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A priori estimates in terms of the maximum norm for the solutions of the Navier–Stokes equations

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Abstract

In this paper, we consider the Cauchy problem for the incompressible Navier–Stokes equations with bounded initial data and derive a priori estimates of the maximum norm of all derivatives of the solution in terms of the maximum norm of the initial velocity field. For illustrative purposes, we first derive corresponding a priori estimates for certain parabolic systems. Because of the pressure term, the case of the Navier–Stokes equations is more difficult, however.

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1. Introduction

In this paper, we consider the Cauchy problem for the 3D incompressible Navier–Stokes equations

$$u_t + u \cdot \nabla u + \nabla p = \Delta u, \quad \nabla \cdot u = 0, \tag{1.1}$$

with initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

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We will assume that

$$f \in L^\infty, \quad \nabla \cdot f = 0. \tag{1.3}$$

Here $\nabla \cdot f = 0$ holds in the sense of distributions.

If instead of $f \in L^\infty$ one assumes $f \in L^q$ for some q with $3 \leqslant q < \infty$, then it is well known that there is a unique strong solution in some maximal time interval $0 \leqslant t < T(f)$ where $0 < T(f) \leqslant \infty$. (The pressure is unique if one requires $p(x,t) \to 0$ as $|x| \to \infty$.) See, for example, [5,8] for the case $q = 3$ and [1] for $3 < q < \infty$. The solution is C^∞ for $0 < t < T(f)$.

If $f \in L^\infty$ then existence of a regular solution follows from [2]. The solution is only unique if one puts some growth restrictions on the pressure as $|x| \to \infty$. A simple example of non-uniqueness where u is bounded and $|p(x,t)| \leqslant C|x|$ is given in [6]. On the other hand, an estimate $|p(x,t)| \leqslant C(1 + |x|^\sigma)$ with $\sigma < 1$ (see [3]) or the assumption $p \in L^1_{\text{loc}}(0, T; BMO)$ (announced in [4]) imply uniqueness. For completeness, we briefly outline the construction of a regular solution, with bounded initial data, in an appendix.

Our main interest in this paper is to prove a priori estimates of the maximum norm of the derivatives of u in terms of the maximum norm of the initial function, $u(x,0) = f(x)$, *assuming* the solution to exist and to be C^∞ for $0 < t < T(f)$.

For illustration we also consider parabolic systems

$$u_t = \Delta u + D_i g(u), \; x \in \mathbb{R}^N, \quad t \geqslant 0 \tag{1.4}$$

with initial condition

$$u(x,0) = f(x) \quad \text{where } f \in L^\infty. \tag{1.5}$$

Here $u(x,t)$ takes values in $\mathbb{R}^n$,

$$D_i = \partial / \partial x_i$$

and $g : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be quadratic in u. The maximal interval of existence is again $0 \leqslant t < T(f)$. We will prove estimates of the maximum norm of the derivatives of the solution in terms of the maximum norm of the initial data, which we denote by

$$|f|_\infty = \sup_x |f(x)| \quad \text{with } |f(x)|^2 = \sum f_i^2(x).$$

To formulate the result, let

$$D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_N)$$

and $|\alpha| = \sum \alpha_i$. For any $j = 0, 1, \dots$, we set

$$|\mathcal{D}^j u(t)|_\infty = |\mathcal{D}^j u(\cdot, t)|_\infty = \max_{|\alpha| = j} |D^\alpha u(\cdot, t)|_\infty,$$

i.e., $|\mathcal{D}^j u(t)|_\infty$ measures all space derivatives of order j in maximum norm.

Theorem 1.1. *Under the above assumptions on f and g the solution of* (1.4), (1.5) *satisfies the following*:

(a) *There is a constant* $c_0 > 0$ *with*

$$T(f) > \frac{c_0}{|f|_\infty^2} \tag{1.6}$$

and

$$|u(\cdot,t)|_\infty \leqslant 2|f|_\infty \quad \text{for } 0 \leqslant t \leqslant \frac{c_0}{|f|_\infty^2}. \tag{1.7}$$

(b) *For every* $j = 1, 2, \ldots,$ *there is a constant* $K_j > 0$ *with*

$$t^{j/2}|\mathcal{D}^j u(\cdot,t)|_\infty \leqslant K_j |f|_\infty \quad \text{for } 0 < t \leqslant \frac{c_0}{|f|_\infty^2}. \tag{1.8}$$

The constants c_0 *and* K_j *are independent of t and f.*

After recalling some elementary estimates for the solution of the heat equation in Section 2, Theorem 1.1 will be shown in Section 3. Then we prove the analogous result for the solution of the Navier–Stokes equations in Section 4. Because of the non-local nature of the pressure, the proof is more complicated, however.

