Applied Mathematics Letters 24 (2011) 1509–1514

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On generalizations of certain summability methods using ideals

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ARTICLE INFO

Article history:
Received 20 January 2011
Accepted 20 March 2011

Keywords:
Ideal
Filter
I-statistical convergence
I-lacunary statistical convergence
Closed subspace

ABSTRACT

In this paper, following the line of Savas and Das (2011) [10], we provide a new approach to two well-known summability methods by using ideals, introduce new notions, namely, I-statistical convergence and I-lacunary statistical convergence, investigate their relationship, and make some observations about these classes.

1. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [1] (see also [2]) as follows. If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$, then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \quad \text{and} \quad \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists, and it is denoted simply by $d(K)$. Clearly, $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if, for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [3] and Šalát [4].

The idea of statistical convergence was further extended to I-convergence in [5] using the notion of ideals of $\mathbb{N}$, with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [6–10].

In another direction, a new type of convergence, called lacunary statistical convergence, was introduced in [11] inspired by the investigations in [12–15] as follows. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to L (or S_θ-convergent to L) if, for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0,$$

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doi:10.1016/j.aml.2011.03.036

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [11], the relation between lacunary statistical convergence and statistical convergence was established, among other things. More results on this convergence can be found in [16].

In this short paper we intend to unify these two approaches, and we use ideals to introduce the concept of I-statistical convergence and I-lacunary statistical convergence, which naturally extend the notions of statistical convergence and lacunary statistical convergence. We try to establish the relation between these two summability notions. The same method was recently adopted by the first two authors, in [10], to introduce another new type of convergence which is associated with I-statistical convergence.

Throughout, by a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ we shall mean a sequence of real numbers.

2. Main results

The following definitions and notions will be needed.

Definition 1. A non-empty family $I \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:

(a) $A, B \in I$ implies $A \cup B \in I$,
(b) $A \in I, B \subset A$ implies $B \in I$.

Definition 2. A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:

(a) $\phi \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F, A \subset B$ implies $B \in F$.

If I is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets $F(I) = \{M \subset \mathbb{N} : \exists A \in I : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 3. A proper ideal I is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Throughout, I will stand for a proper admissible ideal of $\mathbb{N}$, and by a sequence we always mean a sequence of real numbers.

Definition 4 (*See [5]*)**.** Let $I \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be I-convergent to $L \in \mathbb{R}$ if, for each $\epsilon > 0$, the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in I$.
(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be I^*-convergent to $L \in \mathbb{R}$ if there exists $M \in F(I)$ such that $\{x_n\}_{n\in M}$ converges to L.

We now introduce our main definitions.

Definition 5. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be I-statistically convergent to L or $S(I)$-convergent to L if, for each $\epsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in I.$$

In this case, we write $x_k \to L(S(I))$. The class of all I-statistically convergent sequences will be denoted simply by $S(I)$.

Remark 1. Let us take the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, where $\lambda_n = 1$ for $n = 1$ to 10 and $\lambda_n = n - 10$ for all $n \geq 10$, and take $I = I_d$ (the ideal of density zero sets of N); then, let $A = \{1^2, 2^2, 3^2, 4^2, 5^2, \ldots\}$.

Define $x = \{x_k\}_{k\in\mathbb{N}}$ in a normed linear space X by

$$x_k = \begin{cases} ku & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n,\ n \notin A \\ ku & \text{for } n - \lambda_n + 1 \leq k \leq n,\ n \in A \\ \theta & \text{otherwise,} \end{cases}$$

where $u \in X$ is a fixed element with $\|u\| = 1$, and θ is the null element of X. Then, the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is an example of a sequence which is I-statistically convergent (by Theorem 2.3 in [10]) but is not statistically convergent (see Remark 2 in [10]).

Definition 6. Let θ be a lacunary sequence. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be I-lacunary statistically convergent to L or $S_\theta(I)$-convergent to L if, for any $\epsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in I.$$

In this case, we write $x_k \to L(S_\theta(I))$. The class of all I-lacunary statistically convergent sequences will be denoted by $S_\theta(I)$.

It can be checked as in the case of statistically and lacunary statistically convergent sequences that both $S(I)$ and $S_\theta(I)$ are linear subspaces of the space of all real sequences.

As the proofs for both the assertions are similar, we present the proof for $S_\theta(I)$ only.

