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Remarks on Finite Rank Projections

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If every n-dimensional subspace of X^* is the range of a projection of norm less than C, then every subspace of X with codimension n is the range of a projection having norm less than $1 + C$. Also, projection constants of finite-dimensional spaces are determined by finite-dimensional superspaces. It is further demonstrated that spheres cannot, in general, be nicely embedded into unit balls of finite-dimensional spaces.

This note is primarily concerned with the solution of some problems, stated in the paper of Cheney and Price [1], on projections of finite rank (that is, having finite-dimensional range) in Banach spaces. We see in Section 1 that a sphere cannot always be embedded nicely into the unit ball of a finite-dimensional space: In particular, if f_1, f_2, and f_3 are in $l_\infty^{(3)}$ and if for x in $l_1^{(3)}$, $\{f_1(x)^2 + f_2(x)^2 + f_3(x)^2\}^{1/2} \geqslant \| x \|$, we must have $\| f_i \| > 1$ for some i. This gives a negative solution to part of problem 6 of [1].

The "principle of local reflexivity" of Lindenstrauss and Rosenthal [7] is extended, in the second section, to show that finite rank projections on a conjugate space X^* are, in a certain sense, near adjoints of finite rank projections on X. From this one easily deduces that if every n-dimensional subspace of X^* is complemented with norm $< c_n$, then every subspace of X having deficiency n is complemented with norm $< 1 + c_n$ (this gives an affirmative solution to problem 8 of [1]). From an unpublished result of Kadec to the effect that every n-dimensional subspace of every Banach space is complemented with norm $\leqslant n^{1/2}$, it follows that if Y has deficiency n in x and if $\epsilon > 0$, there is a projection of norm $< 1 + n^{1/2} + \epsilon$ of X onto Y. This result and the result of Kadec together with its proof, occur in [2].

Finally, the "compactness argument" of Lindenstrauss (see e.g. [6]) is applied directly to show that if Y is a finite-dimensional subspace of X and if P is a "best" (in terms of norm) projection of X onto Y, then $\| P \| = \sup \| R \|$ where the sup is over all "best" projections of Z onto Y, Z is finite-dimensional and $Y \subset Z \subset X$. This answers problem 9 of [1].

1

We show that, in general, spheres cannot be efficiently inscribed in unit balls of finite-dimensional spaces. Suppose that X is an n-dimensional space with norm $\| \cdot \|$. Suppose that there exist functionals $f_1, \ldots, f_n$ in the ball of $X^*(B_{X^*})$ such that

$$\left\{ \sum_{j=1}^{n} f_j(x)^2 \right\}^{1/2} = \| x \|_2 \geqslant \| x \|$$

for every x in X. There must be a vector x_k in $\bigcap_{i=1, i\neq k}^{n} \ker(f_i)$ such that $f_k(x_k) = 1 = \| x_k \|_2 \geqslant \| x_k \|$. Since $\| f_k \| \leqslant 1$, it follows that $\| x_k \| = \| f_k \| = 1$ for $k = 1, 2, \ldots, n$ and that $f_i(x_j) = \delta_{ij}$. The system $(x_i ; f_i)$ is called a *normal basis* for X and must satisfy the condition that $\operatorname{sp}\{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\}$ is parallel to the supporting hyperplane to B_X at x_k (that is, $\{f_k(x) = 1\}$).

We are now able to show that the ball of $l_1^{(3)}$ has no such inscribed sphere.

THEOREM 1. *If* $\{f_1, f_2, f_3\}$ *are in*

$$l_\infty^{(3)} \ (= (l_1^{(3)})^*) \qquad \text{and} \qquad \{f_1(x)^2 + f_2(x)^2 + f_3(x)^2\}^{1/2} \geqslant \| x \|$$

for every x *in* $l_1^{(3)}$, *then* $\| f_i \| > 1$ *for some* i.

Proof. Suppose that there is a normal basis (as above) with $\| x \|_2 \geqslant \| x \|$ always. Then, notice that $x_i = (a_{i1}, a_{i2}, a_{i3})$ must have $| a_{ij} |$ different from zero for each i, j. This is due to the fact that since $\{\| x \|_2 = 1\}$ is tangent to B_X at x_1, x_2 and x_3 and since $\{\| x \|_2 = 1\} \subset B_X$ these are smooth points of the ball of $l_1^{(3)}$. We may as well assume that a_{11}, a_{12}, and a_{13} are all positive. Then $f_1 = (1, 1, 1)$. Since $(x_i ; f_i)$ is a normal basis, we can conclude that $a_{21} + a_{22} + a_{23} = a_{31} + a_{32} + a_{33} = 0$. For definiteness, assume that $a_{21} > 0$, $a_{22} > 0$ and $a_{23} < 0$ (the argument will apply to all legitimate choices of sign for the a_{ij}'s). This condition forces $f_2 = (1, 1, -1)$. In turn, $a_{31} + a_{32} - a_{33} = 0$, so that $a_{33} = 0$. This is impossible in our situation, and proves the theorem.