As we will also discuss in Section 4, estimate (1.8) implies that $|\mathcal{D}^j u|_\infty$ can be bounded in terms of $|u|_\infty^{j+1}$, which is consistent with the scale invariance of the Navier–Stokes equations. It does not seem to be known under what assumptions a converse bound of $|u|_\infty^{j+1}$ in terms of $|\mathcal{D}^j u|_\infty$ can be established.

2. Auxiliary results for the heat equation

Let $f \in L^\infty(\mathbb{R}^N)$. The solution of

$$u_t = \Delta u, \quad u = f \text{ at } t = 0, \tag{2.1}$$

is denoted by

$$u(\cdot,t) = u(t) = e^{\Delta t} f.$$

It is well-known that

$$|e^{\Delta t} f|_\infty \leqslant |f|_\infty, \quad t \geqslant 0 \tag{2.2}$$

and

$$|\mathcal{D}^j e^{\Delta t} f|_\infty \leqslant C_j t^{-j/2} |f|_\infty, \quad t>0, \quad j=1,2,\ldots \tag{2.3}$$

Here, and in the following, C, C_j, c, etc. denote positive constants that are independent of t and f.

If $F \in L^\infty(\mathbb{R}^N \times [0,T])$ then the solution of

$$u_t = \Delta u + F(x,t), \; u=0 \quad \text{at } t=0, \tag{2.4}$$

can be written as

$$u(t) = \int_0^t e^{\Delta(t-s)} F(s)\, ds.$$

One obtains

$$\begin{aligned} |u(t)|_\infty &\leqslant \int_0^t |F(s)|_\infty\, ds \\ &= \int_0^t s^{-1/2} s^{1/2} |F(s)|_\infty\, ds \\ &\leqslant 2t^{1/2} \max_{0\leqslant s\leqslant t} \{s^{1/2}|F(s)|_\infty\}. \end{aligned} \tag{2.5}$$

To estimate the solution of the equation

$$u_t = \Delta u + D_i F(x,t), \; u=0 \quad \text{at } t=0,$$

we note that D_i commutes with the heat semi-group. Using (2.3) with $j=1$ we have

$$\begin{aligned} |u(t)|_\infty &\leqslant C \int_0^t (t-s)^{-1/2} |F(s)|_\infty\, ds \\ &= \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |F(s)|_\infty\, ds \\ &\leqslant C \max_{0\leqslant s\leqslant t} \{s^{1/2}|F(s)|_\infty\}. \end{aligned} \tag{2.6}$$

3. Estimates for parabolic systems: proof of Theorem 1.1

In this section we consider the system $u_t = \Delta u + D_i g(u)$ with initial condition $u=f$ at $t=0$ where $f \in L^\infty$. It is well-known that the solution is C^∞ in a maximal interval $0<t<T(f)$ where $0<T(f)\leqslant\infty$. We set

$$F(x,t) = g(u(x,t)) \quad \text{for } x \in \mathbb{R}^N, \quad 0\leqslant t<T(f)$$

and consider u as the solution of the inhomogeneous heat equation $u_t = \Delta u + D_i F$. Recall the assumption that $g(u)$ is quadratic in u. Therefore, there is a constant $C_g > 0$ with

$$|g(u)| \leqslant C_g |u|^2, \quad |g_u(u)| \leqslant C_g |u| \quad \text{for all } u \in \mathbb{R}^n. \tag{3.1}$$

All second u–derivatives of g are constant.

We first estimate the maximum norm of u.

