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Geometry of Directing Modules over Tame Algebras*

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Throughout the paper K denotes a fixed algebraically closed field. By an algebra we mean a finite dimensional K-algebra (associative, with an identity) and by a module a finite dimensional left A-module.

The class of algebras may be divided into two disjoint subclasses. One class consists of tame algebras for which the indecomposable modules occur, in each dimension, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two noncommuting endomorphisms, for which the classification up to isomorphism is a well-known unsolved problem. Hence, we can realistically hope to describe modules only for tame algebras. Given an algebra A and a nonnegative vector $\mathbf{d}$ in the Grothendieck group $K_0(A)$ of A, it is an interesting task to study the affine variety $\mathrm{mod}_A(\mathbf{d})$ of A-modules of dimension-vector $\mathbf{d}$ and the action of the corresponding product $G(\mathbf{d})$ of general linear groups. For example, we may ask when the variety $\mathrm{mod}_A(\mathbf{d})$ is irreducible, smooth, complete intersection, Gorenstein, Cohen–Macaulay, normal, ...

The main aim of this paper is to describe the geometry of module varieties $\mathrm{mod}_A(\mathbf{d})$ for the dimension-vectors $\mathbf{d}$ of arbitrary directing modules over tame algebras. Recall that following [20] an indecomposable A-module M is said to be directing if it does not belong to a cycle $M \to M_1 \to \cdots \to M_r \to M$ of nonzero nonisomorphisms between indecomposable A-modules. Directing modules have played an important role

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in the representation theory of algebras: the preprojective components and the preinjective components in general and the connecting components of tilted algebras consist entirely of directing modules. C. M. Ringel has proved in [20] that if M is a directing A-module then its support algebra is a tilted algebra and the dimension-vector of M is a root of the associated Euler (equivalently, Tits) integral quadratic form. We also mention that a detailed structure of the category of modules over tame tilted algebras has been established in [13, 17, 20]. On the other hand, the knowledge of geometric properties of modules over tame tilted algebras is relatively poor.

In our paper [1] we apply the main results proved here to describe the geometry of modules over tame quasi-tilted algebras (investigated in [10, 22]) whose dimension-vectors are the dimension-vectors of arbitrary indecomposable modules.

The paper is organized as follows. In Section 1 we present our main results and recall the related background. Section 2 is devoted to some geometric preliminary results on affine varieties of modules. In Section 3 we recall some facts on the module categories of tame tilted algebras. Section 4 is devoted to the proofs of our main results. The final Section 5 contains some examples illustrating different cases appearing in our considerations.

1. THE MAIN RESULTS AND THE RELATED BACKGROUND

Throughout the paper K will denote a fixed algebraically closed field. By an algebra we mean an associative finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and connected. For such an algebra A, there exists an isomorphism $A \simeq KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of KQ, generated by a (finite) system of forms $\sum_{1 \leq j \leq t} \lambda_j \alpha_{m_j, j} \cdots \alpha_{1, j}$ (called K-linear relations), where $\lambda_1, \dots, \lambda_t$ are elements of K and $\alpha_{m_j, j} \cdots \alpha_{1, j}$, $1 \leq j \leq t$, are paths of length ≥ 2 in Q having a common source and a common end. Denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q, and by $s, e\colon Q_1 \to Q_0$ the maps which assign to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its end $e(\alpha)$, respectively. The category mod_A of all finite dimensional (over K) left A-modules is equivalent to the category $\operatorname{rep}_K(Q, I)$ of all finite dimensional K-vector spaces and $\varphi_\alpha\colon V_{s(\alpha)} \to V_{e(\alpha)}$, $\alpha \in Q_1$, are K-linear maps, satisfying the equations $\sum_{1 \leq j \leq t} \lambda_j \varphi_{\alpha_{m_j, j}} \cdots \varphi_{\alpha_{1, j}} = 0$ for all K-linear relations $\sum_{1 \leq j \leq t} \lambda_j \alpha_{m_j, j} \cdots \alpha_{1, j} \in I$ (see [9, Sect. 4] for details). We shall identify mod_A with $\operatorname{rep}_K(Q, I)$ and call the finite dimensional left A-modules shortly A-modules. The Grothendieck group $K_0(A)$ of A is then identified with the group $\mathbb{Z}^{Q_0}$, and we may assign to each A-module $V = (V_i, \varphi_\alpha)$ its dimension-vector $\mathbf{dim}\, V = (\dim_K V_i)_{i \in Q_0}$. Finally, A is said to be tame if, for any dimension d, there exists a finite number of A-$K[X]$-bimodules M_i, $1 \le i \le n_d$, which are finite rank free right modules over the polynomial algebra $K[X]$ in one variable and all but finitely many isomorphism classes of indecomposable A-modules of dimension d are of the form $M_i \otimes_{K[X]} K[X]/(X - \lambda)$ for some $\lambda \in K$ and some i.