Theorem 1. *$S_\theta(I) \cap l_\infty$ is a closed subset of l_∞ where l_∞ stands for the space of all bounded sequences of real numbers.*

Proof. Suppose that $\{x^n\}_{n\in\mathbb{N}} \subseteq S_\theta(I) \cap l_\infty$ is a convergent sequence and that it converges to $x \in l_\infty$. We need to prove that $x \in S_\theta(I) \cap l_\infty$. Assume that $x^n \to L_n(S_\theta(I))$, $\forall n \in \mathbb{N}$. Take a positive strictly decreasing sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ where $\epsilon_n = \frac{\epsilon}{2^n}$ for a given $\epsilon > 0$. Clearly $\{\epsilon_n\}_{n\in\mathbb{N}}$ converges to 0. Choose a positive integer n such that $\|x - x^n\|_\infty < \frac{\epsilon_n}{4}$. Let $0 < \delta < 1$. Then

$$A = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^n - L_n| \geq \frac{\epsilon_n}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(I)$$

$$\text{and} \quad B = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^{n+1} - L_{n+1}| \geq \frac{\epsilon_{n+1}}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(I).$$

Since $A \cap B \in F(I)$ and $\phi \notin F(I)$, we can choose $r \in A \cap B$.

$$\text{Then } \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^n - L_n| \geq \frac{\epsilon_n}{4}\right\}\right| < \frac{\delta}{3}$$

$$\text{and } \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^{n+1} - L_{n+1}| \geq \frac{\epsilon_{n+1}}{4}\right\}\right| < \frac{\delta}{3}$$

$$\text{and so } \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^n - L_n| \geq \frac{\epsilon_n}{4} \vee |x_k^{n+1} - L_{n+1}| \geq \frac{\epsilon_{n+1}}{4}\right\}\right| < \delta < 1.$$

Hence, there exists a $k \in I_r$ for which $|x_k^n - L_n| < \frac{\epsilon_n}{4}$ and $|x_k^{n+1} - L_{n+1}| < \frac{\epsilon_{n+1}}{4}$.

Then, we can write

$$\begin{aligned}|L_n - L_{n+1}| &\leq |L_n - x_k^n| + |x_k^n - x_k^{n+1}| + |x_k^{n+1} - L_{n+1}| \\ &\leq |x_k^n - L_n| + |x_k^{n+1} - L_{n+1}| + \|x - x^n\|_\infty + \|x - x^{n+1}\|_\infty \\ &\leq \frac{\epsilon_n}{4} + \frac{\epsilon_{n+1}}{4} + \frac{\epsilon_n}{4} + \frac{\epsilon_{n+1}}{4} \leq \epsilon_n.\end{aligned}$$

This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and so there is a real number L such that $L_n \to L$, as $n \to \infty$. We need to prove that $x \to L(S_\theta(I))$. For any $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $\epsilon_n < \frac{\epsilon}{4}$, $\|x - x^n\|_\infty < \frac{\epsilon}{4}$, $|L_n - L| < \frac{\epsilon}{4}$. Then

$$\begin{aligned}\frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| &\leq \frac{1}{h_r}|\{k \in I_r : |x_k^n - L_n| + \|x_k - x_k^n\|_\infty + |L_n - L| \geq \epsilon\}| \\ &\leq \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^n - L_n| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \geq \epsilon\right\}\right| \\ &\leq \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^n - L_n| \geq \frac{\epsilon}{2}\right\}\right|.\end{aligned}$$

This implies that $\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| < \delta\right\} \supseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : |x_k^n - L_n| \geq \frac{\epsilon}{2}\right\}\right| < \delta\right\} \in F(I)$. So $\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| < \delta\right\} \in F(I)$, and so $\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in I$. This gives that $x \to L(S_\theta(I))$, and this completes the proof of the theorem. □

In the following, we investigate the relationship between I-statistical and I-lacunary statistical convergence. However, to prove Theorem 3 which describes the above mentioned relation we will need the following result, which gives an alternative characterization of I-lacunary statistical convergence of bounded real sequences similar to the characterization of lacunary statistical convergence given in [11].

Definition 7 (*cf.* [12,13])**.** Let θ be a lacunary sequence. Then $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $N_\theta(I)$-convergent to L if, for any $\epsilon > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}|x_k - L| \geq \epsilon\right\} \in I.$$

This convergence is denoted by $x_k \to L(N_\theta(I))$, and the class of such sequences will be denoted simply by $N_\theta(I)$.

Theorem 2. *Let $\theta = \{k_r\}_{r\in\mathbb{N}}$ be a lacunary sequence. Then*

(i) (a) $x_k \to L(N_\theta(I)) \Rightarrow x_k \to L(S_\theta(I))$, *and*

(b) $N_\theta(I)$ *is a proper subset of* $S_\theta(I)$.
(ii) $x \in l_\infty$ *and* $x_k \to L(S_\theta(I)) \Rightarrow x_k \to L(N_\theta(I))$,
(iii) $S_\theta(I) \cap l_\infty = N_\theta(I) \cap l_\infty$.