Lemma 3.1. *Let C_g denote the constant in* (3.1) *and let C denote the constant in* (2.6); *set $c_0 = \frac{1}{16C^2\, C_g^2}$. Then we have $T(f) > c_0/|f|_\infty^2$ and*

$$|u(t)|_\infty < 2|f|_\infty \quad \textit{for } 0 \leqslant t < \frac{c_0}{|f|_\infty^2}. \tag{3.2}$$

Proof. If estimate (3.2) does not hold, then denote by t_0 the smallest time with $|u(t_0)|_\infty = 2|f|_\infty$. By assumption, $t_0 < c_0/|f|_\infty^2$. Using (3.1) we have $|F(s)|_\infty \leqslant C_g |u(s)|_\infty^2$. Therefore, by (2.2) and (2.6),

$$\begin{aligned} 2|f|_\infty &= |u(t_0)|_\infty \\ &\leqslant |f|_\infty + C C_g t_0^{1/2} \max_{0 \leqslant s \leqslant t_0} |u(s)|_\infty^2 \\ &= |f|_\infty + C C_g t_0^{1/2} 4 |f|_\infty^2. \end{aligned}$$

This yields

$$1 \leqslant 4 C C_g t_0^{1/2}\, |f|_\infty,$$

thus $t_0 \geqslant 1/(16 C^2 C_g^2 |f|_\infty^2) = c_0/|f|_\infty^2$. This contradiction implies that (3.2) holds. The estimate $T(f) > c_0/|f|_\infty^2$ is valid since $\limsup_{t \to T(f)} |u(t)|_\infty = \infty$ if $T(f)$ is finite. □

We now prove estimate (1.8) by induction in j. Let $j \geqslant 1$ and assume

$$t^{k/2} |\mathcal{D}^k u(t)|_\infty \leqslant K_k |f|_\infty \quad \text{for } 0 \leqslant t \leqslant \frac{c_0}{|f|_\infty^2} \quad \text{and} \quad 0 \leqslant k \leqslant j-1. \tag{3.3}$$

Here c_0 is the constant defined in the previous lemma.

It will be convenient to denote any space derivative $D^\alpha = D_1^{\alpha_1} \ldots D_N^{\alpha_N}$ simply by D^l if $|\alpha| = l$. Apply D^j to the equation $u_t = \Delta u + D_i g(u)$ to obtain

$$v_t = \Delta v + D^{j+1} g(u), \quad v := D^j u,$$

$$v(t) = D^j e^{\Delta t} f + \int_0^t e^{\Delta(t-s)} D^{j+1} g(u(s))\, ds.$$

Using (2.3) we have

$$t^{j/2}|v(t)|_\infty \leqslant C|f|_\infty + t^{j/2}\left|\int_0^t e^{\Delta(t-s)} D^{j+1} g(u(s))\, ds\right|_\infty. \tag{3.4}$$

Split the integral into

$$\int_0^{t/2} + \int_{t/2}^t =: I_1(t) + I_2(t)$$

and obtain

$$\begin{aligned} |I_1(t)|_\infty &= \left|\int_0^{t/2} D^{j+1} e^{\Delta(t-s)} g(u(s))\, ds\right|_\infty \\ &\leqslant C|f|_\infty^2 \int_0^{t/2} (t-s)^{-(j+1)/2}\, ds \\ &\leqslant C|f|_\infty^2 t^{(1-j)/2}. \end{aligned}$$

When estimating $I_2(t)$, only one derivative is moved from $D^{j+1}g(u)$ to the heat semi-group. (If one moves two or more derivatives, then the singularity at $s = t$ becomes non-integrable.) We have

$$\begin{aligned} |I_2(t)|_\infty &= \left|\int_{t/2}^t D e^{\Delta(t-s)} D^j g(u(s))\, ds\right|_\infty \\ &\leqslant C \int_{t/2}^t (t-s)^{-1/2} |D^j g(u(s))|_\infty\, ds. \end{aligned} \tag{3.5}$$

Recall that $g(u)$ is quadratic in u. Therefore,

$$|D^j g(u)|_\infty \leqslant C|u|_\infty |\mathcal{D}^j u|_\infty + C\sum_{k=1}^{j-1} |\mathcal{D}^k u|_\infty |\mathcal{D}^{j-k} u|_\infty.$$

By the induction hypothesis (3.3), the above sum is bounded by $Cs^{-j/2}|f|_\infty^2$. Thus the corresponding part of the integral in (3.5) is bounded by

$$C|f|_\infty^2 \int_{t/2}^t (t-s)^{-1/2} s^{-j/2}\, ds \leqslant C|f|_\infty^2 t^{(1-j)/2}.$$