Fix now a vector $\mathbf{d} = (d_i) \in K_0(A) = \mathbb{Z}^{Q_0}$ with nonnegative coordinates. Denote by $\operatorname{mod}_A(\mathbf{d})$ the set of all representations $V = (V_i, \varphi_i)$ in $\operatorname{rep}_K(Q, I)$ with $V_i = K^{d_i}$ for all $i \in Q_0$. A representation V in $\operatorname{mod}_A(\mathbf{d})$ is given by $d_{e(\alpha)} \times d_{s(\alpha)}$-matrices $V(\alpha)$ determining the maps $\varphi_\alpha\colon K^{s(\alpha)} \to K^{e(\alpha)}$, $\alpha \in Q_1$, in the canonical bases of K^{d_i}, $i \in Q_0$. Moreover, the matrices $V(\alpha)$, $\alpha \in Q_1$, satisfy the equations

$$\sum_{1 \le j \le t} \lambda_j V(\alpha_{m_j, j}) \cdots V(\alpha_{1, j}) = 0$$

for all K-linear relations $\sum_{1 \le j \le t} \lambda_j \alpha_{m_j, j} \cdots \alpha_{1, j} \in I$. Therefore, $\operatorname{mod}_A(\mathbf{d})$ is a closed subset of $\prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ in the Zariski topology, and so $\operatorname{mod}_A(\mathbf{d})$ is an affine variety. We note that $\operatorname{mod}_A(\mathbf{d})$ is not necessarily irreducible. Clearly, it is the case when $I = 0$. The affine (reductive) algebraic group $G(\mathbf{d}) = \prod_{i \in Q_0} \operatorname{Gl}_{d_i}(K)$ acts on the variety $\operatorname{mod}_A(\mathbf{d})$ by conjugation,

$$(gV)(\alpha) = g_{e(\alpha)} V(\alpha) g_{s(\alpha)}^{-1}$$

for $g = (g_i) \in G(\mathbf{d})$, $V \in \operatorname{mod}_A(\mathbf{d})$, $\alpha \in Q_1$. We shall identify an A-module V of dimension-vector $\mathbf{d}$ with the corresponding point of the variety $\operatorname{mod}_A(\mathbf{d})$. The $G(\mathbf{d})$-orbit $G(\mathbf{d})M$ of a module M in $\operatorname{mod}_A(\mathbf{d})$ will be denoted by $\mathscr{O}(M)$. Observe that two A-modules M and N are isomorphic if an only if $\mathscr{O}(M) = \mathscr{O}(N)$. For $M, N \in \operatorname{mod}_A(\mathbf{d})$, we say that N is a degeneration of M if N belongs to the Zariski closure $\overline{\mathscr{O}(M)}$ of $\mathscr{O}(M)$ in $\operatorname{mod}_A(\mathbf{d})$. If $N \in \overline{\mathscr{O}(M)}$ implies $\mathscr{O}(N) = \mathscr{O}(M)$, the orbit $\mathscr{O}(N)$ is said to be maximal. Clearly, an orbit in $\operatorname{mod}_A(\mathbf{d})$ of maximal dimension is maximal, but the converse is not true in general. It is known that the union of all $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ of maximal dimension is an open subset of $\operatorname{mod}_A(\mathbf{d})$, called its open sheet (see [14, 15]).

