Levi condition for hyperbolic equations with oscillating coefficients

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Abstract

In the present paper we explain new Levi conditions of $C^\infty$ type for second-order hyperbolic Cauchy problems. Our goal is to explain the special influence of oscillations in the coefficients. It turns out that such oscillations have an essential influence coupled with the asymptotic behavior of characteristics around multiple points.

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1. Introduction

We are interested in the backward hyperbolic Cauchy problem

\begin{equation}
\begin{cases}
(\partial_t^2 + p(t, D_x)^2 + i q(t, D_x)) u = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\
u(T, x) = \phi(x), & u_t(T, x) = \psi(x), & x \in \mathbb{R}^n,
\end{cases}
\end{equation}

where $0 < T \leq 1$, $D = -i \partial_\zeta$, $u = u(t, x)$, $p(t, \zeta)$ and $q(t, \zeta)$ are real valued for $\zeta \in \mathbb{R}^n$ and of first order with respect to $\zeta$. We will formulate the precise assumptions to the

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coefficients later, now we only explain two of them.

- The Cauchy problem is strictly hyperbolic for $t \in (0, T]$, and the two characteristics may coincide only for $t = 0$.
- The operators $p(t, D_x)$ and $q(t, D_x)$ are continuous for $t \in (0, T]$, and they may oscillate very fast for $t \to 0$.

Thus we have to consider these two different singular effects if $t$ goes to 0.

If one is interested in well-posedness results, then one is forced to formulate Levi conditions; these are relations between the coefficients in the principal part and the lower order term. We are only interested in Levi conditions of $C^\infty$ type.

We restrict ourselves to the case that the operators $p(t, D_x)$ and $q(t, D_x)$ are represented in the following way:

$$p(t, D_x) = \lambda(t) \sqrt{a(t, D_x)}, \quad (1.2)$$

$$q(t, D_x) = \lambda(t) \left( \frac{\lambda(t)}{\Lambda(t)} \right)^{\delta_1} \left( \log \frac{1}{\Lambda(t)} \right)^{\delta_2} b(t, D_x), \quad (\delta_1, \delta_2 \geq 0), \quad (1.3)$$

where

$$a(t, D_x) = \sum_{j,k=1}^{n} a_{jk}(t) D_x^j D_x^k, \quad b(t, D_x) = \sum_{j=1}^{n} b_j(t) D_x^j,$$

$\Lambda(t) > 0$ for $t > 0$, $\Lambda(t) = \int_0^t \lambda(s) \, ds$, the matrix $\{a_{jk}(t)\}_{jk}$ is real, symmetric and positive definite uniformly with respect to $t$. Here $\lambda(t)$ describes the order of touch of both characteristics, $a(t, \cdot)$ and $b(t, \cdot)$ describe only the oscillating behavior.

**Remark 1.1.** The orders $\delta_1$ and $\delta_2$ appearing in $q$ seem to be artificial, but we restrict ourselves only to this case. Indeed, the most important and interesting phenomenon can be observed and described by this equation, and we will find it from the following examples.

Let us introduce two examples which are described by (1.1) together with (1.2), (1.3) and without oscillations; $a_{jk}(t)$ and $b_{j}(t)$ are constants.

**Example 1.1 (Ivrii and Petkov [10]).** Let us consider the Cauchy problem

$$\left( \partial_t^2 + t^{2l}D_x^2 + it^k D_x \right) u = 0, \quad u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x). \quad (1.4)$$

The Levi condition reads as follows: $k \geq l - 1$, where $l \geq 1$.

**Example 1.2 (Tarama [12], Yagdjian [14]).** Let us consider the Cauchy problem

$$\left( \partial_t^2 + e^{-2t} D_x^2 + it^{-\beta} e^{-t} D_x \right) u = 0, \quad u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x). \quad (1.5)$$

The Levi condition reads as follows: $\beta \leq \alpha + 1$, where $\alpha > 0$. 
Both examples explain us that the coefficient \( \lambda(t)^2 \) in the principal part implies as sharp Levi condition the same asymptotic behavior as \( \lambda'(t) \) for \( t \to +0 \) in the lower order term (in these examples we have \( \lambda(t) = t^1 \) and \( \lambda(t) = e^{-t^2} \)). This observation can be generalized to higher order hyperbolic Cauchy problems [14]. Both examples show on the one hand that \( \lambda_1 \leq 1 \) in (1.1) and on the other hand that \( \lambda_2 \) has a meaning only if \( \lambda_1 = 1 \). In this moment the reader should be astonished about \( \lambda_2 > 0 \) if \( \lambda_1 = 1 \) because he might see a contradiction to the sharpness of Levi conditions of \( C^\infty \) type (see [14]). It turns out that such a choice is possible only if we have an oscillating behavior of \( b(t, D_x) \) in \( t \).

The coefficients of the next examples without lower order terms possess crucial oscillations.

**Example 1.3** (Hirosawa [5]). Let us consider the Cauchy problem

\[
\left( \frac{\partial^2}{\partial t^2} + \left( 2 + \cos \left( \left( \log \frac{1}{t} \right)^z \right) \right) \right) D_x^2 \) u = 0, \ u(T, x) = \phi(x), \ u_t(T, x) = \psi(x). \tag{1.6}
\]

Then this Cauchy problem is \( C^\infty \) well-posed if and only if \( z \leq 2 \).

**Example 1.4** (Tarama [13]). Let us consider the Cauchy problem

\[
\left( \frac{\partial^2}{\partial t^2} + e^{-2t^{-z}} \left( 2 + \cos \left( \frac{1}{t} \right) \right) \right) D_x^2 \) u = 0, \ u(T, x) = \phi(x), \ u_t(T, x) = \psi(x). \tag{1.7}
\]

Then this Cauchy problem is \( C^\infty \) well-posed if and only if \( z \geq 1/2 \).

Both examples explain us that an interplay between the oscillating behavior of the coefficient and its asymptotic behavior for \( t \to +0 \) should be satisfied. This interplay is described by the condition

\[
|a'(t)|^2 + |a''(t)| \leq C \left( \frac{\lambda(t)}{\lambda'(t)} \left( \log \frac{1}{\lambda(t)} \right)^{\gamma_a} \right)^2,
\]

if we pose the Cauchy problem

\[
\left( \frac{\partial^2}{\partial t^2} + \lambda(t)^2 a(t) D_x^2 \right) u = 0, \ u(T, x) = \phi(x), \ u_t(T, x) = \psi(x), \tag{1.8}
\]

\( C^\infty \) well-posedness holds if and only if \( \gamma_a \leq 1 \). In Examples 1.3 and 1.4 we choose \( \lambda(t) \equiv 1, \ a(t) = 2 + \cos((\log \frac{1}{t})^2), \) and \( \lambda(t) = e^{-t^2}, \ a(t) = 2 + \cos(\frac{1}{t}), \) respectively. Both examples satisfy condition (1.8) with \( \gamma_a = 1 \) if we restrict ourselves to \( z \leq 2 \) in Example 1.3 and \( z \geq 1/2 \) in Example 1.4.

Finally, the results of paper [4] allow us to study the following example.
Example 1.5 (Colombini et al. [4]). Let us consider the following Cauchy problem:

\[
\left( \partial_t^2 + a(t) D_x^2 + i \frac{1}{t} \log \frac{1}{t} b(t) D_x \right) u = 0, \\
u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x),
\]

(1.10)

where \( a(t) \) satisfies (1.8) with \( \lambda(t) \equiv 1 \) and \( \gamma_a = 1 \), \( b(t) \) satisfies the condition

\[
|b(t)| \leq b_1, \quad |b'(t)| \leq b_2 \frac{1}{t},
\]

(1.11)

with non-negative constants \( b_1 \) and \( b_2 \). Moreover, the additional condition

\[
\frac{1}{t} \log \frac{1}{t} b(t) \in L^1(0, T)
\]

(1.12)

was proposed in [4]. Under these assumptions we have \( C^\infty \) well-posedness.