The remaining part of the integral in (3.5) is bounded by

$$\begin{aligned} \int_{t/2}^t (t-s)^{-1/2} |u(s)|_\infty |\mathcal{D}^j u(s)|_\infty\, ds &\leqslant C|f|_\infty \int_{t/2}^t (t-s)^{-1/2} s^{-j/2} s^{j/2} |\mathcal{D}^j u(s)|_\infty\, ds \\ &\leqslant C|f|_\infty t^{(1-j)/2} \max_{0\leqslant s\leqslant t} \{s^{j/2} |\mathcal{D}^j u(s)|_\infty\}. \end{aligned}$$

We use these bounds for the integral in (3.4) and recall the definition $v = D^j u$. Then, maximizing the resulting estimate of $t^{j/2}|D^j u(t)|_\infty$ over all derivatives D^j of order j and setting

$$\phi(t) = t^{j/2}|\mathcal{D}^j u(t)|_\infty,$$

we have shown the estimate

$$\phi(t) \leqslant C|f|_\infty + Ct^{1/2}|f|_\infty^2 + C|f|_\infty t^{1/2} \max_{0\leqslant s\leqslant t} \phi(s) \quad \text{for } 0\leqslant t\leqslant \frac{c_0}{|f|_\infty^2}.$$

Since $t^{1/2}|f|_\infty \leqslant \sqrt{c_0}$ the second term on the right-hand side of the above estimate is bounded by $C|f|_\infty$. Therefore,

$$\phi(t) \leqslant C_j|f|_\infty + C_j|f|_\infty t^{1/2} \max_{0\leqslant s\leqslant t} \phi(s) \quad \text{for } 0\leqslant t\leqslant \frac{c_0}{|f|_\infty^2}. \tag{3.6}$$

For the remainder of the proof, let the constant C_j be fixed so that the above estimate holds. Set

$$c_j = \min\left\{c_0, \frac{1}{4C_j^2}\right\}.$$

We first claim that

$$\phi(t) < 2C_j|f|_\infty \quad \text{for } 0\leqslant t < \frac{c_j}{|f|_\infty^2}.$$

Otherwise, let $0 < t_0 < c_j/|f|_\infty^2$ denote the smallest time with $\phi(t_0) = 2C_j|f|_\infty$. Then we obtain from (3.6),

$$2C_j|f|_\infty = \phi(t_0) \leqslant C_j|f|_\infty + 2C_j^2|f|_\infty^2 t_0^{1/2},$$

thus

$$1 \leqslant 2C_j|f|_\infty t_0^{1/2}, \quad \text{i.e. } t_0 \geqslant \frac{c_j}{|f|_\infty^2}.$$

This contradiction proves the estimate

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \leqslant 2C_j|f|_\infty \quad \text{for } 0\leqslant t\leqslant \frac{c_j}{|f|_\infty^2}. \tag{3.7}$$

If

$$T_j := \frac{c_j}{|f|_\infty^2} < t \leqslant \frac{c_0}{|f|_\infty^2} =: T_0 \tag{3.8}$$

then we start the corresponding estimate at $t - T_j$. By the previous lemma we have $|u(t-T_j)|_\infty \leqslant 2|f|_\infty$ and obtain

$$T_j^{j/2}|\mathcal{D}^j u(t)|_\infty \leqslant 4C_j|f|_\infty. \tag{3.9}$$

Finally, for any t with (3.8),

$$t^{j/2} \leqslant T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2},$$

and (3.9) yields

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \leqslant 4C_j\left(\frac{c_0}{c_j}\right)^{j/2}|f|_\infty.$$

This completes the proof of Theorem 1.1. □

4. Estimates for the Navier–Stokes equations

We write the Navier–Stokes equations as

$$u_t = \Delta u + Q, \quad \nabla \cdot u = 0, \quad u = f \quad \text{at } t = 0,$$

with

$$\begin{aligned} Q &= -\nabla p - u \cdot \nabla u \\ &= -\nabla p - \sum_j D_j(u_j u). \end{aligned}$$

Here the pressure is determined by the Poisson equation

$$\begin{aligned} -\Delta p &= \sum_{i,j} D_i D_j(u_i u_j) \\ &= \sum_{i,j} (D_i u_j)(D_j u_i). \end{aligned}$$

