \subseteq \bar{I}\}$.

Lemma 3.3. *Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$ and let J be an m-primary ideal. Then*

$$r_J(I_1^s, \dots, I_n^s) \leqslant sr_J(I_1, \dots, I_n)$$
$$r_{J^s}(I_1, \dots, I_n) \geqslant \frac{1}{s} r_J(I_1, \dots, I_n)$$

for all integer $s \geqslant 1$.

Proof. For the first inequality, set $r = r_J(I_1, \dots, I_n)$. Thus $\sigma(I_1, \dots, I_n) = e(I_1 + J^r, \dots, I_n + J^r)$. It is enough to prove that $\sigma(I_1^s, \dots, I_n^s) = e(I_1^s + J^{rs}, \dots, I_n^s + J^{rs})$:

$$\begin{aligned} e(I_1^s + J^{rs}, \dots, I_n^s + J^{rs}) &= e(\overline{I_1^s + J^{rs}}, \dots, \overline{I_n^s + J^{rs}}) = e(\overline{(I_1 + J^r)^s}, \dots, \overline{(I_n + J^r)^s}) \\ &= e((I_1 + J^r)^s, \dots, (I_n + J^r)^s) = s^n e(I_1 + J^r, \dots, I_n + J^r) \\ &= s^n \sigma(I_1, \dots, I_n) = \sigma(I_1^s, \dots, I_n^s), \end{aligned}$$

where last equality comes from [4, Lemma 2.6].

The second inequality comes directly from the definition of $r_{J^s}(I_1, \dots, I_n)$. □

It is easy to find examples of ideals I and J such that $r_J(I_1, \dots, I_n) \neq r(I_1, \dots, I_n)$ in general. This fact motivates the following definition.

Definition 3.4. Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. Let J be an m-primary ideal of R. We define the *Łojasiewicz exponent of* $I_1, \dots, I_n$ *with respect to* J, denoted by $\mathcal{L}_J(I_1, \dots, I_n)$, as

$$\mathcal{L}_J(I_1, \dots, I_n) = \inf_{s \geqslant 1} \frac{r_J(I_1^s, \dots, I_n^s)}{s}. \tag{6}$$

If I is an m-primary ideal of R, then we denote by $\mathcal{L}_J(I)$ the number $\mathcal{L}_J(I, \dots, I)$, where I is repeated n times.

Remark 3.5. Under the conditions of the previous definition, we observe that $\mathcal{L}_J(I_1, \dots, I_n)$ can be seen as a limit inferior:

$$\mathcal{L}_J(I_1, \dots, I_n) = \liminf_{s \to \infty} \frac{r_J(I_1^s, \dots, I_n^s)}{s}.$$

Set $\ell = \mathcal{L}_J(I_1, \dots, I_n)$. In order to prove the equality above, it is enough to see that for all $\epsilon > 0$ and all $p \in \mathbb{Z}_+$, there exists an integer $m \geqslant p$ such that

$$\frac{r_J(I_1^m, \dots, I_n^m)}{m} \leqslant \ell + \epsilon.$$

Let us fix an $\epsilon > 0$ and an integer $p \in \mathbb{Z}_+$. By definition, there exists $q \in \mathbb{Z}_+$ such that

$$\frac{r_J(I_1^q, \dots, I_n^q)}{q} \leqslant \ell + \epsilon.$$

Let $s \in \mathbb{Z}_+$ such that $sq \geqslant p$. Then, from Lemma 3.3 we obtain that

$$\frac{r_J(I_1^{sq}, \dots, I_n^{sq})}{sq} \leqslant \frac{r_J(I_1^q, \dots, I_n^q)}{q} \leqslant \ell + \epsilon.$$

If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denotes an analytic map germ such that $g^{-1}(0) = \{0\}$ and J is an ideal of $\mathcal{O}_n$ of finite colength, then we denote the number $\mathcal{L}_J(I)$, where I is the ideal generated by the component functions of g, by $\mathcal{L}_J(g)$. A straightforward reproduction of the argument in the proof of Theorem 3.1 consisting of replacing the powers of the maximal ideal by the powers of a given ideal of finite colength leads to the following result, which is analogous to Theorem 3.1.

Theorem 3.6. *Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n)$ is finite and let J be a monomial ideal of $\mathcal{O}_n$ of finite colength. Then the sequence $\{\frac{r_J(I_1^s,\dots,I_n^s)}{s}\}_{s\geqslant 1}$ attains a minimum and if $g \in \mathcal{S}_0(I_1, \dots, I_n)$ then*

$$\mathcal{L}_J(g) = \mathcal{L}_J(I_1, \dots, I_n) = \min_{s\geqslant 1} \frac{r_J(I_1^s, \dots, I_n^s)}{s}. \tag{7}$$

Lemma 3.7. *Under the hypothesis of Lemma 3.3 we have*

$$\mathcal{L}_J(I_1^s, \dots, I_n^s) = s\mathcal{L}_J(I_1, \dots, I_n)$$
$$\mathcal{L}_{J^s}(I_1, \dots, I_n) = \frac{1}{s}\mathcal{L}_J(I_1, \dots, I_n)$$

for all $s \in \mathbb{Z}_{\geqslant 1}$.

Proof. For the first equality

$$\mathcal{L}_J(I_1^s, \dots, I_n^s) = \inf_{p\geqslant 1} \frac{r_J(I_1^{sp}, \dots, I_n^{sp})}{p} = s\inf_{p\geqslant 1} \frac{r_J(I_1^{sp}, \dots, I_n^{sp})}{sp} \geqslant s\mathcal{L}_J(I_1, \dots, I_n).$$

On the other hand, by Lemma 3.3 we obtain

$$\inf_{p\geqslant 1} \frac{r_J(I_1^{sp}, \dots, I_n^{sp})}{p} \leqslant s\inf_{p\geqslant 1} \frac{r_J(I_1^{p}, \dots, I_n^{p})}{p} = s\mathcal{L}_J(I_1, \dots, I_n).$$

