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Matrix representation of quaternions

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Abstract

We establish that there are a total of 48 distinct ordered sets of three 4×4 (skew-symmetric) signed permutation matrices which will serve as the basis of an algebra of quaternions.

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1. Introduction

Let h, j, k denote the imaginary units of the algebra of quaternions, then a typical quaternion may be written as

$$q = a + bh + cj + dk,$$

where a, b, c, d are real numbers. Now, let I be the 4×4 identity matrix and let H, J, K be 4×4 matrices with real elements. Then our typical quaternion may be written in matrix form as

$$Q = aI + bH + cJ + dK$$

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and the rules of quaternion addition and multiplication will follow from those of matrix addition and multiplication provided that the matrices H, J, K satisfy the 'Hamiltonian conditions'

$$
\begin{aligned}
&HH = -I, \quad JJ = -I, \quad KK = -I, \\
&HJ = K, \quad JK = H, \quad KH = J, \\
&JH = -K, \quad KJ = -H, \quad HK = -J.
\end{aligned}
$$

2. Fundamental results

Since the 4×4 matrices H, J, K, represent the imaginary units of the system, it seems appropriate to assume that they take the form of signed permutation matrices with one nonzero element in each row and one nonzero element in each column, the nonzero elements taking the values of plus one or minus one.

Let P be a 4×4 signed permutation matrix, then it satisfies the orthogonality condition $P'P = I$. Further, in the present context, we assume that it also satisfies the 'skew involutory condition' $PP = -I$, and we deduce that it is necessarily skew-symmetric $P' = -P$.

Let P be a 'normalised' 4×4 skew-symmetric signed permutation matrix with a plus one in its first row. Then this nonzero element may occur in any of three positions excluding the diagonal. Further, there will be an element of minus one in the corresponding position in the first column. Deleting the rows and columns associated with these nonzero elements, there remains a 2×2 submatrix with nonzero elements in its off-diagonal positions that may be assigned in one of two distinct ways. There are thus a total of six skew-symmetric signed permutation matrices with an element of plus one in their first rows.

Now, these six signed permutation matrices arrange themselves naturally to form two Hamiltonian systems. The first comprises:

$$
H_1 = B \otimes D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
$$

$$
J_1 = E \otimes B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},
$$

$$
K_1 = B \otimes C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},
$$

with their negations, whilst the second system comprises:

$$H_2 = D \otimes B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$J_2 = B \otimes E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$K_2 = C \otimes B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

with their negations, where $\otimes$ denotes Kronecker multiplication and where the 2×2 matrices B, C, D, E are defined by

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. How many distinct triplets?

Arbitrarily selecting any one of the matrices in either system to be our first normalised matrix, then we have two choices of a second normalised matrix from the same system and either of these matrices may be negated. Thus, for each of the two systems, we have a total of eight ordered combinations with the same normalised matrix (possibly negated) in first position. Moreover, each of these triplets may be rotated in the selected order, $(H\ J\ K)$, $(J\ K\ H)$, $(K\ H\ J)$, to yield a total of 24 triplets in each system, or 48 triplets in all.

In view of its importance, it is perhaps worth establishing this result by a different route. Selecting either of the above systems and explicitly augmenting the three given matrices with their negations, we find that there are a total of six possible choices for H. Then, deleting the chosen matrix and its negation, we have four choices for J, leaving K to be determined by $K = HJ$. Thus, we again have a total of 24 choices in each system, or 48 in all.

4. Completeness of the solution

The nontrivial multiplications of pairs of matrices within each system are defined by the Hamiltonian conditions of Section 1. It therefore remains to consider the effect of choosing one matrix from each system. Suppose that H_3 is selected from the first system and J_3 from the second. Then $K_3 = H_3 J_3$ and its negation must both be members of one of these systems if they are to be of interest to us as skew-symmetric signed permutation matrices.

Suppose, for instance, that K_3 is a member of the first system, then $J_3 = -H_3 K_3$, as the product of two members of the first system, must also be a member of the same system, contrary to hypothesis. Alternatively, suppose that K_3 is a member of the second system, then $H_3 = -K_3 J_3$, as the product of two members of the second system, must also be a member of this system. Further, since each of these alternatives lead to a contradiction, we may deduce that we have established that the 48 systems defined in the previous section are the only ones that can serve as the basis of an algebra of quaternions based on 4×4 signed permutation matrices.

5. Generating solutions

In a more general context, the 48 combinations identified in Section 3 are not the only possible solutions to our problem. For, if U is a real 4×4 nonsingular matrix, then the matrices

$$H_* = UHU^{-1}, \quad J_* = UJU^{-1}, \quad K_* = UKU^{-1}$$

also satisfy the Hamiltonian conditions of Section 1. Indeed, given any one solution to our problem, we may generate a full set of 48 solutions (with repetitions) by setting the 4×4 matrix U equal in turn to each of the $4 \cdot 3 \cdot 2 \cdot 2^3 = 192$ normalised signed permutation matrices with plus ones in their first rows.

Alternatively, having chosen any triplet $(H\ J\ K)$ satisfying the Hamiltonian conditions, we may negate any pair of elements to increase the number of solutions with the same normalised matrices in the same order to four; we may then negate the first element of the triplet and interchange the other two elements to increase the number of solutions with the same normalised matrix in first position to eight; then, as in Section 3, we may rotate these eight triplets in the selected order to increase the number of solutions in the chosen system to 24. Finally, we may obtain the 24 solutions in the other system by interchanging the roles of the 2×2 matrices within the Kronecker products of Section 2, so that $\pm H_1$ becomes $\pm H_2$, $\pm J_1$ becomes $\pm J_2$, $\pm K_1$ becomes $\pm K_2$, and vice versa.

6. Concluding remarks

Clearly, each of the 48 solutions based on skew-symmetric signed permutation matrices involves an algebra of 4×4 matrices with four different entities or their negations in the four rows and four columns of the matrix representing each quaternion. In this context, the elements in any one row or any one column may be used as a code defining the corresponding quaternion. This was the approach adopted by Groß et al. [1].

Finally, we note that higher-dimensional solutions to the Hamiltonian problem may readily be found amongst the m-fold Kronecker products of the 2×2 matrices B, C, D, E. For example, when $m = 3$ we have the 8×8 matrices:

$$H_4 = B \otimes C \otimes D, \quad J_4 = D \otimes B \otimes C, \quad K_4 = C \otimes D \otimes B$$

and

$$H_5 = B \otimes E \otimes D, \quad J_5 = E \otimes B \otimes C, \quad K_5 = B \otimes B \otimes B.$$

Reference

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