Proof. (i) (a) If $\epsilon > 0$ and $x_k \to L(N_\theta(I))$, we can write

$$\sum_{k\in I_r} |x_k - L| \geq \sum_{k\in I_r, |x_k - L|\geq\epsilon} |x_k - L| \geq \epsilon |\{k \in I_r : |x_k - L| \geq \epsilon\}|$$

and so $$\frac{1}{\epsilon . h_r} \sum_{k\in I_r} |x_k - L| \geq \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}|.$$

Then, for any $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| \geq \epsilon . \delta\right\} \in I.$$

This proves the result.
(b) In order to establish that the inclusion $N_\theta(I) \subseteq S_\theta(I)$ is proper, let θ be given, and define x_k to be $1, 2, \ldots, [\sqrt{h_r}]$ for the first $[\sqrt{h_r}]$ integers in I_r and $x_k = 0$ otherwise, for all $r = 1, 2, 3, \ldots$. Then, for any $\epsilon > 0$,

$$\frac{1}{h_r} |\{k \in I_r : |x_k - 0| \geq \epsilon\}| \leq \frac{[\sqrt{h_r}]}{h_r},$$

and for any $\delta > 0$ we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - 0| \geq \epsilon\}| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{[\sqrt{h_r}]}{h_r} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to I, it follows that $x_k \to 0(S_\theta(I))$.

On the other hand,

$$\frac{1}{h_r} \sum_{k\in I_r} |x_k - 0| = \frac{1}{h_r} . \frac{[\sqrt{h_r}]([\sqrt{h_r}] + 1)}{2}.$$

Then $\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - 0| \geq \frac{1}{4}\right\} = \left\{r \in \mathbb{N} : \frac{[\sqrt{h_r}]([\sqrt{h_r}]+1)}{h_r} \geq \frac{1}{2}\right\} = \{m, m + 1, m + 2, \ldots\}$ for some $m \in \mathbb{N}$ which belongs to $F(I)$, since I is admissible. So $x_k \nrightarrow 0(N_\theta(I))$.
(ii) Suppose that $x_k \to L(S_\theta(I))$ and $x \in l_\infty$. Then there exists an $M > 0$ such that $|x_k - L| \leq M \ \forall k \in \mathbb{N}$. Given $\epsilon > 0$, we have

$$\frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = \frac{1}{h_r} \sum_{k\in I_r, |x_k - L|\geq\frac{\epsilon}{2}} |x_k - L| + \frac{1}{h_r} \sum_{k\in I_r, |x_k - L|<\frac{\epsilon}{2}} |x_k - L| \leq \frac{M}{h_r} \left|\left\{k \in I_r : |x_k - L| \geq \frac{\epsilon}{2}\right\}\right| + \frac{\epsilon}{2}.$$

Consequently, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| \geq \epsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in I_r : |x_k - L| \geq \frac{\epsilon}{2}\right\}\right| \geq \frac{\epsilon}{2M}\right\} \in I.$$

This proves the result.
(iii) Follows from (i) and (ii). □

Theorem 3. *For any lacunary sequence* θ, *I-statistical convergence implies I-lacunary statistical convergence if and only if* $\liminf_r q_r > 1$. *If* $\liminf_r q_r = 1$, *then there exists a bounded sequence* $\{x_n\}_{n\in\mathbb{N}}$ *which is I-statistically convergent but not I-lacunary statistically convergent.*

Proof. Suppose first that $\liminf_r q_r > 1$. Then there exists $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1 + \alpha}.$$

Since $x_k \to L(S(I))$, for every $\epsilon > 0$, and for sufficiently large r, we have

$$\begin{aligned} \frac{1}{k_r} |\{k \leq k_r : |x_k - L| \geq \epsilon\}| &\geq \frac{1}{k_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \\ &\geq \frac{\alpha}{1 + \alpha} . \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}|. \end{aligned}$$

Then, for any $\delta > 0$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}|\{k \leq k_r : |x_k - L| \geq \epsilon\}| \geq \frac{\delta\alpha}{(1+\alpha)}\right\} \in I.$$

This proves the sufficiency.

Conversely, suppose that $\liminf_r q_r = 1$. Proceeding as in [1, p-510], we can select a subsequence $\{k_{r_j}\}$ of the lacunary sequence θ such that

$$\frac{k_{r_j}}{k_{r_{j-1}}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r_{j-1}}}{k_{r_{j-1}}} > j, \quad \text{where } r_j \geq r_{j-1} + 2.$$

Define a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ by

$$\begin{aligned} x_i &= 1 \quad \text{if } i \in I_{r_j} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then, for any real L,

$$\frac{1}{h_{r_j}} \sum_{k \in I_{r_j}} |x_i - L| = |1 - L| \quad \text{for } j = 1, 2, \ldots$$

$$\text{and} \quad \frac{1}{h_r} \sum_{k \in I_r} |x_i - L| = |L| \quad \text{for } r \neq r_j.$$

Then it is quite clear that x does not belong to $N_\theta(I)$. Since x is bounded, Theorem 2(iii) implies that $x \nrightarrow L(S_\theta(I))$.