Let us see the second equality. Applying Lemma 3.3 we have

$$\mathcal{L}_{J^s}(I_1, \dots, I_n) = \inf_{p\geqslant 1} \frac{r_{J^s}(I_1^{p}, \dots, I_n^{p})}{p} \geqslant \frac{1}{s}\inf_{p\geqslant 1} \frac{r_J(I_1^{p}, \dots, I_n^{p})}{p} = \frac{1}{s}\mathcal{L}_J(I_1, \dots, I_n).$$

Let us denote the number $r_{J^s}(I_1^p, \dots, I_n^p)$ by r_p, for all $p \geqslant 1$. Then

$$\sigma(I_1^p, \dots, I_n^p) > e(I_1^p + J^{s(r_p-1)}, \dots, I_n^p + J^{s(r_p-1)}).$$

In particular

$$r_J(I_1^p, \dots, I_n^p) > s(r_p - 1)$$

for all $p \geqslant 1$. Dividing the previous inequality by p and taking $\liminf_{p\to\infty}$ we obtain by Remark 3.5, that

$$\mathcal{L}_J(I_1, \dots, I_n) = \liminf_{p\to\infty} \frac{r_J(I_1^p, \dots, I_n^p)}{p} \geqslant s\liminf_{p\to\infty}\left(\frac{r_p - 1}{p}\right) = s\mathcal{L}_{J^s}(I_1, \dots, I_n). \quad \square$$

Lemma 3.8. *Let (R, m) be a quasi-unmixed Noetherian local ring of dimension n. Let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. If J_1, J_2 are m-primary ideals of R then*

$$\mathcal{L}_{J_1}(I_1, \dots, I_n) \leqslant \mathcal{L}_{J_1}(J_2)\mathcal{L}_{J_2}(I_1, \dots, I_n).$$

Proof. By (5) we have that

$$r_{J_1}(J_2) = \min\left\{r \geqslant 1 : e(J_2) = e(J_2 + J_1^r)\right\}.$$

Given an integer $r \geqslant 1$, the condition $e(J_2) = e(J_2 + J_1^r)$ is equivalent to saying that $J_1^r \subseteq \overline{J_2}$, by the Rees multiplicity theorem (see [8, p. 222]). Therefore, an elementary computation shows that

$$r_{J_1}(I_1, \dots, I_n) \leqslant r_{J_1}(J_2)r_{J_2}(I_1, \dots, I_n). \tag{8}$$

By the generality of the previous inequality, we have

$$r_{J_1}(I_1^s, \dots, I_n^s) \leqslant r_{J_1}(J_2^p)r_{J_2^p}(I_1^s, \dots, I_n^s) \tag{9}$$

for all integers $p, s \geqslant 1$. The inequality (9) shows that

$$\begin{aligned}\mathcal{L}_{J_1}(I_1, \dots, I_n) &= \inf_{s\geqslant 1} \frac{r_{J_1}(I_1^s, \dots, I_n^s)}{s} \leqslant \inf_{s\geqslant 1} \frac{r_{J_1}(J_2^p)r_{J_2^p}(I_1^s, \dots, I_n^s)}{s} \\ &= r_{J_1}(J_2^p)\mathcal{L}_{J_2^p}(I_1, \dots, I_n) = r_{J_1}(J_2^p)\frac{1}{p}\mathcal{L}_{J_2}(I_1, \dots, I_n)\end{aligned}$$

for all integer $p \geqslant 1$, where the last equality comes from Lemma 3.7. Then

$$\mathcal{L}_{J_1}(I_1, \dots, I_n) \leqslant \left(\inf_{p\geqslant 1} \frac{r_{J_1}(J_2^p)}{p}\right)\mathcal{L}_{J_2}(I_1, \dots, I_n) = \mathcal{L}_{J_1}(J_2)\mathcal{L}_{J_2}(I_1, \dots, I_n). \quad \square$$

We recall the following two results, which will be applied in the next section.

Proposition 3.9 ([4]). *Let (R, m) be a Noetherian local ring of dimension n. For each $i = 1, \dots, n$ let us consider ideals I_i and J_i such that $I_i \subseteq J_i$. Let suppose that $\sigma(I_1, \dots, I_n) < \infty$ and that $\sigma(I_1, \dots, I_n) = \sigma(J_1, \dots, J_n)$. Then*

$$\mathcal{L}_0(I_1, \dots, I_n) \leqslant \mathcal{L}_0(J_1, \dots, J_n). \tag{10}$$

Let us denote the canonical basis in $\mathbb{R}^n$ by $e_1, \dots, e_n$.

Proposition 3.10 ([2]). *Let J be an ideal of finite colength of $\mathcal{O}_n$ and set $r_i = \min\{r : re_i \in \Gamma_+(J)\}$, for all $i = 1, \dots, n$. Then*

$$\max\{r_1, \dots, r_n\} \leqslant \mathcal{L}_0(J)$$

and equality holds if $\overline{J}$ is a monomial ideal.

4. Weighted homogeneous filtrations

Let us fix a vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$. We will usually refer to w as the *vector of weights*. Let $h \in \mathcal{O}_n$, $h \neq 0$, the *degree of h with respect to w*, or *w-degree* of h, is defined as

$$d_w(h) = \min\{\langle k, w\rangle : k \in \operatorname{supp}(h)\},$$

where $\langle\, ,\rangle$ stands for the usual scalar product. In particular, if $x_1, \dots, x_n$ denotes a system of coordinates in $\mathbb{C}^n$ and $x_1^{k_1} \cdots x_n^{k_n}$ is a monomial in $\mathcal{O}_n$, then $d_w(x_1^{k_1} \cdots x_n^{k_n}) = w_1k_1 + \cdots + w_nk_n$. By convention, we set $d_w(0) = +\infty$. If $h \in \mathcal{O}_n$ and $h = \sum_k a_kx^k$ is the Taylor expansion of h around the origin, then we define the *principal part of h with respect to w* as the polynomial given by the sum of those terms a_kx^k such that $\langle k, w\rangle = d_w(h)$. We denote this polynomial by $p_w(h)$.