It seems to be reasonable to generalize the second condition from (1.11) to

\[
|b'(t)| \leq C \left( \frac{\lambda(t)}{\Lambda(t)} \right)^{k_b} \left( \log \frac{1}{\Lambda(t)} \right)^{\gamma_b}
\]

(1.13)

if we are interested to study the Cauchy problem

\[
\left\{ \begin{array}{l}
\left( \partial_t^2 + \lambda(t)^2 a(t) D_x^2 + i \lambda(t) \left( \frac{\dot{\lambda}(t)}{\Lambda(t)} \right)^{\delta_1} \left( \log \frac{1}{\Lambda(t)} \right)^{\delta_2} b(t) D_x \right) u = 0, \\
u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x).
\end{array} \right.
\]

(1.14)

In the next section we will formulate and discuss the main result of this paper. Moreover, we will explain connections to already known results and will give new applications.

2. Main result

Consider the following Cauchy problem:

\[
\left\{ \begin{array}{l}
\left( \partial_t^2 + \lambda(t)^2 a(t, D_x) + i \lambda(t) \left( \frac{\dot{\lambda}(t)}{\Lambda(t)} \right)^{\delta_1} \left( \log \frac{1}{\Lambda(t)} \right)^{\delta_2} b(t, D_x) \right) u = 0, \\
(t, x) \in [0, T] \times \mathbb{R}^n, \quad u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x), \quad x \in \mathbb{R}^n.
\end{array} \right.
\]

(2.1)

Suppose the following conditions:

\[
\lambda(t) \in C^2((0, T]), \quad \lambda'(t) \geq 0, \quad \lambda(t) > 0 \ (t \in (0, T)), \\
\left( \frac{\lambda''(t)}{\lambda(t)} \right)^2 + \left( \frac{\dot{\lambda}(t)}{\lambda(t)} \right)^2 \leq C \left( \frac{\dot{\lambda}(t)}{\Lambda(t)} \right)^2, \quad \log \left( \frac{\lambda(t)}{\Lambda(t)} \right) \leq C_{\gamma_0} \left( \log \frac{1}{\Lambda(t)} \right)^{\gamma_0}.
\]

(2.2)
\[
\begin{cases}
0 < a_0 |\xi|^2 \leq a(t, \xi) \leq a_1 |\xi|^2, |b(t, \xi)| \leq b_1 |\xi|, \\
a_{jk}(t) \in C^2((0, T)), b_j(t) \in C^1((0, T)),
\end{cases}
\]
(2.3)

\[
\begin{align*}
\left| a'_{jk}(t) \right|^2 + \left| a''_{jk}(t) \right| &\leq C \left( \frac{\lambda(t)}{\Lambda(t)} \right) \left( \log \frac{1}{\Lambda(t)} \right)^{\gamma_a}, \\
\left| b_j(t) \right| &\leq C \left( \frac{\lambda(t)}{\Lambda(t)} \right)^{kb} \left( \log \frac{1}{\Lambda(t)} \right)^{\gamma_b}.
\end{align*}
\]
(2.4)

**Theorem 1.** Let \( p(t, \eta) \) and \( q(t, \eta) \) are given by (1.2) and (1.3). Additionally to the assumptions (2.1)–(2.4) we suppose for all \( \eta \in \mathbb{R}^n \) with \( |\eta| = 1 \) the relations
\[
\left| \int^t_0 q(s, \eta) \, ds \right| \leq C \lambda(t),
\]
(2.5)
\[
\left| \int^T_t q(s, \eta) \, ds \right| \leq C \left( \log \frac{1}{\lambda(t)} \right)^{\delta_3}
\]
(2.6)

for any \( t \in (0, T) \). Moreover, if \( \delta_3 > 0 \), then we assume that for any given \( t \in (0, T) \) and \( \eta \in [\mathbb{R}^n ; |\eta| = 1] \) there exists a monotone decreasing sequence of positive real numbers \( \{t_j = t_j(t, \eta)\}_{j=0}^N \) with \( t_0 = T \) and \( t_N = t \) such that
\[
\sum_{j=1}^N \left| \int_{t_j}^{t_{j-1}} \frac{q(s, \eta)}{p(s, \eta)} \, ds \right| - \left| \int^T_t \frac{q(s, \eta)}{p(s, \eta)} \, ds \right| \leq C
\]
(2.7)
and
\[
\max_j \left\{ \int_{t_j}^{t_{j-1}} \frac{q(s, \eta)}{p(s, \eta)} \, ds - \int_{t_j}^{t_{j-1}} \frac{q(s, \eta)}{p(s, \eta)} \, ds \right\} \leq C,
\]
(2.8)

where \( C \) is independent of \( t \) and \( \eta \). Then there exists a unique solution \( u \) of (2.1) satisfying the following energy estimate:
\[
\sqrt{\lambda(t)} \left\| \left( \lambda(t) \nabla u(t, \cdot), u_t(t, \cdot) \right) \right\|_{H^s}^2 \\
\leq C_0 \sqrt{\lambda(T)} \left\| \exp \left( C_1 (\log(D))^{-1/2} \right) (\nabla \phi(\cdot), \psi(\cdot)) \right\|_{H^s}^2
\]
(2.9)

uniformly for any \( t \in [0, T] \), where \( \gamma \geq 0 \) is given as follows:

(i) if \( \delta_1 = \kappa_b = 1 \), then \( \gamma = \max \left\{ \gamma_0, \gamma_a, \frac{\gamma_b + \delta_2}{2}, \delta_2, \delta_3 \right\} \);

(ii) if \( \delta_1 = 1 \) and \( \kappa_b < 1 \), then \( \gamma = \max \left\{ \gamma_0, \gamma_a, \delta_2, \delta_3 \right\} \);

(iii) if \( \delta_1 < 1 \) and \( \kappa_b + \delta_1 = 2 \), then \( \gamma = \max \left\{ \gamma_0, \gamma_a, \frac{\gamma_b + \delta_2}{2}, \delta_3 \right\} \);

(iv) if \( \delta_1 < 1 \) and \( \kappa_b + \delta_1 < 2 \), then \( \gamma = \max \left\{ \gamma_0, \gamma_a, \delta_3 \right\} \).
Here the data are supposed to satisfy the regularity \( \exp((\log(D))^{\gamma})) (\nabla \phi(\cdot), \psi(\cdot)) \in H^s, s \geq 0. \)

2.1. Discussion of assumptions

- Assumption (2.2) contains reasonable assumptions to the auxiliary function \( \lambda = \lambda(t) \) for \( t > 0 \). This function describes the asymptotical behavior of characteristic roots for \( t \to +0 \). The Cauchy problem is supposed to be strictly hyperbolic for \( t > 0 \). It may become weakly hyperbolic only for \( t = 0 \) if \( \lambda(0) = 0 \). The regularity of \( \lambda \) corresponds to the number of steps of diagonalization procedure we have to carry out.

- Assumption (2.3) explains that the coefficients \( a_{jk}(t) \) and \( b_j(t) \) describe the oscillating behavior of coefficients without having an additional influence on the asymptotical behavior for \( t \to +0 \). The assumed regularity corresponds to the number of steps of diagonalization procedure.

- Assumption (2.4) describes the interplay between oscillating and asymptotic behavior of coefficients. The exact description of this interplay has an important influence on the \( C^\infty \) well-posedness and the so-called loss of regularity \( \gamma \), which is described in Theorem 1 (cf. with Examples 1.3 and 1.4).

- Assumptions (2.5)–(2.8) describe the non-standard Levi conditions taking account not only the asymptotical behavior of coefficients for \( t \to +0 \) but also the oscillating behavior. In particular, conditions (2.6)–(2.8) describe a balance in the oscillating behavior of coefficients which is necessary to guarantee \( C^\infty \) well-posedness (cf. with the counter-example from [9]). If we study the Cauchy problem (1.14) under the assumption that the coefficient \( b = b(t) \) has a fixed sign on \((0, T]\), then \( \delta_2 = 0 \) and \( \delta_1 \leq 1 \) in (2.5). Thus the lower order term has the same asymptotical behavior as \( \lambda'(t) \) for \( t \to +0 \) (cf. with Examples 1.1 and 1.2).