Dropping the t–dependence in our notation, we have

$$p(x) = \frac{1}{4\pi}\sum_{i,j}\int |x-y|^{-1} D_i D_j u_i u_j(y)\,dy. \tag{4.1}$$

Remark. The Calderon–Zygmund theory of singular integrals guarantees that $p \in BMO$, the space of functions with bounded mean oscillation. See, for example, [7]. In general, $p \notin L^\infty$. For the global part, p_{gl}, of p (see below), we will only need maximum norm estimates of *derivatives*. The *BMO* norm of p will not be used. See the appendix for an elementary discussion of integral (4.1).

We decompose p into a local and a global part, $p = p_{\rm lc} + p_{\rm gl}$, as follows: Choose a C^∞ cut-off function $\phi(r)$ with

$$\phi(r) = 1 \quad \text{for } 0 \leqslant r \leqslant 1, \quad \phi(r) = 0 \quad \text{for } r \geqslant 2.$$

Then, for $\delta > 0$, define

$$p_{\rm lc}(x) = \frac{1}{4\pi} \sum_{i,j} \int |x-y|^{-1} D_i D_j \big(\phi(\delta^{-1}|x-y|) u_i(y) u_j(y)\big)\, dy. \tag{4.2}$$

The global part, $p_{\rm gl} = p - p_{\rm lc}$, is determined correspondingly with ϕ replaced by $1-\phi$. It is clear that $p_{\rm lc}(x)$ depends only on the values $u(y)$ for $|x-y| < 2\delta$. Correspondingly, $p_{\rm gl}(x)$ depends only on the values $u(y)$ for $|x-y| > \delta$. The decomposition $p = p_{\rm lc} + p_{\rm gl}$ depends on ϕ and on δ, which is suppressed in our notation. Later we will choose $\delta = \sqrt{t}$.

We first estimate the pressure in terms of u. The estimates are valid at each time t where $0 < t < T(f)$.

Lemma 4.1. *There is a constant $C > 0$, independent of t, δ, and f, so that the following holds*:

$$|p_{\rm lc}|_\infty \leqslant C(|u|_\infty^2 + \delta |u|_\infty |\mathcal{D}u|_\infty), \tag{4.3}$$

$$|\mathcal{D}p_{\rm lc}|_\infty \leqslant C(\delta^{-1}|u|_\infty^2 + \delta |\mathcal{D}u|_\infty^2), \tag{4.4}$$

$$|\mathcal{D}p_{\rm gl}|_\infty \leqslant C\delta^{-1}|u|_\infty^2. \tag{4.5}$$

Proof. The argument of ϕ, ϕ', etc. is always $\delta^{-1}|x-y|$, which we suppress in our notation. Integrating by parts in formula (4.2) for $p_{\rm lc}$, we have

$$|p_{\rm lc}(x)| \leqslant C \sum_{i,j} \int |x-y|^{-2} |D_i(\phi u_i u_j)|\, dy.$$

Clearly,

$$|D_i(\phi u_i u_j)| \leqslant C(\delta^{-1}|u|_\infty^2 + |u|_\infty |\mathcal{D}u|_\infty).$$

(The constant C depends on the maximum norm of ϕ and ϕ'.) Since

$$\int_{|x-y|\leqslant 2\delta} |x-y|^{-2}\,dy \leqslant C\delta$$

we obtain (4.3).

To estimate $|\mathcal{D}p_{\mathrm{lc}}|_\infty$, we first apply $D_{k,x} = \partial/\partial x_k$ under the integral sign in (4.2). Note that

$$|D_{k,x}|x-y|^{-1}| \leqslant |x-y|^{-2}$$

and

$$|D_{k,x}\phi| \leqslant \delta^{-1}|\phi'|_\infty.$$

When estimating the term

$$T_1 = \sum_{i,j} \int |x-y|^{-2} D_i D_j(\phi u_i u_j)\,dy$$

it is important to note that

$$\sum D_i D_j(u_i u_j) = \sum (D_i u_j)(D_j u_i),$$

i.e., 2nd derivatives of u are not needed to bound T_1. One obtains

$$|T_1| \leqslant C(\delta^{-1}|u|_\infty^2 + \delta|\mathcal{D}u|_\infty^2).$$

The term

$$T_2 = \sum_{i,j} \int |x-y|^{-1} D_i D_j((D_{k,x}\phi)u_i u_j))\,dy$$

is treated similarly, without integration by parts, and (4.4) follows.

