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A Tauberian theorem for the discrete M_φ summability method

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ABSTRACT

The object of this work is to retrieve the convergence of a series from its discrete M_φ summability under certain conditions. We obtain as a corollary a Tauberian theorem for the discrete logarithmic summability method.

1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers with partial sums (s_n). The Cesàro means of (s_n) are defined by

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k \quad (n = 0, 1, \ldots).$$

The identity

$$s_n - \sigma_n = V_n \quad (n = 0, 1, \ldots), \tag{1}$$

where $V_n = \frac{1}{n+1}\sum_{k=0}^{n} ka_k$, is known as the Kronecker identity.

Definition 1.1. If $\sum_{n=0}^{\infty} a_n x^n$ converges for $0 < x < 1$ and tends to s as $x \to 1^-$ we say that (s_n) is Abel summable to s and write $s_n \to s\ (A)$.

Denote the space of analytic functions in $0 < x < 1$ by $\mathcal{A}$ and the class of kernels of the integral transforms of functions in $\mathcal{A}$ by Φ. The following properties of functions φ in Φ are needed.

(1) There exists a number $\alpha_0 = \alpha_0(\Phi) \in (0, 1)$ such that every $\varphi \in \Phi$ is analytical in $[\alpha_0, 1)$.
(2) For every $\varphi \in \Phi$, $\varphi(x) \to \infty, x \to 1^-$.
(3) Each $\varphi \in \Phi$ is zero-free in $[\alpha_0, 1)$.
(4) For each $m \geq 1$,

$$\frac{\varphi_m(x)}{\varphi_{m-1}(x)} = o(1), x \to 1^-,$$

where $\varphi_0 = \varphi$ and $\varphi_m(x) = \int_{\alpha_0}^{x} \varphi_{m-1}(t)dt$.

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For every f in $\mathcal{A}$ and $\varphi \in \Phi$ we define

$$M(f, \varphi, x) = \frac{\int_{\alpha_0}^{x} f(t)\varphi(t)dt}{\varphi_1(x)}$$

if $x \neq \alpha_0$ and

$$\lim_{x \to \alpha_0} M(f, \varphi, x) = f(\alpha_0)$$

if $x = \alpha_0$.

Definition 1.2. If

$$\lim_{x \to 1^-} M(f, \varphi, x) = s \tag{2}$$

then we say that (s_n) is M_φ summable to s and write $s_n \to s\ (M_\varphi)$.

It is plain that Abel summability implies M_φ summability, but the converse statement is not always true. If $\varphi(x) = \frac{1}{(1-x)^2}$ for $x \in (0, 1)$ then the corresponding M_φ summability method reduces to the (A, 1) summability method defined in [1]. In this case, $M(f, \varphi, x) = (1 - x) \sum_{n=0}^{\infty} \sigma_n x^n$.

Discrete summability methods for power series have been extensively studied by a number of authors including Armitage and Maddox [2], and Watson [3,4].

We assume that (λ_n) satisfies $1 \leq \lambda_0 < \lambda_1 < \cdots \to \infty$. Define the sequence (x_n) by $x_n = 1 - \frac{1}{\lambda_n}$.

Definition 1.3. If $M(f, \varphi, x_n)$ exists for all n and $\lim_{n\to\infty} M(f, \varphi, x_n) = s$ then we say that (s_n) is summable by the discrete (M_φ) method and write $s_n \to s\ (M_\varphi)_\lambda$.

It is clear that $(M_\varphi)_\lambda$ includes (M_φ) from Definitions 1.2 and 1.3. If (s_n) converges to s, then (s_n) is $(M_\varphi)_\lambda$ summable to s. But the converse is satisfied under some additional conditions, which are so called Tauberian conditions. In this work we recover the convergence of (s_n) from its $(M_\varphi)_\lambda$ summability under some Tauberian conditions. We obtain as a corollary a Tauberian theorem for the discrete logarithmic summability method. The proof of our theorem mimics the proofing techniques of Theorem 3 in Ishiguro [5].

Theorem 1.4. *Suppose that:*

(i) $\gamma_1 \leq \frac{\lambda_n}{n} \leq \gamma_2$ *for some positive constants* γ_1 *and* γ_2.
(ii) $n\frac{\varphi_2(x_n)}{\varphi_1(x_n)} = O(1),\ n \to \infty$.
(iii) $s_n \to s\ (M_\varphi)$.
(iv) $na_n = o(1),\ n \to \infty$.

Then $s_n \to s$ *as* $n \to \infty$.