Definition 4.1. We say that a function $h \in \mathcal{O}_n$ is *weighted homogeneous of degree d with respect to w* if $\langle k, w\rangle = d$, for all $k \in \operatorname{supp}(h)$. The function h is said to be *semi-weighted homogeneous of degree d with respect to w* when $p_w(h)$ has an isolated singularity at the origin. Note that $p_w(h)$ is weighted homogeneous with respect to w.

It is well known that, if h is a semi-weighted homogeneous function, then h has an isolated singularity at the origin and that h and $p_w(h)$ have the same Milnor number (see for instance [1, Section 12]). Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be an analytic map germ, let us denote the map $(p_w(g_1), \dots, p_w(g_n))$ by $p_w(g)$. The map g is said to be *semi-weighted homogeneous with respect to w* when $(p_w(g))^{-1}(0) = \{0\}$.

If I is an ideal of $\mathcal{O}_n$, then we define the *degree of I with respect to w*, or *w-degree of I*, as

$$d_w(I) = \min\{d_w(h) : h \in I\}.$$

If $g_1, \dots, g_r$ constitutes a generating system of I, then it is straightforward to see that $d_w(I) = \min\{d_w(g_1), \dots, d_w(g_r)\}$.

Let $r \in \mathbb{Z}_+$, then we denote by $\mathcal{B}_r$ the set of all $h \in \mathcal{O}_n$ such that $d_w(h) \geqslant r$ (therefore $0 \in \mathcal{B}_r$). We observe that

(a) $\mathcal{B}_r$ is an integrally closed monomial ideal of finite colength, for all $r \geqslant 1$;
(b) $\mathcal{B}_r\mathcal{B}_s \subseteq \mathcal{B}_{r+s}$, $r, s \geqslant 1$;
(c) $\mathcal{B}_0 = \mathcal{O}_n$.

The family of ideals $\{\mathcal{B}_r\}_{r\geqslant 1}$ is called the *weighted homogeneous filtration induced by w*. We denote by $\mathcal{A}_r$ the ideal of $\mathcal{O}_n$ generated by the monomials x^k such that $d_w(x^k) = r$. If there is not any monomial x^k such that $d_w(x^k) = r$ then we set $\mathcal{A}_r = 0$. Given an integer $r \geqslant 1$, we observe that $\mathcal{A}_r \subseteq \mathcal{B}_r$ and that $\overline{\mathcal{A}_r} \neq \mathcal{B}_r$ in general. Moreover it follows easily that $\overline{\mathcal{A}_r} = \mathcal{B}_r$ if and only if $\mathcal{A}_r$ is an ideal of finite colength of $\mathcal{O}_n$.

If $r_1, \dots, r_n \in \mathbb{Z}_{\geqslant 1}$, then it is not true in general that $\sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n}) < \infty$, even if $\mathcal{A}_{r_i} \neq 0$, for all $i = 1, \dots, n$. However $\sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) < \infty$, since $\mathcal{B}_{r_i}$ has finite colength, for all $i = 1, \dots, n$. For instance, let us consider the vector $w = (3, 1)$. Then we have

$$\mathcal{A}_4 = \langle xy, y^4\rangle, \qquad \mathcal{A}_5 = \langle xy^2, y^5\rangle.$$

We observe that the ideal $\mathcal{A}_4 + \mathcal{A}_5$ does not have finite colength, therefore $\sigma(\mathcal{A}_4, \mathcal{A}_5)$ is not finite (see Remark 2.3).

Proposition 4.2. *Let $r_1, \dots, r_n \in \mathbb{Z}_{\geqslant 1}$. If $\sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n}) < \infty$ then $\sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) < \infty$ and*

$$\sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n}) = \sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}.$$

Proof. By Proposition 2.2, there exists a sufficiently general element $(h_1, \dots, h_n) \in \mathcal{B}_{r_1} \oplus \cdots \oplus \mathcal{B}_{r_n}$ such that

$$\sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) = e(h_1, \dots, h_n). \tag{11}$$

The condition $\sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n}) < \infty$ implies that $\mathcal{A}_{r_i} \neq 0$, for all $i = 1, \dots, n$. The ideal $\mathcal{A}_{r_i}$ is generated by the monomials of w-degree r_i, for all $i = 1, \dots, n$, then h_i can be written as $h_i = g_i + g_i'$, for all $i = 1 \dots, n$, where $(g_1, \dots, g_n)$ is a sufficiently general element of $\mathcal{A}_{r_1} \oplus \cdots \oplus \mathcal{A}_{r_n}$ and $g_i' \in \mathcal{O}_n$ verifies that $d_w(g_i') > r_i$, for all $i = 1, \ldots, n$. Therefore $p_w(h_i) = g_i$, for all $i = 1, \ldots, n$.

Let g denote the map $(g_1, \ldots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. The condition $\sigma(\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}) < \infty$ and the genericity of g imply that g is finite, that is, $g^{-1}(0) = \{0\}$ and $\sigma(\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}) = e(g_1, \ldots, g_n)$. Consequently the map $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is semi-weighted homogeneous with respect to w. By [1, Section 12] (see also [7] for a more general phenomenon), this implies that

$$e(h_1, \ldots, h_n) = e(g_1, \ldots, g_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}.$$

Then the result follows. □

Definition 4.3. Let $J_1, \ldots, J_n$ be a family of ideals of $\mathcal{O}_n$ and let $r_i = d_w(J_i)$, for all $i = 1, \ldots, n$. We say that $J_1, \ldots, J_n$ *admits a w-matching* if there exists a permutation τ of $\{1, \ldots, n\}$ and an index $i_0 \in \{1, \ldots, n\}$ such that

(a) $w_{i_0} = \min\{w_1, \ldots, w_n\}$,
(b) $r_{\tau(i_0)} = \max\{r_1, \ldots, r_n\}$ and
(c) the pure monomial $x_i^{r_{\tau(i)}/w_i}$ belongs to $J_{\tau(i)}$, for all $i \neq i_0$.

Remark 4.4. If $r \in \mathbb{Z}_{\geqslant 1}$ then we observe that $\mathcal{A}_r$ has finite colength if and only if w_i divides r, for all $i = 1, \ldots, n$. Let $r_1, \ldots, r_n \in \mathbb{Z}_{\geqslant 1}$ such that $\mathcal{A}_{r_i}$ has finite colength, for all $i = 1, \ldots, n$. Then condition (c) of the above definition is not a restriction in this case and therefore $\mathcal{A}_{r_1}, \ldots, \mathcal{A}_{r_n}$ admits a w-matching.