To estimate $|\mathcal{D}p_{\mathrm{gl}}|_\infty$, we write

$$p_{\mathrm{gl}}(x) = \frac{1}{4\pi}\sum_{i,j}\int (D_i D_j|x-y|^{-1})(1-\phi)u_i u_j\,dy$$

and apply $D_{k,x}$ under the integral sign. Using the estimates

$$\int_{|x-y|\geqslant \delta} |x-y|^{-4}\,dy \leqslant C\delta^{-1}$$

and, if ϕ is differentiated,

$$\int_{2\delta \geqslant |x-y| \geqslant \delta} |x-y|^{-3}\, dy \leqslant C$$

bound (4.5) is obtained. $\square$

Recall that

$$u_t = \Delta u + Q, \quad Q = -\nabla p - u \cdot \nabla u, \quad u = f \quad \text{at } t = 0.$$

We write $Q = Q_{\text{lc}} + Q_{\text{gl}}$ with

$$Q_{\text{lc}} = -\nabla p_{\text{lc}} - \sum_j D_j(u_j u),$$
$$Q_{\text{gl}} = -\nabla p_{\text{gl}}.$$

Using the estimates of the previous lemma and the heat equation estimates (2.2), (2.5), and (2.6), we will prove the following.

Lemma 4.2. *Set*

$$V(t) = |u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty, \quad 0 < t < T(f). \tag{4.6}$$

There is a constant $C > 0$, *independent of t and f, so that*

$$V(t) \leqslant C|f|_\infty + Ct^{1/2} \max_{0 \leqslant s \leqslant t} V^2(s), \quad 0 < t < T(f). \tag{4.7}$$

Proof. Using the previous lemma with $\delta = t^{1/2}$, we have

$$|p_{\text{lc}}|_\infty + |u_j u|_\infty \leqslant C(|u|^2_\infty + t^{1/2}|u|_\infty|\mathcal{D}u|_\infty), \tag{4.8}$$

$$|Q_{\text{lc}}|_\infty \leqslant C(t^{-1/2}|u|^2_\infty + t^{1/2}|\mathcal{D}u|^2_\infty), \tag{4.9}$$

$$|Q_{\text{gl}}|_\infty \leqslant Ct^{-1/2}|u|^2_\infty. \tag{4.10}$$

Since $u_t = \Delta u + Q_{\mathrm{lc}} + Q_{\mathrm{gl}}$ and since Q_{lc} is obtained by applying one space derivative to the terms p_{lc} and $u_j u$, we obtain from (2.2), (4.8), (2.6), (4.10), (2.5),

$$\begin{aligned} |u(t)|_\infty &\leqslant |f|_\infty + C \max_{0\leqslant s\leqslant t} (s^{1/2}|u(s)|_\infty^2 + s|u(s)|_\infty |\mathcal{D}u(s)|_\infty) + Ct^{1/2} \max_{0\leqslant s\leqslant t} |u(s)|_\infty^2 \\ &\leqslant |f|_\infty + Ct^{1/2} \max_{0\leqslant s\leqslant t} (|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2) \\ &\leqslant |f|_\infty + Ct^{1/2} \max_{0\leqslant s\leqslant t} V^2(s). \end{aligned}$$

For $v(t) = D_k u(t)$ we have

$$v_t = \Delta v + D_k Q$$

with

$$|Q|_\infty \leqslant C(t^{-1/2}|u|_\infty^2 + t^{1/2}|\mathcal{D}u|_\infty^2).$$

Therefore, by (2.3) with $j = 1$ and by (2.6),

$$\begin{aligned} t^{1/2}|v(t)|_\infty &\leqslant C|f|_\infty + Ct^{1/2} \max_{0\leqslant s\leqslant t} (|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2) \\ &\leqslant C|f|_\infty + Ct^{1/2} \max_{0\leqslant s\leqslant t} V^2(s) \end{aligned}$$

The lemma is proved. □

Lemma 4.2 allows us to estimate $|u(t)|_\infty$ and $|\mathcal{D}u(t)|_\infty$ in terms of $|f|_\infty$ in a small time interval.