- Conditions (2.7) and (2.8) have only a meaning in the case if \( q(t, \eta)/p(t, \eta) \) changes its sign; otherwise the conditions hold with the constant \( C = 0 \) by the choice of \( \{t_j\}_{j=0}^N \) with \( N = 1 \). For instance, (2.6)–(2.8) hold if the condition

\[
\int_t^T \left| \frac{q(s, \eta)}{p(s, \eta)} \right| ds \leq C \left( \log \frac{1}{\Lambda(\eta)} \right)^{\delta_3}
\]

is satisfied, but using (2.10) we cannot feel any new good effect from changing the sign of \( q(t, \eta)/p(t, \eta) \). Our conditions (2.6)–(2.8) are set between (2.10) and (2.6), and we can actually feel a new effect which is never come up on assumption (2.10).

2.2. Relations to previous results

To Example 1.1: Suppose that \( \lambda(t) = t^1 \), \( a(t) = b(t) \equiv 1 \), \( \delta_1 = l - k \) with \( l \geq 1 \) and \( k \geq 0 \), and \( \delta_2 = 0 \). It follows that \( \gamma_0 = \gamma_a = \gamma_b = \kappa_b = 0 \). Condition (2.5) holds
without any restriction to $l$ and $k$, but (2.6) holds for $\delta_1 = l - k = 1$ with $\delta_3 = 1$ or $\delta_1 = l - k < 1$ with $\delta_3 = 0$. Therefore, we have

$$k = l - 1, \quad (ii) \implies \gamma = \max \left\{ \gamma_0, \gamma_a, \frac{\gamma_b + \delta_2}{2}, \delta_2, \delta_3 \right\} = \max\{0, 0, 0, 0, 1\} = 1$$

and

$$k > l - 1, \quad (iii) \implies \gamma = \max \left\{ \gamma_0, \gamma_a, \delta_2, \delta_3 \right\} = 0.$$

To Example 1.2: Suppose that $\lambda(t) = e^{-t^\alpha}$, $a(t) = b(t) \equiv 1$ and $\delta_1 = 1$. It follows that $\gamma_0 = \gamma_a = 0$. Condition (2.5) holds for any $\delta_2$ with $x > 0$. On the other hand, (2.6) holds for $\delta_2 \leq \delta_3 - 1$. Actually, $\delta_2 \leq 0$ is a necessary and sufficient condition for the $C^\infty$ well-posedness. Therefore, we have from (ii) that

$$\gamma = \max\{\gamma_0, \gamma_a, \delta_2, \delta_3\} = \max\{\varepsilon, 0, 0, \delta_2, \delta_2 + 1\} = \delta_2 + 1.$$

Consequently, $\delta_2 = 0$ implies a finite loss of regularity $\gamma$. This is satisfied in Example 1.2. But if we consider instead of the Cauchy problem (1.5) the Cauchy problem

$$(\partial_t^2 + e^{-2t^{-2}} D_x^2 + it^{-\alpha - 1 - 2}\partial_x e^{-t^{-2}} D_x) u = 0, \quad u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x),$$

with $-1 < \delta_2 < 0$, then the loss is arbitrarily small.

To Example 1.3: Suppose that $b(t, D_x) \equiv 0$ and $\lambda(t) \equiv 1$. It follows that $\gamma_0 = 0$. Noting $\kappa_b = \gamma_b = \delta_1 = \delta_2 = \delta_3 = 0$, we have from (iv) that

$$\gamma = \max\{\gamma_0, \gamma_a, \delta_3\} = \{0, \gamma_a, 0\} = \gamma_a.$$

To Example 1.4 (Hirosawa [7], Tarama [13], Yagdjian [14]): Suppose that $\lambda(t) = e^{-t^{-2}}$ with $x > 0$ and $a(t) = 2 + \cos(\frac{1}{t})$ and $b(t, D_x) \equiv 0$. It follows that $\gamma_0 = \gamma_a > 0$, $\gamma_a = \frac{1}{x} - 1$ and $\kappa_b = \gamma_b = \delta_1 = \delta_2 = \delta_3 = 0$. Thus we have from (iv) that

$$\gamma = \max\{\gamma_0, \gamma_a, \delta_3\} = \max\{\varepsilon, \frac{1}{x} - 1\}.$$

Example 2.1 (Hirosawa and Reissig [9]). Suppose that $\lambda(t) \equiv 1$. It follows that $\gamma_0 = 0$, $\kappa_b = 1$, $\gamma_a = \gamma_b$, $\delta_1 = 1$ and $\delta_2 = \gamma_a$ (or $\gamma_b$). Then we easily see that (2.10) holds for $\delta_3 = \gamma_a + 1$. Therefore, we have from (i) that

$$\gamma = \max\left\{ \gamma_0, \gamma_a, \frac{\gamma_b + \delta_2}{2}, \delta_2, \delta_3 \right\} = \max\{0, \gamma_a, \gamma_a, \gamma_a + 1\} = \gamma_a + 1.$$
In Corollary 2 from [9] we succeeded to construct an example of a pair of coefficients \( (a(t), b(t)) \) satisfying (2.4)–(2.6) such that the solution really loses the regularity \( \exp(C_1 \log(D))^{\gamma_a + 1} \). Moreover, in Remark 8 from [9] we gave an example for coefficients \( (a(t), b(t)) \) satisfying (2.4)–(2.6) with the above \( \gamma_0, \gamma_a, \gamma_b, \kappa_b, \delta_1 \) and \( \delta_2 \), but with \( \delta_3 = 0 \).

**Example 2.2** (New effect from the oscillations of \( b(t) \)). Suppose that \( \lambda(t) = t^l \) and \( a(t) \equiv 1 \). It follows that \( \gamma_0 = \gamma_a = 0 \). For a \( C^2 \) and periodic function \( \Phi \) we define \( b(t) \) by

\[
\frac{b(t)}{b'} = \Phi'(o(t)), \quad \text{where } o(t) = (\log t^{-1})^{\gamma_b + 1}.
\]

Then \( b(t) \) satisfies (2.4) with \( \kappa_b = 1 \), and

\[
\left| \int_0^t s^l o'(s)\Phi'(o(s)) \, ds \right| \leq Ct^l \max_t \{ |\Phi(o(t))| \} \leq Ct^l,
\]

and

\[
\left| \int_t^T o'(s)\Phi'(o(s)) \, ds \right| \leq C.
\]

It follows that (2.5) and (2.6) hold with \( \delta_2 = \gamma_b \) and \( \delta_3 = 0 \). Therefore, by (i) we have

\[
\gamma = \max \left\{ \gamma_0, \gamma_a, \gamma_b + \frac{\delta_2}{2}, \delta_2, \delta_3 \right\} = \max \{ 0, 0, \gamma_b, \gamma_b, 0 \} = \gamma_b.
\]

Thus the Cauchy problem

\[
\left( \tilde{\gamma}_t^2 + t^{2l} D_x^2 + it^{l-1} \left( \log t^{-1} \right)^\gamma_b \left( \Phi'(o(t)) D_x \right) \right) u = 0,
\]

\[
u(T, x) = \phi(x), \quad u_t(T, x) = \psi(x),
\]

is \( C^\infty \) well-posed for \( \gamma_b \leq 1 \).