Let us consider the case $n = 2$ of the previous definition. Therefore, let $r_1, r_2 \in \mathbb{Z}_{\geqslant 1}$ with $r_1 > r_2$ and let us suppose that $w_1 < w_2$. Let J_1, J_2 be ideals of $\mathcal{O}_2$ such that $d_w(J_i) = r_i$, $i = 1, 2$. Then J_1, J_2 admits a w-matching if and only if $y^{r_2/w_2} \in J_2$.

Example 4.5. Set $w = (1, 2, 3, 4)$ and $r_1 = 10, r_2 = 9, r_3 = 8, r_4 = 6$. The family of ideals given by

$$J_1 = \langle x_1 x_3^3 \rangle, \qquad J_2 = \langle x_3^3, x_1 x_4^2 \rangle, \qquad J_3 = \langle x_4^2, x_1^2 x_3^2 \rangle, \qquad J_4 = \langle x_2^3, x_2 x_4 \rangle,$$

admits a w-matching. Observe that here $i_0 = 1$ and the permutation τ is defined by $\tau(1) = 1, \tau(2) = 4, \tau(3) = 2, \tau(4) = 3$.

Let us observe that, if $J_1, \ldots, J_n$ admits a w-matching, then it is always possible to reorder the ideals J_i in such a way that $\tau(i_0) = i_0$, and therefore one could restrict to the case $\tau = \mathrm{id}$ after a permutation of the ideals J_i. But the permutation τ is specially relevant when considering ideals coming from the gradient of a function f (see Example 4.12).

Lemma 4.6. *Let $r_1, \ldots, r_n \in \mathbb{Z}_{\geqslant 1}$ and let $I_1, \ldots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $d_w(I_i) = r_i$, for all $i = 1, \ldots, n$, and $\sigma(I_1, \ldots, I_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}$. Let J be an ideal of $\mathcal{O}_n$ such that $\overline{J} = \overline{\langle x_1^{r\alpha_1}, \ldots, x_n^{r\alpha_n} \rangle}$, for some $r \geqslant 1$, where $\alpha_i = \frac{\overline{w}}{w_i}$ and $\overline{w} = w_1 \cdots w_n$. Then*

$$e(I_1 + J, \ldots, I_n + J) = \frac{\min\{r_1, \overline{w}r\} \cdots \min\{r_n, \overline{w}r\}}{\overline{w}}. \tag{12}$$

Proof. Let $A = \{i : r_i < r\overline{w}\}$. After a reordering of the integers $r_1, \ldots, r_n$ we can assume that $A = \{1, \ldots, s\}$, for some $s \geqslant 1$. Then, since $\overline{J} = \mathcal{B}_{r\overline{w}}$ we conclude that $e(I_1 + J, \ldots, I_n + J) = e(I_1 + J, \ldots, I_s + J, J, \ldots, J)$.

By Proposition 2.2, there exist an element $(g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n$ such that $d_w(g_i) = r_i$, for all $i = 1, \ldots, n$, and

$$e(g_1, \ldots, g_n) = \sigma(I_1, \ldots, I_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}. \tag{13}$$

Let us denote by R the quotient ring $\mathcal{O}_n / \langle p_w(g_1), \ldots, p_w(g_s) \rangle$ and let H denote the ideal of $\mathcal{O}_n$ generated by $x_1^{r\alpha_1}, \ldots, x_n^{r\alpha_n}$.

Relation (13) implies, by [6, Theorem 3.3], that the ideal generated by $p_w(g_1), \ldots, p_w(g_n)$ has finite colength. In particular, these elements form a regular sequence and then $\dim(R) = n - s$. Hence there exists a sufficiently general element $(h_1, \ldots, h_{n-s}) \in H \oplus \cdots \oplus H$ such that the images of the h_i in R generate a reduction of the image of J in R, by the theorem of existence of reductions (see [8, p. 166]). In particular, the ideal $K = \langle p_w(g_1), \ldots, p_w(g_s), h_1, \ldots, h_{n-s} \rangle$ has finite colength.

Since h_i is a generic $\mathbb{C}$-linear combination of $x_1^{r\alpha_1}, \ldots, x_n^{r\alpha_n}$, for all $i = 1, \ldots, n$, we have that $p_w(h_i) = h_i$, for all $i = 1, \ldots, n$. Then $K = \langle p_w(g_1), \ldots, p_w(g_s), p_w(h_1), \ldots, p_w(h_{n-s}) \rangle$. Therefore

$$e(K) = \frac{r_1 \cdots r_s (\overline{w}r)^{n-s}}{w_1 \cdots w_n} = \frac{\min\{r_1, \overline{w}r\} \cdots \min\{r_n, \overline{w}r\}}{\overline{w}}, \tag{14}$$

where the first equality comes from [1, Section 12] (see also [6, Theorem 3.3]).

Since I_i is a monomial ideal, for all $i = 1, \ldots, n$, we have that $p_w(g_i) \in I_i$, for all $i = 1, \ldots, n$. In particular we have $e(K) \geqslant e(I_1 + J, \ldots, I_n + J)$, by Lemma 2.4. Then

$$e(K) \geqslant e(I_1 + H, \ldots, I_n + H) \geqslant \frac{\min\{r_1, \overline{w}r\} \cdots \min\{r_n, \overline{w}r\}}{\overline{w}}, \tag{15}$$

where the second inequality follows from [6, Theorem 3.3].

The hypothesis $\bar{J} = \overline{H}$ implies that

$$e(I_1 + J, \dots, I_n + J) = e(I_1 + H, \dots, I_n + H). \tag{16}$$

Then the result follows by joining (14)–(16). □

Theorem 4.7. *Let $r_1, \dots, r_n \in \mathbb{Z}_{\geqslant 1}$ such that $\sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n}) < \infty$. Let $J_1, \dots, J_n$ be a set of ideals of $\mathcal{O}_n$ with $d_w(J_i) = r_i$, for all $i = 1, \dots, n$, and $\sigma(J_1, \dots, J_n) = \sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n})$. Then*

$$\mathcal{L}_0(J_1, \dots, J_n) \leqslant \mathcal{L}_0(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) \leqslant \frac{\max\{r_1, \dots, r_n\}}{\min\{w_1, \dots, w_n\}} \tag{17}$$

and the above inequalities turn into equalities if $J_1, \dots, J_n$ admit a w-matching.