2

Let us recall some elementary facts and notation which will be used here. If R is a finite rank projection on X, then R: $X \to X$ and R^* is a finite rank projection on X^*. If $\{x_1, \ldots, x_n\}$ are in X, then $[x_1, \ldots, x_n]$ is to denote the linear span the x_i's in X. If T is a map from X to Y and W is a subspace of X denote the norm of $T \mid W$ by $\| T \|_W$.

Theorem 2 below is a modification of the "principle of local reflexivity" of Lindenstrauss and Rosenthal [7]. The author has recently learned that similar versions of this principle occur in [4] and [5]. One change in the proof is the use of the following lemma (suggested to the author by J. Daneman) instead of the separation lemma of Klee [3].

LEMMA. *Let $C_1, \ldots, C_n$ be open convex subsets of a Banach space X, and suppose $\bigcap \bar{C}_i^{w*}$ has a nonempty core. Then $\bigcap C_i \neq \varnothing$. (For a set A in X, $\bar{A}^{w*}$ denotes its weak* closure in X^{**}).*

Proof. By induction, first consider $n = 2$ (the case $n = 1$ is trivial and the second case provides the proof of the lemma). Suppose $C_1 \cap C_2 = \varnothing$ so there is an f in X^* and a scalar α such that $f(C_1) < \alpha < f(C_2)$, then, $f(\bar{C}_1^{w*}) \leqslant \alpha \leqslant f(\bar{C}_2^{w*})$. Let Ψ in X^{**} be such that $\Psi(f) = 1$. Since there is a core point φ of $\bar{C}_1^{w*} \cap \bar{C}_2^{w*}$, there is $\delta > 0$ such that $|\lambda| < \delta$ implies $\varphi + \lambda\Psi$ is in $\bar{C}_1^{w*} \cap \bar{C}_2^{w*}$. This is incompatible with $(\varphi + \lambda\Psi)(f) = \alpha$ for all such λ, giving the desired contradiction. Now, assuming the conclusion for $n - 1$, let $C_1, \ldots, C_n$ satisfy the hypotheses so that $\varnothing \neq D = \bigcap_2^n C_j$. Let $\varphi \in$ core $\bar{C}_2^{w*} \cap \cdots \cap \bar{C}_n^{w*}$ and $\varphi \notin \bar{D}^{w*}$. Then there is an f in X^* such that $\varphi(f) > 1$ and $f(d) \leqslant 1$ for all d in D. However, letting $B_i = C_i \cap \{x \mid f(x) > 1\}$ for $i = 2, 3, \ldots, n$, we see that the hypotheses for the case $n - 1$ apply to give $\varnothing \neq \bigcap_2^n B_j \subset D$ which is a contradiction. Thus core $\bar{C}_2^{w*} \cap \cdots \cap \bar{C}_n^{w*} \subset \bar{D}^{w*}$ so that $\bar{C}_1^{w*} \cap \bar{D}^{w*}$ has a core. Now apply the argument for $n = 2$ to the pair C_1, D to see that $\varnothing \neq C_1 \cap D = \bigcap_{j=1}^n C_j$.

THEOREM 2. *Let P be a finite rank projection on X^* and let $\epsilon > 0$. Let V be any finite-dimensional subspace of X^*. Then there is a finite rank projection R on X such that $R^*(X^*) = P(X^*)$, $\| P - R^* \|_V < \epsilon$ and $\| R \| < \| P \| + \epsilon$.*

Proof. Let

$$Pf = \sum_{i=1}^{n} \varphi_i(f) f_i \qquad \text{with} \qquad \varphi_i(f_j) = \delta_{ij},$$

where $\{\varphi_1, \ldots, \varphi_n\} \subset X^{**}$. Next choose $\{f_{n+1}, \ldots, f_m\}$ in $[\varphi_1, \ldots, \varphi_n]_\perp$ so that $\{f_1, \ldots, f_m\}$ is a basis for $\mathrm{sp}\{f_1, \ldots, f_n, V\} = [f_1, \ldots, f_n, V]$. Now for $\delta > 0$ and $\eta > 0$ (to be determined later), let $\{\Psi_i \mid 1 \leqslant i \leqslant p\}$ be a δ-net on the unit sphere of $[\varphi_1, \ldots, \varphi_n]$ in X^{**}. Define the following open convex subsets of $X^n (= X \times \cdots \times X)$, for $i = 1, 2, \ldots, p$:

$$K_i = \left\{ (x_1, \ldots, x_n) \,\middle|\, \left\| \sum_{j=1}^{n} \Psi_i(f_j)\, x_j \right\| < 1 + \delta \right\}$$

and

$$D = \left\{(x_1, ..., x_n) \,\Big|\, \sum_{i=1}^{m} \sum_{j=1}^{n} |f_i(x_j) - \delta_{ij}| < \eta\right\}.$$