To Example 1.5: Suppose that \( \lambda(t) \equiv 1 \), \( \gamma_a = 1 \), \( \kappa_b = \gamma_b = 1 \), \( \delta_1 = \delta_2 = 1 \). All assumptions (2.2)–(2.4) are satisfied. But the additional assumption (which neglects any strong oscillating behavior)

\[
\frac{1}{t} \log \frac{1}{t} b(t, \cdot) = q(t, \cdot) \in L^1((0, T)) \),
\]

actually implies that \( \delta_1 = \delta_2 = 0 \) in (2.5) and (2.6). Thus we have from (iv) that

\[
\gamma = \max \{ \gamma_0, \gamma_a, \delta_3 \} = \{ 0, 1, 0 \} = 1.
\]
To Example 2.2: The functions $q(t, \eta)$ and $q(t, \eta)/p(t, \eta)$ of Example 2.2 are given by

$$q(t, \eta) = t^{l-1} \left( \log t^{-1} \right)^{\gamma_b} \Phi'(\omega(t)), \quad \frac{q(t, \eta)}{p(t, \eta)} = t^{-1} \left( \log t^{-1} \right)^{\gamma_b} \Phi'(\omega(t)), \quad |\eta| = 1,$$

but the asymptotic behavior for $t \to +0$ seems to be very bad for $\gamma_b > 0$ from the up to now known point of view of Levi conditions, which are usually described by the order of Lebesgue integrals over $q(t, \eta)$ and $q(t, \eta)/p(t, \eta)$. Indeed, in general we cannot expect $C^\infty$ well-posedness for such a Cauchy problem. On the other hand, our main result tells us, that not only the asymptotic behavior of $q(t, \eta)$ (of course in relation to the asymptotic behavior of $p(t, \eta)$) for $t \to +0$ is essential, but also the Riemann integrals over $q(t, \eta)$ and $q(t, \eta)/p(t, \eta)$ are important. A suitable oscillating behavior of $\Phi'(\omega(t))$ may guarantee that Levi conditions (2.5)–(2.8) of $C^\infty$ type are satisfied.

Remark 2.1. The results of [9] show that the solution of the Cauchy problem

$$\begin{cases}
\left( \xi_t^2 + 2ic(t)\lambda(t)D_x\xi_t + a(t)^2\lambda(t)^2D_x^2 \right) v(t, \xi) = 0, \\
u(T, x) = \hat{\phi}(x), \quad u_t(T, x) = \psi(x),
\end{cases}$$

has, in general, an infinite loss of regularity if the condition from (2.4) for $a(t)$ is satisfied for $c(t)$, too, with any small parameter $\gamma_a$. This is one motivation for the structure of our model (1.1) with principal part $\xi_t^2 + p(t, D_x)^2$. Thus if we are interested in $H^\infty$ well-posedness results for higher order equations, then we are not able to formulate assumptions as (2.4) with $\gamma_a \leq 1$, but we are forced to choose $\gamma_a = 0$. This is done in [1,2]. A lot of examples show that the statement of our result is optimal if we have an infinite loss of derivatives. In recent papers [3,7] and results of optimality are given for strictly hyperbolic Cauchy problems with non-regular coefficients in $t$ if a finite loss or an arbitrary small loss appears.

3. Proof of Theorem 1

After partial Fourier transformation the equation of (1.1) is rewritten as follows:

$$\begin{cases}
\left( \xi_t^2 + \lambda(t)^2a(t, \zeta) + i\lambda(t) \left( \frac{\lambda(t)}{\Lambda(t)} \right)^{\delta_1} \left( \log \frac{1}{\Lambda(t)} \right)^{\delta_2} b(t, \zeta) \right) v(t, \zeta) = 0, \\
(t, \zeta) \in [0, T] \times \mathbb{R}^n, \quad v(T, \zeta) = \hat{\phi}(\zeta), \quad v_t(T, \zeta) = \hat{\psi}(\zeta), \quad \zeta \in \mathbb{R}^n.
\end{cases}$$

(3.1)
For a positive parameter $N$ we define the hyperbolic zone $Z_H(N)$ and the pseudo-differential zone $Z_{\Psi}(N)$ of the extended phase space $[0, T] \times \mathbb{R}^n$ by

$$Z_H(N) = \left\{ (t, \zeta) ; \Lambda(t)|\zeta| \geq N \left( \log \frac{1}{\Lambda(t)} \right)^\gamma \right\},$$

$$Z_{\Psi}(N) = \left\{ (t, \zeta) ; \Lambda(t)|\zeta| \leq N \left( \log \frac{1}{\Lambda(t)} \right)^\gamma \right\}.$$

Here we denote by $t_\zeta$ the solution of

$$\Lambda(t_\zeta)|\zeta| = N \left( \log \frac{1}{\Lambda(t_\zeta)} \right)^\gamma.$$

### 3.1. Estimates in $Z_{\Psi}(N)$

In $Z_{\Psi}(N)$ we define $\tilde{p}(t, \zeta)$ and $Q(t, \zeta)$ by

$$\tilde{p}(t, \zeta) := \left( \frac{\log \frac{1}{\Lambda(t_\zeta)}}{t_\zeta|\zeta|} + \frac{\lambda(t)\Lambda(t)^{\frac{1}{2}}}{\Lambda(t_\zeta)^{\frac{1}{2}}} \right)|\zeta| \quad \text{and} \quad Q(t, \zeta) = \int_0^t q(s, \zeta) \, ds.$$

Introducing $w(t, \zeta)$ by

$$w(t, \zeta) := \exp \left( i \int_0^t Q(s, \zeta) \, ds \right) v(t, \zeta),$$

and noting

$$\tilde{c}_t^2 v(t, \zeta) = \exp \left( -i \int_0^t Q(s, \zeta) \, ds \right) \left( \tilde{c}_t^2 - 2i Q(t, \zeta) \tilde{c}_t - i q(t, \zeta) - Q(t, \zeta)^2 \right) w(t, \zeta),$$

Eq. (3.1) is rewritten as follows:

$$\begin{cases}
\left( \tilde{c}_t^2 + p(t, \zeta)^2 - Q(t, \zeta)^2 - 2i Q(t, \zeta) \tilde{c}_t \right) w(t, \zeta) = 0, \quad (t, \zeta) \in [0, T] \times \mathbb{R}^n, \\
w(T, \zeta) = e^{i \int_0^T Q(s, \zeta) ds} \tilde{\phi}(\zeta), \\
w_t(T, \zeta) = e^{i \int_0^T Q(s, \zeta) ds} \left( \tilde{\psi}(\zeta) - i Q(T, \zeta) \tilde{\phi}(\zeta) \right), \quad \zeta \in \mathbb{R}^n.
\end{cases}$$

(3.2)

We define $W(t, \zeta)$ by

$$W(t, \zeta) := (-i w_t(t, \zeta), \tilde{p}(t, \zeta) w(t, \zeta))^T.$$
Then (3.2) can be transformed to the system

\[
\begin{aligned}
(\partial_t - B(t, \zeta)) W(t, \zeta) &= 0, \quad (t, \zeta) \in [0, t_\zeta] \times \mathbb{R}^n, \\
W(t_\zeta, \zeta) &= \left(-i w_\zeta(t_\zeta, \zeta), \tilde{p}(t_\zeta, \zeta) w(t_\zeta, \zeta)\right)^T,
\end{aligned}
\]  

(3.3)

where

\[
B(t, \zeta) := \begin{pmatrix}
2i Q(t, \zeta) & i(p(t, \zeta)^2 - Q(t, \zeta)^2) \\
i \tilde{p}(t, \zeta) & i \tilde{p}(t, \zeta) / p(t, \zeta)
\end{pmatrix}.
\]

Noting the estimates

\[
\int_0^{t_\zeta} \tilde{p}(t, \zeta) dt = \left(\log \frac{1}{\Lambda(t_\zeta)}\right)^\gamma + \frac{2}{3} \Lambda(t_\zeta) |\zeta| \leq C \left(\log \frac{1}{\Lambda(t_\zeta)}\right)^\gamma,
\]

\[
\int_0^{t_\zeta} \left| \frac{\partial_t \tilde{p}(t, \zeta)}{\tilde{p}(t, \zeta)} \right| dt = \log \left(1 + \frac{N t_\zeta \lambda(t_\zeta)}{\Lambda(t_\zeta)}\right) \leq C \left(\log \frac{1}{\Lambda(t_\zeta)}\right)^\gamma \leq C \left(\log \frac{1}{\Lambda(t_\zeta)}\right)^\gamma,
\]

\[
\int_0^{t_\zeta} \frac{p(t, \zeta)^2}{\tilde{p}(t, \zeta)} dt \leq C \Lambda(t_\zeta)^{1/2} |\zeta| \int_0^{t_\zeta} \lambda(t) \Lambda(t)^{-1/2} dt \leq C \Lambda(t_\zeta) |\zeta| \leq C \left(\log \frac{1}{\Lambda(t_\zeta)}\right)^\gamma
\]

and

\[
\int_0^{t_\zeta} \frac{Q(t, \zeta)^2}{\tilde{p}(t, \zeta)} dt \leq C \Lambda(t_\zeta)^{1/2} |\zeta| \int_0^{t_\zeta} \lambda(t) \Lambda(t)^{-1/2} dt \leq C \Lambda(t_\zeta) |\zeta| \leq C \left(\log \frac{1}{\Lambda(t_\zeta)}\right)^\gamma
\]

by (2.5), we obtain the following estimate:

\[
|W(t, \zeta)| \leq C \exp \left(\mathcal{C} (\log(\zeta))^\gamma\right) |W(t_\zeta, \zeta)|
\]

(3.4)

for any \( t \in [0, t_\zeta] \), where the constant \( C \) is independent of \( \zeta \).