Proof. The condition $\sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n}) < \infty$ and the equality $\sigma(J_1, \dots, J_n) = \sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n})$ imply that

$$\sigma(J_1, \dots, J_n) = \sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n},$$

by Proposition 4.2. Then we can apply Proposition 3.9 to deduce that

$$\mathcal{L}_0(J_1, \dots, J_n) \leqslant \mathcal{L}_0(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}).$$

Let us denote $\max\{r_1, \dots, r_n\}$ and $\min\{w_1, \dots, w_n\}$ by p and q, respectively. Let us see that $\mathcal{L}_0(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) \leqslant \frac{p}{q}$.

Let us denote by $\overline{w}$ the product $w_1 \cdots w_n$ and let us consider the ideal $J = \langle x_1^{\alpha_1}, \dots, x_n^{\alpha_n} \rangle$, where $\alpha_i = \frac{\overline{w}}{w_i}$, for all $i = 1, \dots, n$. Since $\sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) < \infty$, it makes sense to compute the number $r_J(\mathcal{B}^s_{r_1}, \dots, \mathcal{B}^s_{r_n})$, for all $s \geqslant 1$:

$$\begin{aligned} r_J(\mathcal{B}^s_{r_1}, \dots, \mathcal{B}^s_{r_n}) &= \min\left\{r \geqslant 1 : \sigma(\mathcal{B}^s_{r_1}, \dots, \mathcal{B}^s_{r_n}) = e(\mathcal{B}^s_{r_1} + J^r, \dots, \mathcal{B}^s_{r_n} + J^r)\right\} \\ &= \min\left\{r \geqslant 1 : \frac{sr_1 \cdots sr_n}{\overline{w}} = \frac{\min\{sr_1, \overline{w}r\} \cdots \min\{sr_n, \overline{w}r\}}{\overline{w}}\right\} \\ &= \min\left\{r \geqslant 1 : \overline{w}r \geqslant \max\{sr_1, \dots, sr_n\}\right\} \\ &= \min\left\{r \geqslant 1 : r \geqslant \frac{\max\{sr_1, \dots, sr_n\}}{\overline{w}}\right\} = \left\lceil \frac{\max\{sr_1, \dots, sr_n\}}{\overline{w}} \right\rceil, \end{aligned}$$

where $\lceil a \rceil$ denotes the least integer greater than or equal to a, for any $a \in \mathbb{R}$, and the second equality is a direct application of Lemma 4.6. Therefore

$$\begin{aligned} \mathcal{L}_J(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) &= \inf_{s \geqslant 1} \frac{r_J(\mathcal{B}^s_{r_1}, \dots, \mathcal{B}^s_{r_n})}{s} \leqslant \inf_{a \geqslant 1} \frac{r_J(\mathcal{B}^{a\overline{w}}_{r_1}, \dots, \mathcal{B}^{a\overline{w}}_{r_n})}{a\overline{w}} \\ &= \inf_{a \geqslant 1} \frac{1}{a\overline{w}} \left\lceil \frac{\max\{a\overline{w}r_1, \dots, a\overline{w}r_n\}}{\overline{w}} \right\rceil = \frac{\max\{r_1, \dots, r_n\}}{\overline{w}}. \end{aligned}$$

Moreover, by Proposition 3.10 we have

$$\mathcal{L}_0(J) = \max\{\alpha_1, \dots, \alpha_n\} = \frac{\overline{w}}{\min\{w_1, \dots, w_n\}},$$

since J is a monomial ideal. Therefore, by Lemma 3.8 we obtain

$$\begin{aligned} \mathcal{L}_0(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) &\leqslant \mathcal{L}_0(J)\mathcal{L}_J(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) \\ &\leqslant \frac{\overline{w}}{\min\{w_1, \dots, w_n\}} \frac{\max\{r_1, \dots, r_n\}}{\overline{w}} = \frac{\max\{r_1, \dots, r_n\}}{\min\{w_1, \dots, w_n\}}. \end{aligned}$$

Let us prove that $\mathcal{L}_0(J_1, \dots, J_n) \geqslant \frac{p}{q}$ supposing that $J_1, \dots, J_n$ admit a w-matching. This inequality holds if and only if

$$\frac{r(J_1^s, \dots, J_n^s)}{s} \geqslant \frac{p}{q}$$

for all $s \geqslant 1$. By Lemma 3.3 we have that $qr(J_1^s, \dots, J_n^s) \geqslant r(J_1^{sq}, \dots, J_n^{sq})$, for all $s \geqslant 1$. Therefore it suffices to show that

$$r(J_1^{sq}, \dots, J_n^{sq}) > sp - 1, \tag{18}$$

for all $s \geqslant 1$. Let us fix an integer $s \geqslant 1$, then relation (18) is equivalent to saying that

$$\sigma(J_1^{sq}, \dots, J_n^{sq}) > e(J_1^{sq} + m^{sp-1}, \dots, J_n^{sq} + m^{sp-1}). \tag{19}$$

Since $J_1, \dots, J_n$ admits a w-matching, let us consider a permutation τ of $\{1, \dots, n\}$ such that

(a) $w_{i_0} = \min\{w_1, \dots, w_n\}$,
(b) $r_{\tau(i_0)} = \max\{r_1, \dots, r_n\}$ and
(c) the pure monomial $x_i^{r_{\tau(i)}/w_i}$ belongs to $J_{\tau(i)}$ for all $i \neq i_0$.