Lemma 4.3. *Let $C > 0$ denote the constant in estimate* (4.7) *and set*

$$c_0 = \frac{1}{16C^4}.$$

Then $T(f) > c_0/|f|_\infty^2$ and

$$|u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty < 2C|f|_\infty \quad \text{for } 0 \leqslant t < \frac{c_0}{|f|_\infty^2}. \tag{4.11}$$

Proof. Recall the definition of $V(t)$ in (4.6). If (4.11) does not hold, then denote by t_0 the smallest time with $V(t_0) = 2C|f|_\infty$. Using (4.7) we have

$$\begin{aligned} 2C|f|_\infty &= V(t_0) \\ &\leqslant C|f|_\infty + Ct_0^{1/2} 4C^2 |f|_\infty^2, \end{aligned}$$

thus

$$1 \leqslant 4C^2 t_0^{1/2}\, |f|_\infty,$$

thus $t_0 \geqslant c_0/|f|_\infty^2$. This contradiction proves (4.11), and $T(f) > c_0/|f|_\infty^2$ follows. □

Lemma 4.3 proves bound (1.8) for the solution of the Navier–Stokes equations for $j = 0$ and 1. By an induction argument as in the proof of Theorem 1.1 one obtains the following.

Theorem 4.1. *Consider the Cauchy problem for the Navier–Stokes equations*, (1.1) *and* (1.2), *where* $f \in L^\infty$, $\nabla \cdot f = 0$. *There is a constant* $c_0 > 0$ *and for every* $j = 0, 1, \ldots,$ *there is a constant* K_j *so that*

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \leqslant K_j |f|_\infty \quad \text{for } 0 < t \leqslant \frac{c_0}{|f|_\infty^2}. \tag{4.12}$$

The constants c_0 *and* K_j *are independent of t and f.*

Remarks. We can apply estimate (4.12) for

$$\frac{c_0}{2|f|_\infty^2} \leqslant t \leqslant \frac{c_0}{|f|_\infty^2} \tag{4.13}$$

and obtain

$$|\mathcal{D}^j u(t)|_\infty \leqslant C_j |f|_\infty^{j+1} \tag{4.14}$$

in interval (4.13). Starting the estimate at $t_0 \in [0, T(f))$ we have

$$|\mathcal{D}^j u(t_0 + t)|_\infty \leqslant C_j |u(t_0)|_\infty^{j+1} \tag{4.15}$$

for

$$\frac{c_0}{2|u(t_0)|_\infty^2} \leqslant t \leqslant \frac{c_0}{|u(t_0)|_\infty^2}. \tag{4.16}$$

Then, if t_1 is fixed with

$$\frac{c_0}{2|f|_\infty^2} \leqslant t_1 < T(f),$$

we can maximize both sides of (4.15) over $0 \leqslant t_0 \leqslant t_1$ and obtain

$$\max\left\{ |\mathcal{D}^j u(t)|_\infty : \frac{c_0}{2|f|_\infty^2} \leqslant t \leqslant t_1 + \tau \right\} \leqslant C_j \max\{ |u(t)|_\infty^{j+1} : 0 \leqslant t \leqslant t_1 \} \tag{4.17}$$

with

$$\tau = \frac{c_0}{|u(t_1)|_\infty^2}.$$

Estimate (4.17) says, essentially, that the maximum of the jth derivatives of u, measured by $|\mathcal{D}^j u|_\infty$, can be bounded in terms of $|u|_\infty^{j+1}$. Clearly, a time interval near $t = 0$ has to be excluded on the left-hand side of (4.17) for smoothing to become effective. The positive value of τ on the left-hand side of (4.17) shows that $|u|_\infty^{j+1}$ controls $|\mathcal{D}^j u|_\infty$ for some time into the future.

As is well known, if u, p solve the Navier–Stokes equations and $\lambda > 0$ is any scaling parameter, then the functions u_λ, p_λ defined by

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

also solve the Navier–Stokes equations. Clearly,

$$|u_\lambda(t)|_\infty = \lambda |u(\lambda^2 t)|_\infty, \quad |\mathcal{D}^j u_\lambda(t)|_\infty = \lambda^{j+1} |\mathcal{D}^j u(\lambda^2 t)|_\infty.$$

Therefore, $|\mathcal{D}^j u|_\infty$ and $|u|_\infty^{j+1}$ both scale like λ^{j+1}, which is, of course, consistent with the estimate (4.17). We do not know under what assumptions $|u|_\infty^{j+1}$ can conversely be estimated in terms of $|\mathcal{D}^j u|_\infty$.