We set

\[
V(t, \zeta) := \left(-i v_\zeta(t, \zeta), p(t, \zeta) v(t, \zeta)\right)^T
\]
in $\mathcal{Z}_\Psi$. Noting the equality
\[
|W(t, \xi)|^2 = \left( Q(t, \xi)^2 + \tilde{p}(t, \xi)^2 \right) |v(t, \xi)|^2 + |v_t(t, \xi)|^2,
\]
and the estimates
\[
Q(t, \xi)^2 + \tilde{p}(t, \xi)^2 \geq \left( \frac{\log \frac{1}{\Lambda(t, \xi)}}{t_n^2 |\xi|^2} \right)^{2\gamma} \lambda(t, \xi)^2 \exp \left( -2 \log \left( t_n^2 \lambda(t, \xi) \right) \right) \\
\geq \frac{\lambda(t, \xi)^2}{N^2} \exp \left( -C (\log \langle \xi \rangle)^{\gamma_0} \right)
\]
and
\[
Q(t, \xi)^2 + \tilde{p}(t, \xi)^2 \leq C |\xi|^2 \left( \lambda(t, \xi)^2 + \left( \frac{\log \frac{1}{\Lambda(t, \xi)}}{t_n^2 |\xi|} \right) \right)^2 \\
\leq C \lambda(t, \xi)^2 |\xi|^2 \left( 1 + \frac{\Lambda(t, \xi)}{t_n^2 \lambda(t, \xi)} \right)^2 \leq 2C \lambda(t, \xi)^2 |\xi|^2,
\]
which follow from (2.2) and (2.5), we have
\[
|W(t, \xi)| \leq C |V(t, \xi)| \quad \text{and} \quad \exp \left( -C (\log \langle \xi \rangle)^{\gamma_0} \right) |V(t, \xi)| \leq C |W(t, \xi)|
\]
in $\mathcal{Z}_\Psi(N)$. Therefore, we obtain
\[
|V(t, \xi)| \leq C_0 \exp \left( C_1 (\log \langle \xi \rangle)^{\gamma} \right) |V(t, \xi)|
\]
in $\mathcal{Z}_\Psi(N)$.

3.2. Estimates in $Z_{\mathcal{H}}(N)$

Let us start from problem (3.1), that is, from
\[
\begin{cases}
\left( \partial_t^2 + p(t, \xi)^2 + iq(t, \xi) \right) v(t, \xi) = 0, \quad (t, \xi) \in [0, T] \times \mathbb{R}^n, \\
v(T, \xi) = \hat{\phi}(\xi), \quad v_t(T, \xi) = \hat{\psi}(\xi), \quad \xi \in \mathbb{R}^n.
\end{cases}
\] (3.6)

We define $V_0(t, \xi)$ by
\[
V_0(t, \xi) := (-iv_t(t, \xi), p(t, \xi)v(t, \xi))^T.
\]
Then (3.6) can be transformed to
\[
\begin{cases}
(\partial_t - B_0(t, \zeta)) V_0(t, \zeta) = 0, & (t, \zeta) \in [t_\zeta, T] \times \mathbb{R}^n, \\
V_0(T, \zeta) := (-i v_\zeta(T, \zeta), p(T, \zeta)v(T, \zeta))^T, & \zeta \in \mathbb{R}^n,
\end{cases}
\tag{3.7}
\]
where
\[
B_0(t, \zeta) := \begin{pmatrix} 0 & ip(t, \zeta) - \frac{q(t, \zeta)}{p(t, \zeta)} \\ ip(t, \zeta) & \frac{\dot{v}_\zeta p(t, \zeta)}{p(t, \zeta)} \end{pmatrix}.
\]

3.2.1. First step of diagonalization procedure

With the matrix
\[
M_1 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]
we have
\[
M_1^{-1} B_0 M_1 = \begin{pmatrix} ip + \frac{p_t}{2p} & 0 \\ 0 & -ip + \frac{p_t}{2p} \end{pmatrix} + \begin{pmatrix} -\frac{q}{2p} & \frac{q}{2p} - \frac{p_t}{2p} \\ -\frac{q}{2p} - \frac{p_t}{2p} & \frac{q}{2p} \end{pmatrix} =: A_1 + B_1.
\]
Thus it follows that
\[
M_1^{-1} (\partial_t - B_0) M_1^{-1} V_0 = (\partial_t - A_1 - B_1) M_1^{-1} V_0 = 0. \tag{3.8}
\]

3.2.2. Second step of diagonalization procedure

Let us denote the \((j, k)\)th element of the matrix \(B_1\) by \(B_{jk}\). We define the matrix \(M_2 = M_2(t, \zeta)\) by
\[
M_2 := \begin{pmatrix} 1 & -\frac{B_{12}}{2ip} \\ \frac{B_{21}}{2ip} & 1 \end{pmatrix}, \quad M_2^{-1} := \frac{1}{\det M_2} \begin{pmatrix} 1 & \frac{B_{12}}{2ip} \\ -\frac{B_{21}}{2ip} & 1 \end{pmatrix}.
\]
Here we note that
\[
\det M_2 = 1 - \frac{1}{4p^2} \left( \frac{p_t}{2p} - \frac{q}{2p} \right) \left( \frac{p_t}{2p} + \frac{q}{2p} \right) = 1 - \frac{1}{4p^2} \left( \left( \frac{p_t}{2p} \right)^2 - \left( \frac{q}{2p} \right)^2 \right)
\geq 1 - \frac{1}{4p^2} \left( \frac{p_t}{2p} \right)^2 \geq 1 - \frac{C}{\lambda(t)} \frac{(\dot{\lambda}(t))^2}{\Lambda(t)} \left( \log \frac{1}{\Lambda(t)} \right)^{\gamma_a} \geq 1 - \frac{C}{\Lambda(t)^2 |\zeta|^2} \geq 1 - \frac{C}{N}
\]
thus \(M_2\) is invertible for large \(N\).
Let us introduce \((A_1)_{jj} =: \tau_j\), it follows that
\[
M_2 := \begin{pmatrix}
1 & -B_{12} \\
\tau_1 - \tau_2 & -1
\end{pmatrix}.
\]