Next, let $k_{r_j-1} \leq n \leq k_{r_{j+1}-1}$. Then, from Theorem 2.1 in [1], we can write

$$\frac{\epsilon}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| \leq \frac{1}{n}\sum_{i=1}^{n}|x_i| \leq \frac{k_{r_j-1} + h_{r_j}}{k_{r_j-1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is I-statistically convergent for any admissible ideal I.

It is known that [11] lacunary statistical convergence implies statistical convergence if and only if $\lim_r \sup q_r < \infty$ (i.e., when $I = I_{\text{fin}}$ is the ideal of finite subsets of $\mathbb{N}$). However, for arbitrary admissible ideal I, this is not clear, and we leave it as an open problem. □

Remark 2. For any lacunary sequence θ, with $\liminf_r q_r > 1$, the sequence given in Remark 1 is an example of a sequence which is I-lacunary statistically convergent.

Problem 1. When does I-lacunary statistical convergence imply I-statistical convergence?

Recall [5,8] that an admissible ideal I is said to satisfy condition (AP) if, for any mutually disjoint sequence of sets $\{A_i\}_{i\in\mathbb{N}}$ in I, there exists a sequence $\{B_i\}_{i\in\mathbb{N}}$ in I such that $A_i \Delta B_i$ is finite for all $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} B_i \in I$.

It was observed in [5] (see also [8]) that, for a sequence $\{x_n\}_{n\in\mathbb{N}}$, I-convergence is equivalent to I^*-convergence if and only if the ideal I satisfies the condition (AP). More facts about condition (AP) and its importance can be found in [6]. We are now ready to prove our next result.

Theorem 4. *Let I be an admissible ideal satisfying condition* (AP), *and let $\theta \in F(I)$. If $x \in S(I) \cap S_\theta(I)$, then $S(I) - \lim x = S_\theta(I) - \lim x$.*

Proof. Suppose that $S(I) - \lim x = L$, $S_\theta(I) - \lim x = L'$, and $L \neq L'$. Let $0 < \epsilon < \frac{1}{2}|L - L'|$. Since I satisfies the condition (AP), there exists $M \in F(I)$ (i.e., $\mathbb{N} \setminus M \in I$) such that

$$\lim_{r\to\infty} \frac{1}{m_r}|\{k \leq m_r : |x_k - L| \geq \epsilon\}| = 0, \quad \text{where } M = \{m_1, m_2, m_3, \ldots\}.$$

Let $A = \{k \leq m_r : |x_k - L| \geq \epsilon\}$ and $B = \{k \leq m_r : |x_k - L'| \geq \epsilon\}$. Then $m_r = |A \cup B| \leq |A| + |B|$. This implies that

$$1 \leq \frac{|A|}{m_r} + \frac{|B|}{m_r}. \text{ Since } \frac{|B|}{m_r} \leq 1 \text{ and } \lim_{r\to\infty} \frac{|A|}{m_r} = 0, \text{ so we must have } \lim_{r\to\infty} \frac{|B|}{m_r} = 1.$$

Let $M^* = \{k_{l_1}, k_{l_2}, k_{l_3}, \ldots\} = M \cap \theta \in F(I)$. Then the k_{l_p}th term of the statistical limit expression ${m_r}^{-1}|\{k \leq m_r : |x_k - L'| \geq \epsilon\}|$ is

$$\frac{1}{k_{l_p}} \left| \left\{ k \in \bigcup_{i=1}^{l_p} I_i : |x_k - L'| \geq \epsilon \right\} \right| = \frac{1}{\sum_{i=1}^{l_p} h_i} \sum_{i=1}^{l_p} t_i h_i, \tag{1}$$

where $t_i = {h_i}^{-1}|\{k \in I_i : |x_k - L'| \geq \epsilon\}| \xrightarrow{I} 0$ because $x_k \to L'(S_\theta(I))$. Since θ is a lacunary sequence, (1) is a regular weighted mean transform of t_i's, and therefore it is also I-convergent to zero as $p \to \infty$, and so it has a subsequence which is convergent to zero since I satisfies condition (AP). But since this is a subsequence of $\{n^{-1}|\{k \leq n : |x_k - L'| \geq \epsilon\}|\}_{n \in M}$, we infer that $\{n^{-1}|\{k \leq n : |x_k - L'| \geq \epsilon\}|\}_{n \in M}$ is not convergent to 1, which is a contradiction. This completes the proof of the theorem. □

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