Let us define the ideal

$$H = \left\langle x_i^{\frac{r_{\tau(i)}sq}{w_i}} : i \neq i_0 \right\rangle + \left\langle x_{i_0}^{sp-1} \right\rangle.$$

Then

$$\begin{aligned} e(H) &= e\Big(x_1^{\frac{r_{\tau(1)}sq}{w_1}}, \dots, x_{i_0-1}^{\frac{r_{\tau(i_0-1)}sq}{w_{i_0-1}}}, x_{i_0}^{sp-1}, x_{i_0+1}^{\frac{r_{\tau(i_0+1)}sq}{w_{i_0+1}}}, \dots, x_n^{\frac{r_{\tau(n)}sq}{w_n}}\Big) \\ &= (sq)^{n-1}\frac{r_1\cdots r_n}{r_{\tau(i_0)}}\frac{w_{i_0}}{w_1\cdots w_n}(sp-1). \end{aligned}$$

Since $x_i^{\frac{r_{\tau(i)}}{w_i}} \in J_{\tau(i)}$ for all $i \in \{1, \dots, n\} \setminus \{i_0\}$, and $x_{i_0}^{sp-1} \in m^{sp-1}$, we can apply Lemma 2.4 to conclude that

$$e(H) \geqslant e(J_{\tau(1)}^{sq} + m^{sp-1}, \dots, J_{\tau(n)}^{sq} + m^{sp-1}) = e(J_1^{sq} + m^{sp-1}, \dots, J_n^{sq} + m^{sp-1}). \tag{20}$$

Hence, if we prove that $\sigma(J_1^{sq}, \dots, J_n^{sq}) > e(H)$ then the result follows.

By [4, Lemma 2.6], we have that $\sigma(J_1^{sq}, \dots, J_n^{sq}) = (sq)^n\sigma(J_1, \dots, J_n)$. Then, using the hypothesis $\sigma(J_1, \dots, J_n) = \sigma(\mathcal{A}_{r_1}, \dots, \mathcal{A}_{r_n})$ and Proposition 4.2, we obtain that

$$\sigma(J_1^{sq}, \dots, J_n^{sq}) = (sq)^n\frac{r_1\cdots r_n}{w_1\cdots w_n}. \tag{21}$$

Thus, since we assume that $r_{\tau(i_0)} = p$ and $w_{i_0} = q$, we have that $\sigma(J_1^{sq}, \dots, J_n^{sq}) > e(H)$ if and only if

$$sq > \frac{q}{p}(sp-1),$$

which is to say that $spq > spq - q$. Therefore relation (19) holds for all integer $s \geqslant 1$ and consequently the inequality $\mathcal{L}_0(J_{r_1}, \dots, J_{r_n}) \geqslant \frac{p}{q}$ follows. Thus relation (17) is proven. □

Remark 4.8. We observe that the condition that $J_1, \dots, J_n$ admits a w-matching cannot be removed from the hypothesis of the previous theorem. Let us consider now the weighted homogeneous filtration in $\mathcal{O}_2$ induced by the vector of weights $w = (1, 4)$ and let J_1, J_2 be the ideals of $\mathcal{O}_2$ given by $J_1 = \langle x^4\rangle$, $J_2 = \langle y^2\rangle$. We observe that $d_w(x^4) = 4$, $d_w(y^2) = 8$ and consequently the right hand side of (17) would lead to the conclusion that $\mathcal{L}_0(J_1, J_2) = 8$, which is not the case, since clearly $\mathcal{L}_0(x^4, y^2) = 4$. We also observe that the system of ideals J_1, J_2 does not admit a w-matching.

In order to simplify the exposition, we need to introduce the following definition.

Definition 4.9. If $f \in \mathcal{O}_n$, $f(0) = 0$, then f is termed *convenient* when $\Gamma_+(f)$ intersects each coordinate axis. Let J_i denote the ideal of $\mathcal{O}_n$ generated by all monomials x^k such that $k \in \Gamma_+(\partial f/\partial x_i)$, $i = 1, \dots, n$. Let us fix a vector of weights $w \in \mathbb{Z}^n_{\geqslant 1}$. Then we say that f *admits a w-matching* when the family of ideals $J_1, \dots, J_n$ admits a w-matching (see Definition 4.3).

If a function $f \in \mathcal{O}_n$ is convenient and quasi-homogeneous, then f admits a w-matching. Observe that in this case the monomials x_i^{d/w_i} are in the support of f, for $i = 1, \dots, n$. Then there is a pure monomial in x_i belonging to the support of the partial derivative $\partial f/\partial x_i$ and one could take $\tau = \mathrm{id}$ in the definition of w-matching (see Definition 4.3).

Let us fix a vector of weights $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ and an integer $d \geqslant 1$. Then we denote by $\mathcal{O}(w; d)$ the set of all functions $f \in \mathcal{O}_n$ such that f is semi-weighted homogeneous with respect to w of degree d.

Remark 4.10. From Definition 4.3 we observe that a function $f \in \mathcal{O}(w; d)$ admits a w-matching if and only if $p_w(f)$ admits a w-matching, since the ideals J_i introduced in Definition 4.9 have the same w-degree as the analogous ideals defined for $p_w(f)$.

Corollary 4.11. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a semi-weighted homogeneous function of degree d with respect to the weights $w_1, \dots, w_n$. Then*

$$\mathcal{L}_0(\nabla f) \leqslant \frac{d - \min\{w_1, \dots, w_n\}}{\min\{w_1, \dots, w_n\}} \tag{22}$$

and equality holds iff f admits a w-matching.

Proof. Let J_i denote the ideal of $\mathcal{O}_n$ generated by all monomials x^k such that $k \in \Gamma_+(\partial f/\partial x_i)$, $i = 1, \dots, n$. Since f has an isolated singularity at the origin (that is, the ideal $J(f)$ has finite colength) then $\sigma(J_1, \dots, J_n) < \infty$, by Proposition 2.2. Then Theorem 3.1 shows that $\mathcal{L}_0(\nabla f) = \mathcal{L}_0(J_1, \dots, J_n)$. We observe that $d_w(J_i) = d - w_i$, for all $i = 1, \dots, n$. Then the result arises as a direct application of Theorem 4.7. □

It has been proven recently by Płoski et al. [9] that equality holds in (22) for all weighted homogeneous functions $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that f has an isolated singularity at the origin, under the hypothesis that $2w_i \leqslant d$ for all i.