Noting the following equalities:
\[
[A_1, M_2] = A_1 M_2 - M_2 A_1
= \begin{pmatrix}
\tau_1 & 0 \\
0 & \tau_2
\end{pmatrix} \begin{pmatrix}
1 & -B_{12} \\
\tau_1 - \tau_2 & -1
\end{pmatrix} - \begin{pmatrix}
1 & -B_{12} \\
\tau_1 - \tau_2 & -1
\end{pmatrix} \begin{pmatrix}
\tau_1 & 0 \\
0 & \tau_2
\end{pmatrix}
= \begin{pmatrix}
\tau_1 B_{12} / \tau_2 - B_{12} / \tau_2 \\
- \tau_1 B_{12} / \tau_2 + B_{12} / \tau_2
\end{pmatrix}
= \begin{pmatrix}
0 & -B_{12} \\
B_{12} - B_{11} B_{12} & B_{21} - B_{11} B_{21}
\end{pmatrix},
\]

\[
M_2^{-1} ([A_1, M_2] + B_1 M_2) - \text{diag} B_1
= M_2^{-1} \left( \begin{pmatrix}
B_{11} & 0 \\
0 & B_{22}
\end{pmatrix} + \frac{1}{\tau_1 - \tau_2} \begin{pmatrix}
B_{12} B_{21} & -B_{11} B_{12} \\
B_{21} B_{22} & -B_{12} B_{21}
\end{pmatrix} \right) - \text{diag} B_1
= M_2^{-1} \left( (I - M_2) \begin{pmatrix}
B_{11} & 0 \\
0 & B_{22}
\end{pmatrix} + \frac{1}{\tau_1 - \tau_2} \begin{pmatrix}
B_{12} B_{21} & -B_{11} B_{12} \\
B_{21} B_{22} & -B_{12} B_{21}
\end{pmatrix} \right),
\]

and the estimates
\[
\left| M_2^{-1} \right| \leq \frac{C}{p} \left( \frac{|q|}{p} + \frac{2|p_1|}{p} \right) \leq \frac{C}{\lambda |\xi|} \left( \left( \frac{\lambda}{\Lambda} \right)^{\delta_1} \left( \log \frac{1}{\Lambda} \right)^{\delta_2} + \frac{\lambda}{\Lambda} \left( \log \frac{1}{\Lambda} \right)^{\gamma_a} \right)
\leq \begin{cases}
\frac{C (\log \frac{1}{\Lambda}) \max \left( \gamma_a, \delta_2 \right)}{\lambda |\xi|} & \leq C \left( \log \frac{1}{\Lambda} \right)^{\max \left( \gamma_a, \delta_2 \right)} \leq C \quad (\delta_1 = 1), \\
\frac{C (\log \frac{1}{\Lambda}) \gamma_a}{\lambda |\xi|} & \leq C \left( \log \frac{1}{\Lambda} \right)^{\gamma_a - \gamma} \leq C \quad (\delta_1 < 1),
\end{cases}
\]

we have
\[
\left| (I - M_2) \begin{pmatrix}
B_{11} & 0 \\
0 & B_{22}
\end{pmatrix} + \frac{1}{\tau_1 - \tau_2} \begin{pmatrix}
B_{12} B_{21} & -B_{11} B_{12} \\
B_{21} B_{22} & -B_{12} B_{21}
\end{pmatrix} \right|
\leq C \frac{p_1^2 + |p_1 q| + q^2}{p^3} \leq \frac{C}{\lambda |\xi|} \left( \left( \frac{\lambda}{\Lambda} \right)^{\gamma_a} \right)^2 + \left( \left( \frac{\lambda}{\Lambda} \right)^{\delta_1} \right)^2.
\]
\[
\begin{align*}
\begin{aligned}
&\leq \frac{C\lambda}{\Lambda^2|\xi|} \left\{ \left( \frac{\log \frac{1}{\Lambda}}{\Lambda} \right)^{2\max\{\gamma_a, \delta_2\}} \quad (\delta_1 = 1), \\
&\quad \left( \frac{\log \frac{1}{\Lambda}}{\Lambda} \right)^{2\gamma_a} \quad (\delta_1 < 1), \\
&\leq \frac{C\lambda}{\Lambda^2|\xi|} \left( \frac{1}{\Lambda} \right)^{2\gamma_a}.
\end{aligned}
\end{align*}
\]

Moreover, we have
\[
|\partial_t M_2| \leq \frac{C}{\lambda|\xi|} \left( \left( \frac{\lambda}{A} \right)^{\delta_1 + \kappa_b} \left( \log \frac{1}{\Lambda} \right)^{\gamma_b + \delta_2} + \left( \frac{\lambda}{A} \right)^{1 + \delta_1} \left( \log \frac{1}{\Lambda} \right)^{\gamma_a + \delta_2} \right.
\]
\[
+ \left. \left( \frac{\lambda}{A} \right)^2 \left( \log \frac{1}{\Lambda} \right)^{2\gamma_a} \right) \left\{ \begin{aligned}
&\left( \frac{\log \frac{1}{\Lambda}}{\Lambda} \right)^{\max\{\gamma_a + \delta_2, \gamma_b + \delta_2, 2\gamma_a\}} \quad (\delta_1 = \kappa_b = 1), \\
&\left( \frac{\log \frac{1}{\Lambda}}{\Lambda} \right)^{\max\{\gamma_b + \delta_2, 2\gamma_a\}} \quad (\delta_1 = 1, \ k_b < 1), \\
&\left( \frac{\log \frac{1}{\Lambda}}{\Lambda} \right)^{\max\{\gamma_b + \delta_2, 2\gamma_a\}} \quad (\delta_1 < 1, \ \delta_1 + \kappa_b = 2), \\
&\left( \frac{\log \frac{1}{\Lambda}}{\Lambda} \right)^{2\gamma_a} \quad (\delta_1 < 1, \ \delta_1 + \kappa_b < 2),
\end{aligned} \right\}
\leq \frac{C\lambda}{\Lambda^2|\xi|} \left( \frac{1}{\Lambda} \right)^{2\gamma}.
\]

Therefore, (3.8) can be written as follows:
\[
M_2^{-1} (\partial_t - A_1 - B_1) M_2 M_2^{-1} M_1^{-1} V_0 = (\partial_t - A_1 - \text{diag} B_1 - B_2) M_2^{-1} M_1^{-1} V_0 = 0,
\]
where
\[
B_2 := M_2^{-1}([A_1, M_2] + B_1 M_2) - \text{diag} B_1 - M_2^{-1}(\partial_t M_2)
\]
and
\[
|B_2(t, \tilde{\xi})| \leq C \frac{C\lambda (t)}{\Lambda^2(t)|\xi|} \left( \log \frac{1}{\Lambda} \right)^{2\gamma}.
\]

3.2.3. **An elliptic transformation**

We define the matrices \(M_3 = M_3(t, \tilde{\xi})\) and \(B_3(t, \tilde{\xi})\) as follows:
\[
M_3 := \begin{pmatrix}
\exp \left( -\int_t^T (A_1)_{11}(s, \tilde{\xi}) \, ds \right) & 0 \\
0 & \exp \left( -\int_t^T (A_1)_{22}(s, \tilde{\xi}) \, ds \right)
\end{pmatrix}
\]
and \( B_3 := M_3^{-1}B_2M_3 \). Here we note that
\[
\int_t^T (A_1)_{11}(s, \zeta) \, ds = i \int_t^T p(s, \zeta) \, ds + \frac{1}{2} \log \left( \frac{p(T, \zeta)}{p(t, \zeta)} \right)
\]
and
\[
\int_t^T (A_1)_{22}(s, \zeta) \, ds = -i \int_t^T p(s, \zeta) \, ds + \frac{1}{2} \log \left( \frac{p(T, \zeta)}{p(t, \zeta)} \right).
\]
Hence, it follows that
\[
|B_3(t, \zeta)| = \left| M_3^{-1}(t, \zeta)B_2(t, \zeta)M_3(t, \zeta) \right| \leq C |B_2(t, \zeta)| \leq C \frac{\lambda(t)}{A(t)^2|\zeta|} \left( \log \frac{1}{A(t)} \right)^{2\gamma}. \tag{3.9}
\]
Consequently, we have
\[
M_3^{-1}(\hat{c}_t - A_1 - \text{diag } B_1 - B_2)M_3 = \hat{c}_t - \text{diag } B_1 - B_3 = \hat{c}_t - \frac{q}{2p} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - B_3.
\]
We define \( V_1(t, \zeta) = (v_{11}(t, \zeta), v_{12}(t, \zeta))^T \) by
\[
V_1(t, \zeta) := M_3(t, \zeta)^{-1}M_2(t, \zeta)^{-1}M_1^{-1}V_0(t, \zeta)
\]
in \( Z_H \). Then \( V_1(t, \zeta) \) is a solution of
\[
\left( \hat{c}_t - \frac{q(t, \zeta)}{2p(t, \zeta)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - B_3(t, \zeta) \right) V_1(t, \zeta) = 0, \tag{3.10}
\]
where \( B_3 = B_3(t, \zeta) \) satisfies estimate (3.9).