Proof. We have

$$\begin{aligned}
s_n - M(f, \varphi, x_n) &= \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} s_n \varphi(t)dt - \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} f(t)\varphi(t)dt \\
&= \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=0}^{\infty} (s_n - s_k) x^k \varphi(x)dx \\
&= \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=0}^{n} (s_n - s_k) x^k \varphi(x)dx + \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=n+1}^{\infty} (s_n - s_k) x^k \varphi(x)dx \\
&= I_1 + I_2, \text{ say.}
\end{aligned}$$

It suffices to show that both $I_1 \to 0$ and $I_2 \to 0$ as $n \to \infty$. For I_1 we have

$$\begin{aligned}
|I_1| &= \frac{1}{\varphi_1(x_n)} \left| \int_{\alpha_0}^{x_n} (1-x) \sum_{k=0}^{n} (s_n - s_k) x^k \varphi(x)dx \right| \\
&\leq \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=0}^{n} |s_n - s_k| x^k \varphi(x)dx \\
&\leq \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=0}^{n} |s_n - s_k| \varphi(x)dx
\end{aligned}$$

$$= \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x)\left\{|a_1+a_2+\cdots+a_n| + |a_2+\cdots+a_n| + \cdots + |a_n|\right\} \varphi(x)dx$$
$$= \frac{1}{\varphi_1(x_n)} \left(\sum_{k=0}^{n} k|a_k|\right) \int_{\alpha_0}^{x_n} (1-x)\varphi(x)dx$$
$$= \frac{n}{\varphi_1(x_n)} \left(\frac{1}{n}\sum_{k=0}^{n} k|a_k|\right) \left(\frac{\varphi_1(x_n)}{n} + \varphi_2(x_n)\right)$$
$$= \left(\frac{1}{n}\sum_{k=0}^{n} k|a_k|\right) \left(\frac{1}{\lambda_n} + n\frac{\varphi_2(x_n)}{\varphi_1(x_n)}\right).$$

We have, from the condition (iv), $\frac{1}{n}\sum_{k=0}^{n} k|a_k| = o(1)$ for $n \to \infty$. Hence, by condition (ii), we have

$$I_1 = o(1), \ n \to \infty. \tag{3}$$

Now estimate I_2. By condition (iv), there is an m such that $|na_n| \le \varepsilon$ for all $k \ge m$. Assume that $k > n \ge m$. Then we have

$$|s_k - s_n| \le \varepsilon \left\{\frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{k}\right\} = \varepsilon Q_k, \text{ say.}$$

Now

$$|I_2| \le \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=n+1}^{\infty} \varepsilon Q_k x^k \varphi(x)dx$$
$$\le \frac{1}{\varphi_1(x_n)} \int_{\alpha_0}^{x_n} (1-x) \sum_{k=n+1}^{\infty} \varepsilon Q_k x_n^k \varphi(x)dx$$

and

$$Q_k = \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{k}$$
$$\le \frac{k-n}{n}$$
$$< \frac{k+1}{n}.$$

Since $Q_k < \frac{k+1}{n}$, we get

$$|I_2| \le \frac{\varepsilon}{n\varphi_1(x_n)} \left(\frac{\varphi_1(x_n)}{\lambda_n} + \varphi_2(x_n)\right) \sum_{k=n+1}^{\infty} (k+1)x_n^k$$
$$\le \varepsilon \frac{(\lambda_n)^2}{n\varphi_1(x_n)} \left(\frac{\varphi_1(x_n)}{\lambda_n} + \varphi_2(x_n)\right)$$
$$= \varepsilon \left(\frac{\lambda_n}{n} + \frac{(\lambda_n)^2}{n}\frac{\varphi_2(x_n)}{\varphi_1(x_n)}\right).$$

By the conditions (i) and (ii), we have

$$|I_2| \le \varepsilon C, \tag{4}$$

for large n and some positive constant C.

Finally we have, from (3) and (4), that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} M(f, \varphi, x_n)$$

which completes the proof of Theorem 1.4. □

Using Theorem 1.4 and the Kronecker identity, we obtain the following result.

Corollary 1.5. *Suppose that:*

(i) $\gamma_1 \le \frac{\lambda_n}{n} \le \gamma_2$ *for some positive constants* γ_1 *and* γ_2.
(ii) $n\frac{\varphi_2(x_n)}{\varphi_1(x_n)} = O(1), n \to \infty$.

(iii) $\sigma_n \to s\,(M_\varphi)_\lambda$.
(iv) $V_n = o(1),\, n \to \infty$.

Then $s_n \to s$ *as* $n \to \infty$.

If we choose $\varphi(x) = \frac{1}{1-x}$ for $x \in (0, 1)$ then the corresponding M_φ summability method reduces to the logarithmic summability method. In this case we obtain the following result.

Corollary 1.6. *If* $na_n \to 0$ *and for some positive constants* γ_1 *and* γ_2, $\gamma_1 \le \frac{\lambda_n}{n} \le \gamma_2$, *then* $s_n \to s\,(M_\varphi)_\lambda$ *implies* $s_n \to s$ *as* $n \to \infty$.

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