The result of Corollary 4.11 holds for any number of variables.

Example 4.12. Let us consider the vector of weights $w = (1, 2, 3, 5)$ and the polynomial $f : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ given by $f(x_1, x_2, x_3, x_4) = x_3^9 - x_2^{11}x_4 + x_2x_4^5 + x_1^{27}$. Then f is weighted homogeneous with w-degree 27 and f has an isolated singularity at the origin. The ideals J_i introduced in Definition 4.9 are given by

$$J_1 = \langle x_1^{26} \rangle \qquad J_2 = \langle x_2^{10}x_4, x_4^5 \rangle \qquad J_3 = \langle x_3^8 \rangle \qquad J_4 = \langle x_2^{11}, x_2x_4^4 \rangle.$$

Then we observe that the polynomial f admits w-matching. Here the permutation τ of Definition 4.3 is $\tau(1) = 1$, $\tau(2) = 4$, $\tau(3) = 3$, $\tau(4) = 2$. Then it follows from Corollary 4.11 that $\mathcal{L}_0(\nabla f) = 26$.

Given a vector of weights $w = (w_1, \dots, w_n)$ and a degree d, then it is not always possible to find a weighted homogeneous function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of degree d with respect to w such that f admits a w-matching, as the following example shows.

Example 4.13. Let $w = (1, 2, 3)$ and $d = 16$. Let f be a weighted homogeneous function of degree d with respect to w. Let J_i denote the ideal of $\mathcal{O}_3$ generated by all monomials x^k such that $k \in \Gamma_+(\partial f/\partial x_i)$, for all $i = 1, 2, 3$. As a direct consequence of Definition 4.3, if J_1, J_2, J_3 admits a w-matching, then J_3 contains a pure monomial of x_2 or a pure monomial of x_3, which is impossible since $d_w(J_3) = 13$ and neither 2 nor 3 are divisors of 13.

However we observe that $\mathcal{O}(w; d) \neq \emptyset$, since the function $f(x_1, x_2, x_3) = x_1^{16} + x_2^8 + x_1x_3^5$ belongs to $\mathcal{O}(w; d)$.

Proposition 4.14. *Let $d, w_1, \dots, w_n$ be non-negative integers such that w_i divides d for all $i = 1, \dots, n$. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous function of degree d with respect to the weights $w_1, \dots, w_n$. Let us assume that f has an isolated singularity at the origin. Then there exists a change of coordinates $\mathbf{x}$ in $(\mathbb{C}^n, 0)$ of the form $x_i = y_i + h_i(y_1, \dots, y_n)$, where h_i is a polynomial in $y_1, \dots, y_n$, $i = 1, \dots, n$, such that:*

(1) *the function $f \circ \mathbf{x}$ is convenient;*
(2) *if $h_i \neq 0$, then the polynomial h_i is weighted homogeneous of degree w_i with respect to w and therefore $f \circ \mathbf{x}$ is weighted homogeneous of degree d with respect to w.*

Proof. Since f has an isolated singularity at the origin, for any $i = 1, \dots, n$ we can fix an index $k_i \in \{1, \dots, n\}$ such that $x_i^{m_i}$ appears in the support of $\frac{\partial f}{\partial x_{k_i}}$, where $m_i = \frac{d - w_{k_i}}{w_i}$, which is to say that the monomial $x_{k_i}x_i^{m_i}$ appears in the support of f. Then w_i divides $d - w_{k_i}$ and consequently w_i divides w_{k_i}, since w_i divides d by assumption.

For all $j = 1, \dots, n$, we set $L_j = \{i : k_i = j, i \neq j\}$. Let us define

$$h_j = \begin{cases} \displaystyle\sum_{i \in L_j} a_{j,i} y_i^{w_j/w_i} & \text{if } L_j \neq \emptyset \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

where we suppose that $\{a_{j,i}\}_{j,i}$ is a generic choice of coefficients in $\mathbb{C}$. It is straightforward to see that, given an index $j \in \{1, \dots, n\}$ such that $h_j \neq 0$, the polynomial h_j is weighted homogeneous of degree w_j.

Let us consider the map $\mathbf{x} : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $\mathbf{x}(y_1, \dots, y_n) = (x_1, \dots, x_n)$, given by

$$x_j = y_j + h_j(y) \quad \text{for all } j = 1, \dots, n.$$

We conclude that $\mathbf{x}$ is a local biholomorphism, the function $f \circ \mathbf{x}$ is weighted homogeneous with respect to w of degree d and, by the genericity of the coefficients $a_{j,i}$ in (23), the pure monomial y_i^{d/w_i} appears in the support of $f \circ \mathbf{x}$, for all $i = 1, \dots, n$. Hence the function $f \circ \mathbf{x}$ is convenient. □

Example 4.15. Set $w = (1, 2, 3, 4, 6)$ and $d = 12$. The polynomial $f = x_1^{12} + x_2^4x_4 + x_4^3 + x_3^2x_5 + x_5^2$ is weighted homogeneous of degree 12. Let J_i denote the ideal of $\mathcal{O}_5$ generated by all monomials x^k such that $k \in \Gamma_+(\partial f/\partial x_i)$, $i = 1, \dots, 5$. A straightforward computation shows that

$$J_1 = \langle x_1^{11} \rangle, \qquad J_2 = \langle x_2^3x_4 \rangle, \qquad J_3 = \langle x_3x_5 \rangle, \qquad J_4 = \langle x_2^4, x_4^2 \rangle, \qquad J_5 = \langle x_3^2, x_5 \rangle.$$

Since the ideals J_2 and J_3 do not contain any pure monomial, the family of ideals $\{J_i : i = 1, \dots, 5\}$ does not admit a w-matching.

Following the proof of Proposition 4.14, we consider the coordinate change $\mathbf{x} : (\mathbb{C}^5, 0) \to (\mathbb{C}^5, 0)$, given by: $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_4 + y_2^2$, $x_5 = y_5 + y_3^2$. Let $g = f \circ \mathbf{x}$ and let J_i' denote the ideal of $\mathcal{O}_5$ generated by all monomials y^k such that $k \in \Gamma_+(\partial g/\partial y_i)$, $i = 1, \dots, 5$. Then, as shown in that proof, the function g is convenient and therefore the family of ideals $\{J_i' : i = 1, \dots, 5\}$ admits a w-matching.