3.2.4. An auxiliary transformation

Let us introduce with \( \eta = \frac{\zeta}{|\zeta|} \) the functions
\[
r(t, \eta) = r(t, \zeta) := \frac{q(t, \zeta)}{2p(t, \zeta)} = \frac{q(t, \eta)}{2p(t, \eta)}.
\]
For a fixed \( \zeta \in \mathbb{R}^n \) we take the sequence \( \{t_j\}_{j=0}^N \) such that assumptions (2.7) and (2.8) hold with \( t = t_\zeta \). Then we can suppose from (2.7) that there exists a sequence of
non-negative real numbers \( \{\sigma_j\}_{j=1}^N \) satisfying \( \sum_{j=1}^N \sigma_j \leq C/2 \) with the constant \( C \) from (2.7) such that

\[
\begin{cases}
\int_{t_j}^{t_{j-1}} r(s, \eta) \, ds \geq -\sigma_j & \text{if } \int_t^T r(s, \eta) \, ds \geq 0, \\
\int_{t_j}^{t_{j-1}} r(s, \eta) \, ds \leq \sigma_j & \text{if } \int_t^T r(s, \eta) \, ds \leq 0.
\end{cases}
\tag{3.11}
\]

Indeed, if \( \int_t^T r(s, \eta) \, ds \geq 0 \), then we have from (2.7) that

\[
\sum_{j=1}^N \left| \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds \right| - \int_t^T r(s, \eta) \, ds = 2 \sum_{j \in I_-} \int_{t_j}^{t_{j-1}} -r(s, \eta) \, ds \leq C,
\]

where

\[ I_- := \left\{ 1 \leq j \leq N ; \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds \leq 0 \right\}. \]

Then for

\[ \sigma_j = \begin{cases} -\int_{t_j}^{t_{j-1}} r(s, \eta) \, ds & \text{if } j \in I_- , \\
0 & \text{if } j \in I_+ := \{1, \ldots, N\} \setminus I_- , \end{cases} \]

we have

\[
\sum_{j=1}^N \left| \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds + \sigma_j \right| = \sum_{j \in I_+} \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds \\
+ \sum_{j \in I_-} \left( \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds + \sigma_j \right) \\
+ \sum_{j=1}^N \left( \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds + \sigma_j \right).
\]

Consequently, \( \int_{t_j}^{t_{j-1}} r(s, \eta) \, ds + \sigma_j \) have the same sign for all \( j \). The other case can be proved in an analogous way.

We restrict ourselves to the case that the first estimate of (3.11) holds.
Let $\theta = \theta(t, \xi) \in L_{1,\text{loc}}((t, T); L_{\infty}(\mathbb{R}))$ and let us define the function $\Theta = \Theta(t, \xi)$ by

$$\Theta = \Theta(t, \xi) := \exp \left( \int_{t}^{T} \theta(s, \xi) \, ds \right).$$

Choosing

$$Y = Y(t, \xi) := \begin{pmatrix} \Theta(t, \xi)^{-1} & 0 \\ 0 & \Theta(t, \xi) \end{pmatrix} V_1(t, \xi)$$

allows to rewrite the Cauchy problem (3.10) in the form

$$\left( \dot{t} - (r(t, \xi) + \theta(t, \xi)) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - Q(t, \xi) \right) Y(t, \xi) = 0, \quad Y(T, \xi) = V_1(T, \xi), \quad (3.12)$$

where

$$Q = Q(t, \xi) := \begin{pmatrix} (B_3)_{11}(t, \xi) & (B_3)_{12}(t, \xi) \\ \Theta(t, \xi)^2(B_3)_{21}(t, \xi) & (B_3)_{22}(t, \xi) \end{pmatrix},$$

and $(B_3)_{jk}$ denotes the $(j, k)$th element of $B_3$. Here we can check that $Q$ satisfies estimate (3.9), too.

The auxiliary transformation has to satisfy the following two properties:

- It guarantees that $r(t, \xi) + \theta(t, \xi) \geq 0$ in $Z_H(N)$.
- The transformations $\Theta(t, \xi)$ and $\Theta(t, \xi)^{-1}$ are uniformly bounded in $Z_H(N)$, thus we have no additional contribution to the loss of regularity from these transformations.

3.2.5. Lyapunov and energy functions

We define the Lyapunov function $S(t, \xi)$ and the energy function $E(t, \xi)$ of $Y(t, \xi)$ by

$$S(t, \xi) := -|y_1(t, \xi)|^2 + |y_2(t, \xi)|^2 \quad \text{and} \quad E(t, \xi) := |y_1(t, \xi)|^2 + |y_2(t, \xi)|^2, \quad (3.13)$$

where $(y_1, y_2) = Y^T$. Then we can estimate

$$\dot{t} E(t, \xi) = 2 \text{Re} \left( y_1 \bar{\dot{y}}_1 y_1 \right) + 2 \text{Re} \left( y_2 \bar{\dot{y}}_1 y_2 \right)$$

$$= 2 (r + \theta) S(t, \xi) + 2 \text{Re} \left( y_1 \left( Q_{11} y_1 + Q_{12} y_2 \right) \right)$$

$$+ 2 \text{Re} \left( y_2 \left( Q_{22} y_2 + Q_{21} y_1 \right) \right)$$

$$= 2 (r + \theta) S(t, \xi) + 2 \text{Re}(Q_{11})|y_1|^2 + 2 \text{Re}(Q_{22})|y_2|^2$$

$$+ 2 \text{Re} \left( (Q_{12} y_1 \bar{y}_2) + 2 \text{Re} \left( Q_{21} \bar{y}_1 y_2 \right) \right)$$

$$\leq 2 \left( r + \theta + C \max \{\Theta^2, \Theta^{-2} \} \right) (t) \left( \frac{\lambda(t)}{\lambda(t)} \right)^{2/(2 \epsilon)} E(t, \xi)$$
for all \((t, \zeta) \in \mathbb{Z}_H(N)\). Here we essentially use, that \(r + \theta \geq 0\), otherwise we are not able to estimate the Lyapunov function by the energy function. Consequently, using the properties of \(\Theta, \Theta^{-1}\) we have

\[
E(t, \zeta) \leq C \exp \left( \int_t^T 2 \left( r(s, \zeta) + \theta(s, \zeta) + \langle \zeta \rangle^{-1} \frac{\dot{\lambda}(s)}{\Lambda(s)^2} \right) \right) \left( \log \frac{1}{\Lambda(s)} \right)^{2(y)} ds E(T, \zeta)
\]

for all \(t \in [t^\xi, T]\).

3.2.6. Conclusion

Now we use the Levi condition (2.6). Then we get from (3.14) by the aid of the definition of \(t^\xi\) the estimate

\[
E(t, \zeta) \leq \exp \left( C_1 (\log \langle \zeta \rangle)^{2(y)} \right) \exp \left( \langle \zeta \rangle^{-1} \int_{t^\xi}^T \frac{\dot{\lambda}(s)}{\Lambda(s)^2} \left( \log \frac{1}{\Lambda(s)} \right)^{2(y)} ds \right) E(T, \zeta)
\]

for all \(t \in [t^\xi, T]\). Using the definition of the hyperbolic zone we obtain

\[
\langle \zeta \rangle^{-1} \int_{t^\xi}^T \frac{\dot{\lambda}(s)}{\Lambda(s)^2} \left( \log \frac{1}{\Lambda(s)} \right)^{2(y)} ds \leq C_2 (\log \langle \zeta \rangle)^{2(y)}.
\]

Together with (3.15) we finally arrive at

\[
|Y(t, \zeta)| \leq C_0 \exp \left( C_1 (\log \langle \zeta \rangle)^{2(y)} \right) |Y(T, \zeta)|
\]

in \(\mathbb{Z}_H(N)\). The backward transformation and both estimates (3.5) and (3.16) multiplied by \(\langle \zeta \rangle^s\) yield the a priori estimate of the theorem.