Let K_i^{**} and D^{**} be the similarly defined subsets of $(X^{**})^n$. Then D^{**} is a weak*-open set containing $(\varphi_1, ..., \varphi_n)$ and K_i^{**} is strongly open, containing $(\varphi_1, ..., \varphi_n)$. It follows easily (as in [7; proof of Theor. 3.1]) that $\bar{K}_i^{w*} \supset K_i^{**}$, and $\bar{D}^{w*} \supset D^{**}$. The hypotheses of the lemma are now satisfied for the $p + 1$ sets $K_1, ..., K_p, D$. Therefore, there is some $(x_1, ..., x_n)$ in X^n common to all of these sets. Now define $T: [\varphi_1, ..., \varphi_n] \to [x_1, ..., x_n]$ as the linear extension of $T\varphi_i = x_i$; $i = 1, ..., n$. Let $\psi \in [\varphi_1, ..., \varphi_n]$ have norm one, let ψ_j satisfy $\| \psi - \psi_j \| < \delta$ and suppose $\| T\psi \| = \| T \|$. Then

$$\| T \| \leqslant \| T(\psi_j)\| + \| T \| \| \psi - \psi_j \|.$$

Now, $\| T(\psi_j)\| = \| \sum \psi_j(f_i)\, x_i \| < 1 + \delta$ since $(x_1, ..., x_n)$ is in K_j. It follows from these inequalities that $\| T \| \leqslant (1 + \delta)/(1 - \delta)$. (This argument is similar to the same norm estimate in [7].) Since $(x_1, ..., x_n)$ is in D it follows that the matrix $B = (f_j(x_i) \mid i = 1, ..., n; j = 1, ..., n)$ is invertible for $\eta < 1$ (using the Neumann series). Let $A = (a_{ij})$ be its inverse so that $A = \sum (I - B)^k$. Now set $y_i = \sum_{j=1}^n a_{ij} x_j$ and let Δ be the linear extension of $\Delta x_i = y_i$ to all of $[x_1, ..., x_n]$. It is easy to see that $f_i(y_j) = \delta_{ij}$. Now we estimate the norm of Δ. Let $W = \sum f_i(W)\, y_i$. Then

$$\| W - \Delta^{-1}(W)\| = \left\| \sum f_i(W)(y_i - x_i) \right\| \leqslant \| W \| \sum \| f_i \| \| y_i - x_i \|,$$

$$\left(\sum_{i=1}^{n} \| f_i \| \left\| \left(\sum_{j=1}^{n} \delta_{ij} - a_{ij}\right) x_j \right\|\right) \| W \|$$

$$\leqslant \left(\sum_{i,j=1}^{n} | \delta_{ij} - a_{ij} | \, \| f_i \| \| x_j \|\right) \| W \|,$$

$$\leqslant \left(\left(\frac{\eta}{1 - \eta}\right) \| T \| \sum_{i,j=1}^{n} \| f_i \| \| \varphi_j \|\right) \| W \|,$$

$$\leqslant \left(\left(\frac{\eta}{1 - \eta}\right)\left(\frac{1 + \delta}{1 - \delta}\right) \sum_{i,j=1}^{n} \| f_i \| \| \varphi_j \|\right) \| W \|,$$

$$\leqslant \left(2\left(\frac{\eta}{1 - \eta}\right) \sum_{i,j=1}^{n} \| f_i \| \| \varphi_j \|\right) \| W \|, \qquad \text{for} \quad 0 < \delta \leqslant \tfrac{1}{2}.$$

Therefore, for any $K \in (0, 1)$ there exists $\eta_0 > 0$ such that $0 < \eta < \eta_0$ implies $\| W - \Delta^{-1}(W)\| < K \| W \|$. Thus, $\Delta = \sum (I - \Delta^{-1})^j$ so $\| \Delta \| \leqslant (1 - K)^{-1}$. Now we define $Ru = \sum f_i(u) y_i$. If J is the canonical map from X to X^{**}, one verifies directly that $R = \Delta TP^*J$, so that

$$\| R \| \leqslant (1/(1 - K))((1 + \delta)/(1 - \delta)) \| P \|.$$

By choosing η and δ small we get $\| R \| \leqslant \| P \| + \epsilon$. Further, it is clear that $R^*X^* = PX^*$. Now to check the final assertion,