Corollary 4.16. *Let d, $w_1, \ldots, w_n$ be non-negative integers such that w_i divides d for all i = $1, \ldots, n$. Let f : $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a semi-weighted homogeneous function of degree d with respect to the weights $w_1, \ldots, w_n$. Then*

$$\mathcal{L}_0(\nabla f) = \frac{d - \min\{w_1, \ldots, w_n\}}{\min\{w_1, \ldots, w_n\}}.$$

Proof. Since f is semi-weighted homogeneous, the principal part $p_w(f)$ has an isolated singularity at the origin. Let $\mathbf{x} : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denote the analytic coordinate change obtained in Proposition 4.14 applied to $p_w(f)$. The function $p_w(f) \circ \mathbf{x}$ is weighted homogeneous of degree d with respect to w. Therefore

$$p_w(f) \circ \mathbf{x} = p_w(f \circ \mathbf{x}),$$

which implies that $f \circ \mathbf{x}$ is a semi-weighted homogeneous function. Then, by Proposition 4.14 and Remark 4.10, the function $f \circ \mathbf{x}$ admits a w-matching. Thus we obtain, by Corollary 4.11, that

$$\mathcal{L}_0(\nabla(f \circ \mathbf{x})) = \frac{d - \min\{w_1, \ldots, w_n\}}{\min\{w_1, \ldots w_n\}}.$$

Then the result follows, since the local Łojasiewicz exponent is a bianalytic invariant. □

We remark that in Corollary 4.16 we do not assume $2w_i \leqslant d$ as in [9]. This assumption cannot be eliminated from the main result of [9], as the following example shows. The result in 4.16 holds for any number of variables, but the assumptions are also restrictive, since we are assuming that the weights w_i divide d.

Example 4.17. Let us consider the polynomial f of $\mathcal{O}_3$ given by $f = x_1x_3 + x_2^2 + x_1^2x_2$. We observe that f is weighted homogeneous of degree 4 with respect to the vector of weights $w = (1, 2, 3)$. The Jacobian ideal is $\langle x_1, x_2, x_3 \rangle$ so that $\mathcal{L}_0(\nabla f) = 1 \neq 3$. We remark that it is easy to check that f does not admit a w-matching.

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References

[1] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, Singularities of differentiable maps, in: Vol. I: The Classification of Critical Points, Caustics and Wave Fronts, in: Monogr. Math., vol. 82, Birkhäuser Boston, Inc., Boston, MA, 1985.
[2] C. Bivià-Ausina, Jacobian ideals and the Newton non-degeneracy condition, Proc. Edinb. Math. Soc. 48 (2005) 21–36.
[3] C. Bivià-Ausina, Joint reductions of monomial ideals and multiplicity of complex analytic maps, Math. Res. Lett. 15 (2) (2008) 389–407.
[4] C. Bivià-Ausina, Local Łojasiewicz exponents, Milnor numbers and mixed multiplicities of ideals, Math. Z. 262 (2) (2009) 389–409.
[5] C. Bivià-Ausina, S. Encinas, Łojasiewicz exponents and resolution of singularities, Arch. Math. 93 (3) (2009) 225–234.
[6] C. Bivià-Ausina, T. Fukui, M. Saia, Newton graded algebras and the codimension of non-degenerate ideals, Math. Proc. Camb. Phil. Soc. 133 (2002) 55–75.
[7] J. Briançon, H. Maynadier-Gervais, Sur le nombre de Milnor d'une singularité semi-quasi-homogène, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 317–320.
[8] C. Huneke, I. Swanson, Integral Closure of Ideals, Rings, and Modules, in: London Math. Soc. Lecture Note Series, vol. 336, Cambridge University Press, 2006.
[9] T. Krasiński, G. Oleksik, A. Płoski, The Łojasiewicz exponent of an isolated weighted homogeneous surface singularity, Proc. Amer. Math. Soc. 137 (10) (2009) 3387–3397.
[10] M. Lejeune, B. Teissier, Clôture intégrale des idéaux et equisingularité, with an appendix by J.J. Risler, Ann. Fac. Sci. Toulouse Math. (6) 17 (4) (2008) 781–859.
[11] S. Łojasiewicz, Introduction to Complex Analytic Geometry, Birkhäuser Verlag, 1991.
[12] M. Nagata, Note on a paper of Samuel concerning asymptotic properties of ideals, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. 30 (1957) 165–175.
[13] V.P. Palamodov, The multiplicity of a holomorphic transformation, Funkcional. Anal. i Priložen 1 (3) (1967) 54–65.
[14] D. Rees, Valuations associated with a local ring II, J. Lond. Math. Soc. 31 (1956) 228–235.
[15] D. Rees, Generalizations of reductions and mixed multiplicities, J. Lond. Math. Soc. (2) 29 (1984) 397–414.
[16] O. Saeki, Topological invariance of weights for weighted homogeneous isolated singularities in $\mathbb{C}^3$, Proc. Amer. Math. Soc. 103 (3) (1988) 905–909.
[17] P. Samuel, Some asymptotic properties of powers of ideals, Ann. of Math. 56 (1) (1952) 11–21.
[18] I. Swanson, Multigraded Hilbert functions and mixed multiplicities, in: Syzygies and Hilbert Functions, in: Lect. Notes Pure Appl. Math., vol. 254, Chapman & Hall/CRC, Boca Raton, FL, 2007, pp. 267–280.
[19] B. Teissier, Variétés polaires I. Invariants polaires des singularités d'hypersurfaces, Invent. Math. 40 (3) (1977) 267–292.
[20] B. Teissier, Cycles évanescents, sections planes et conditions of Whitney, Singularités Cargèse Ast. (7–8) (1973) 285–362.
[21] S.S.-T. Yau, Topological types and multiplicities of isolated quasihomogeneous surface singularities, Bull. Amer. Math. Soc. 19 (1988) 447–454.