The case \(\int_t^T r(s, \eta) ds \leq 0\) can be studied in the same way. Changing the order of components in \(V_1\), we get for the new vector \(\bar{V}_1 = \tilde{V}_1(t, \zeta)\):

\[
\left( \partial_t - r \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - B_3 \right) V_1 = \left( \partial_t - \tilde{r} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \tilde{B}_3 \right) \bar{V}_1 = 0,
\]

where \(\tilde{r} = \tilde{r}(t, \zeta) := -r(t, \zeta)\) satisfies the condition of the first case, and \(\tilde{B}_3 = \tilde{B}_3(t, \zeta)\) satisfies estimate (3.9).

3.2.7. Choice of the auxiliary function

It remains to define \(\theta\) and to check both properties which are described at the end of Section 3.2.4.
Let us consider the first case of (3.11); the other case can be studied in an analogous way. We denote by

\[ M_j := \frac{1}{2} \left( \int_{t_j}^{t_{j-1}} (|r(s, \eta)| + r(s, \eta)) \, ds + \sigma_j \right) \]

and

\[ m_j := \frac{1}{2} \left( \int_{t_j}^{t_{j-1}} (|r(s, \eta)| - r(s, \eta)) \, ds - \sigma_j \right). \]

Then we have from (3.11) that \( M_j \geq m_j \geq 0 \) holds uniformly with respect to \( j \). Moreover, we have with the constant \( C \) of (2.8) the inequalities \( m_j \leq C \) for all \( j \).

We define \( \theta(t, \xi) = \theta(t, \eta) \), \( \eta = \xi/|\xi| \), by

\[ \theta(t, \eta) := \frac{1}{2} \left( |r(t, \eta)| - r(t, \eta) - \frac{m_j}{M_j} (|r(t, \eta)| + r(t, \eta)) \right) \]

for \( t \in [t_j, t_{j-1}] \). Then we have

\[ r(t, \eta) + \theta(t, \eta) = \frac{1}{2} (r(t, \eta) + |r(t, \eta)|) \left( 1 - \frac{m_j}{M_j} \right) \geq 0. \]

Finally, we have to check the boundedness of the integral \( \int_t^T \theta(s, \eta) \, ds \) for \( (t, \xi) \in Z_H(N) \). The inequalities \( M_j \geq m_j \) lead for all \( t \in [t_N, t_{N-1}] \) to

\[
\left| \int_t^T \theta(s, \eta) \, ds \right| \leq \frac{1}{2} \sum_{j=1}^{N-1} \sigma_j \left( 1 + \frac{m_j}{M_j} \right) + \left| \int_t^{t_{N-1}} \theta(s, \eta) \, ds \right|
\]

\[
\leq \sum_{j=1}^{N} \sigma_j + \frac{1}{2} \int_t^{t_{N-1}} (|r(s, \eta)| - r(s, \eta)) \, ds - \sigma_N
\]

\[
- \frac{m_N}{M_N} \left( \int_t^{t_{N-1}} (|r(s, \eta)| + r(s, \eta)) \, ds + \sigma_N \right)
\]

\[
\leq \sum_{j=1}^{N} \sigma_j + \frac{1}{2} \left( \int_{t_N}^{t_{N-1}} (|r(s, \eta)| - r(s, \eta)) \, ds - \sigma_N \right)
\]

\[
+ \frac{m_N}{2M_N} \left( \int_{t_j}^{t_{j-1}} (|r(s, \eta)| + r(s, \eta)) \, ds + \sigma_N \right)
\]

\[
\leq \frac{C}{2} + 2m_N \leq \frac{5}{2} C,
\]
where $C$ is the constant from (2.7) and (2.8). This completes the proof of the theorem. □

4. Concluding remarks and a further example

**Remark 4.1.** We discussed which behavior of the coefficients in (2.1) generates a loss of regularity for the solution. We have shown the following influences:

(I) The vanishing order or the asymptotic behavior of the coefficients for $t \to 0$ as it is described by model (2.1) itself, and conditions (2.2) and (2.3).

(II) The rate of oscillations in the coefficients as it is described by condition (2.4).

(III) An interplay between the lower order term and the principal part as it is described by conditions (2.6) to (2.8). This interplay is part of the Levi conditions.

*Levi conditions are understood up to now as a balance of the vanishing order among the coefficients. Such an interpretation is correct for monotonous coefficients, but not sufficient to improve Levi conditions for oscillating coefficients.*

**Remark 4.2.** The main purpose of the present paper is to show that in the case of oscillating coefficients Levi condition should be described by some Riemann integrals taking account of the oscillating behavior of the coefficients as in conditions (2.5) and (2.6). Indeed, we can feel new effects for oscillating coefficients as in Example 2.1. Precisely, according to (2.5) and (2.6) only the integral over the lower order part is important near $t = 0$, but the *interaction between the lower order part and principal part* is important away from $t = 0$. To control these interactions we introduced conditions (2.7) and (2.8).

**Remark 4.3.** In [9, Example 2.1], it is proved that for $\lambda(t) \equiv 1$ condition (2.6) follows from (2.4) for $\delta_3 = \max\{\gamma_a, \gamma_b\} + 1$, and this value is optimal. Precisely, there are examples of $a_{jk}(t)$ and $b_j(t)$ satisfying (2.3)–(2.5), (2.7) and (2.8) such that the reverse inequality of (2.6) holds with $\delta_3 = \max\{\gamma_a, \gamma_b\} + 1$. Moreover, these examples provide the optimality of estimate (2.9) with $\gamma = \delta_3$. Thus from the level of interplay of the coefficients described by (2.6), which is not always restricted by the rates of oscillations $\gamma_a$ and $\gamma_b$, different possibilities of regularity loss may appear between *no loss* (Example 2.2 with $\gamma_b = 0$) and *infinite order loss* (Example 2.1 with $\gamma_a > 0$). (See Example 4.1 below for intermediate cases between two of the examples.)

**Remark 4.4.** We formulated our main result for $C^2$ coefficients with respect to $t$ in the principal part and for $C^1$ coefficients in the term of first order. Papers [7,10], open possibly an opportunity to weaken these regularity conditions in $t$ to a bit more regular as $C^1$ and $C^0$. But this remark should be understood in the moment as a conjecture only.

The following example can be interpreted as an intermediate case between Examples 1.1 and 2.2.
Example 4.1. Let $a(t) \equiv 1$ and

$$
\begin{align*}
b(t) := \Phi'(\omega(t)) + c_0 \left(t \omega'(t)\right)^{-1}
\end{align*}
$$

with a non-negative real number $c_0$, where $\Phi(\omega)$ and $\omega(t)$ are the same functions from Example 2.2. Additionally, we assume that $\Phi'(\omega(t)) = 0$ only at $\{s_j\}$ and $\{\tau_j\}$ satisfying $s_j < \tau_j < s_{j-1}$,

$$
m_0 := \Phi(\omega(\tau_j)) - \Phi(\omega(s_j)) = \Phi(\omega(\tau_j)) - \Phi(\omega(s_{j-1})) > 0
$$

for all $j$. Then we have for $q(t)/p(t) = \omega'(t)(\Phi'(\omega(t)) + c_0 t^{-1}$ that (2.5) and (2.6) hold with $\delta_2 = \gamma_b$ and $\delta_3 = 1$. Moreover, from the above choice of $\Phi$ we also have (2.7) and (2.8) with $C = m_0$ and $\{s_j = t_j\}$. Thus (2.9) holds for

$$
\gamma = \max \left\{ \gamma_0, \gamma_a, \frac{\gamma_b + \delta_2}{2}, \delta_2, \delta_3 \right\} = \max \{0, 0, \gamma_b, \gamma_b, 1\} = \max \{\gamma_b, 1\}.
$$

References