Appendix. The Cauchy problem for the Navier–Stokes equations with bounded initial data

First let $f \in C^\infty \cap L^\infty, \nabla \cdot f = 0$. Define a sequence $u^n(x,t), p^n(x,t)$ of C^∞ functions by

$$-\Delta p^{n+1} = \sum_{i,j} D_i D_j (u_i^n u_j^n) \tag{A.1}$$

$$u_t^{n+1} = \Delta u^{n+1} - u^n \cdot \nabla u^n - \nabla p^{n+1}, \quad u^{n+1}(x,0) = f(x) \tag{A.2}$$

with $u^0 \equiv f$. The Calderon–Zygmund theory of singular integrals can be used to discuss the Poisson equation (A.1). An elementary approach is as follows:

If $\Phi(z) = \frac{1}{4\pi}|z|^{-1}$ and $\Phi_{ij}(z) = D_i D_j \Phi(z)$ then (A.1) yields, formally,

$$p^{n+1}(x) = \sum \int \Phi_{ij}(x-y)(u_i^n u_j^n)(y)\, dy, \tag{A.3}$$

where the dependence on t is suppressed in our notation. Since

$$\Phi_{ij}(z) = |z|^{-3}\Phi_{ij}(z^0), \quad z^0 = z/|z|,$$

the integrals in (A.3) generally do not exist as Lebesgue integrals. However, the (non-integrable) singularity of $\Phi_{ij}(z)$ at $z = 0$ causes no problems since the functions u_i^n are smooth and

$$\int_{|z|=1} \Phi_{ij}\, dS = 0.$$

Also, since $|\mathcal{D}\Phi_{ij}(z)| \leqslant C|z|^{-4}$, we have by the mean–value theorem

$$|\Phi_{ij}(x-y) - \Phi_{ij}(y)| \leqslant \frac{C|x|}{|y|^4} \quad \text{for } |y| \geqslant 3|x| \quad (say),$$

and therefore the limits

$$p_{ij}^{n+1}(x) \coloneqq \lim_{R\to\infty} \int_{|y|\leqslant R} (\Phi_{ij}(x-y) - \Phi_{ij}(y))(u_i^n u_j^n)(y)\, dy$$

can be shown to exist. The function

$$p^{n+1}(x) = \sum p_{ij}^{n+1}(x)$$

solves (A.1). As in Section 4, we can decompose p^{n+1} into a local and a global part, $p^{n+1} = p_{\text{lc}}^{n+1} + p_{\text{gl}}^{n+1}$. In general, $p_{\text{gl}}^{n+1} \notin L^\infty$, but this is not important since only *derivative* estimates of p_{gl}^{n+1} are needed to derive estimates for u^{n+1}; compare Lemma 4.1.

Proceeding as in Section 4, we obtain that

$$t^{j/2}|\mathcal{D}^j u^n(t)|_\infty \leqslant K_j |f|_\infty \quad \text{for } 0 < t \leqslant \frac{c_0}{|f|_\infty^2}, \quad j = 0, 1, \ldots$$

Convergence of $u^n(x,t)$ and its derivatives w.r.t. $|\cdot|_\infty$ follows, as usual, by a Picard contraction argument. As $n \to \infty$, the global part of the pressure, p_{gl}^{n+1}, converges in maximum norm in any bounded set $|x| \leqslant R$, and one obtains a well-defined smooth limit p of $p^{n+1} = p_{\text{lc}}^{n+1} + p_{\text{gl}}^{n+1}$.

If $f \in L^\infty$ is not smooth, one can approximate f by C^∞ functions f^j in maximum norm, $|f - f^j|_\infty \to 0$ as $j \to \infty$. The f^j are not uniformly smooth. However, the existence interval for the initial functions f^j can be chosen uniformly in j since it only depends on $|f^j|_\infty$, which approaches $|f|_\infty$. A simple limit argument, $u^j \to u, p^j \to p$, yields a solution with initial data f.

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