$$\begin{aligned}\Big\| (R^* - P) \sum_{j=1}^{m} \alpha_j f_j \Big\| &= \Big\| \sum_{j=n+1}^{m} \alpha_j R^* f_j \Big\| = \Big\| \sum_{j=n+1}^{m} \alpha_j \sum_{i=1}^{n} f_j(y_i) f_i \Big\|, \\ &= \Big\| \sum_{j=n+1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_j a_{ik} f_j(x_k) f_i \Big\|, \\ &\leqslant \Big(\sum_{j=n+1}^{m} \sum_{k=1}^{n} | f_j(x_k)| \Big) \sum | \alpha_j | \sum | a_{ik} | \sum \| f_i \|,\end{aligned}$$

and the first term (being smaller than η) approaches 0 as $\eta \to 0$ for each $\sum_{j=1}^{m} \alpha_j f_j$. By choosing η smaller if necessary, the conclusion follows.

COROLLARY. *If every n-dimensional subspace of X^* is complemented with norm $< K_n$, then every subspace of X having deficiency n is complemented with norm $< 1 + K_n$.*

Proof. Let $U = [f_1, ..., f_n]_\perp$, $P: X^* \to [f_1, ..., f_n]$ having norm $< K_n$ and $\epsilon < K_n - \| P \|$. If R is the projection of the theorem, then $(I - R)X = (R^*X^*)_\perp = U$ and $\| I - R \| \leqslant 1 + \| R \| \leqslant 1 + \| P \| + \epsilon < 1 + K_n$.

This gives an affirmative solution to Problem 8 of [1].

Kadec has recently shown the following (see [2]): If Y is an n-dimensional subspace of (any Banach space) X, then there is a projection of X onto Y with norm $\leqslant n^{1/2}$. This allows the following refinement of Theorem 6 of [1]. (This result also appears in [2]).

COROLLARY. *If Y has deficiency n in X, and if $\epsilon > 0$ there is a projection of norm $< 1 + n^{1/2} + \epsilon$ of X onto Y.*

It is not known whether every Banach space has "nicely" complemented subspaces of arbitrarily large finite dimension. That is, given X, does there exist a constant M such that for every n there is a subspace U of X having dimension $\geqslant n$ and complemented with norm $\leqslant M$. The next corollary says that one may as well restrict his attention to conjugate spaces in studying this question.

COROLLARY. *If, for X^*, there is a constant M and subspaces V_n of X^* with* $\dim V_n \geqslant n$ *complemented with norm* $\leqslant M$, *then X contains subspaces U_n with* $\dim U_n \geqslant n$ *and complemented with norm* $\leqslant M$.

We must remark that if a finite-dimensional subspace is complemented with norm $\leqslant M + \epsilon$ for every $\epsilon > 0$, then it is complemented with norm $\leqslant M$.

3

The following is a direct application of the Lindenstrauss "compactness argument" (see e.g. [6]).

THEOREM 3. *Let X be an n-dimensional subspace of a Banach space Z and let P be a projection of least norm of Z onto X. Then* $\| P \| = \sup_R \| R \|$ *where R ranges over all "minimum norm" projections from W to X, W finite-dimensional, $X \subset W \subset Z$.*

Proof. Let $\mathscr{B} \subset 2^Z$ be the collection of all finite-dimensional superspaces of X partially ordered by inclusion. For each $B \in \mathscr{B}$, let P_B be a best (in terms of norm) projection of B onto X and extend P_B to all of Z by setting $P_B z = 0$ if $z \in Z \backslash B$. By the Kadec result above, it follows that $\| P_B z \| \leqslant (\dim X)^{1/2} \| z \|$ for every $z \in Z$. Now let

$$W = \prod_{z \in Z} n \| z \| B_X$$

which is compact in the product topology since X is n-dimensional. The net $(P_B(z))_{z \in Z}$ is in W, and thus has a convergent subnet, say $(P_C(z))$. Thus, $P_C(z)$ converges in X for each z in Z. It is clear that, defining $P\colon Z \to X$ by $Pz = \lim P_C z$, P is bounded and $Px = x$ for all x in X. Also, for $z_1, z_2 \in Z$, and all $C \supset [z_1, z_2, X]$, $P_c(\alpha z_1 + \beta z_2) = \alpha P_C(z_1) + \beta P_C(z_2)$, so P is linear. Further $\| Pz \| \leqslant \{\lim_C \| P_C \|\} \| z \|$ giving the desired result.

Let X be n-dimensional. For any superspace W of X let $P(X, W)$ be the norm of the best projection of W onto X. Define

$$P_m(X) = \sup\{P(X, W) \mid \dim W = m\},$$

$$P(X) = \sup\{P(X, W) \mid W \supset X\}.$$

The affirmative solution to problem 9 of [1] is

COROLLARY. $P(X) = \sup P_m(X)$.

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