Assume now that $A = KQ/I$ is of finite global dimension. Then there is a (nonsymmetric) bilinear form $\langle \text{-}, \text{-} \rangle_A$ on $K_0(A)$ such that

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}_A^i(M, N)$$

for all A-modules M, N (see [20, (2.4)]). Then the corresponding quadratic form $\chi_A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle_A$, $\mathbf{x} \in K_0(A)$, is called the Euler form of A. If A is triangular (Q has no oriented cycles), we may also consider the Tits form q_A of A, defined for $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0} = K_0(A)$ as

$$q_A(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{e(\alpha)} + \sum_{i, j \in Q_0} r_{ij} x_i x_j,$$

where r_{ij} is the number of K-linear relations with source i and end j in a minimal set $\mathscr{R}$ of K-linear relations generating the ideal I. If gl.dim $A \leq 2$ then q_A coincides with χ_A [3]. Moreover, we set

$$a(\mathbf{x}) = \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{e(\alpha)} - \sum_{i, j \in Q_0} r_{ij} x_i x_j$$

for $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0}$. We note that all tilted algebras A are triangular of global dimension at most 2, and so the forms χ_A, q_A are defined, and in fact they coincide. A vector $\mathbf{d} \in \mathbb{Z}^{Q_0}$ is called connected if the full subquiver of Q given by its support $\operatorname{supp}(\mathbf{d}) = \{i \in Q_0;\ d_i \neq 0\}$ is connected. Moreover, we say that $\mathbf{d}$ is positive if $\mathbf{d}$ is nonzero and $d_i \geq 0$ for all $i \in Q_0$.

Sometimes it is convenient to consider an algebra $A = KQ/I$ as a finite K-category whose objects are the vertices of Q, and, for any two vertices $x, y \in Q_0$, the space of morphisms from x to y is the quotient space of the K-vector space $KQ(x, y)$ of all K-linear combinations of paths in Q from x to y by the subspace $I(x, y) = I \cap KQ(x, y)$. A full subcategory C of A is said to be convex (in A) if, for any path $a_0 \to a_1 \to \cdots \to a_t$ with a_0 and a_t in C, all vertices a_i, $0 \leq i \leq t$, belong to C. Clearly, if C is a convex subcategory of A, we may identify mod_C with the full subcategory of mod_A given by all representations with the support contained in C. Moreover, if A is triangular and C is a convex subcategory of A, then χ_C and q_C are the restrictions of χ_A and q_A to $K_0(C)$. By the support $\operatorname{supp}(M)$ of an A-module M we mean the full subcategory of A given by the support of its dimension-vector. If $\operatorname{supp}(M) = A$ then the module M is said to be sincere.

It has been proved by K. Bongartz [6, Proposition 3.4; 5, Proposition 2] that if V is a preprojective indecomposable A-module and $\mathbf{d} = \mathbf{dim}\, V$ then $\operatorname{mod}_A(\mathbf{d}) = \overline{\mathscr{O}(V)}$ and is Cohen–Macaulay (in fact even a complete intersection), and moreover $\operatorname{mod}_A(\mathbf{d})$ is normal if A is representation-finite. Our main results presented below concern arbitrary directing modules over tame algebras. For completeness we also present alternative proofs of the mentioned above results due to K. Bongartz.

THEOREM 1. *Let A be a tame algebra, V a directing module, and* $\mathbf{d} = \mathbf{dim}\, V$. *Then*

(i) $\mathrm{mod}_A(\mathbf{d})$ *is a complete intersection of dimension* $a(\mathbf{d})$ *and has at most two irreducible components.*

(ii) *The maximal* $G(\mathbf{d})$*-orbits in* $\mathrm{mod}_A(\mathbf{d})$ *consist of nonsingular modules.*

(iii) $\mathscr{O}(V)$ *is the open sheet of* $\mathrm{mod}_A(\mathbf{d})$.

(iv) *All but finite number of* $G(\mathbf{d})$*-orbits in* $\mathrm{mod}_A(\mathbf{d})$ *have codimension at least two.*

(v) *All* $G(\mathbf{d})$*-orbits in* $\mathrm{mod}_A(\mathbf{d})$ *of codimension one are contained in* $\overline{\mathscr{O}(V)}$.

THEOREM 2. *Let A be a tame algebra, V a directing A-module,* $B = \mathrm{supp}(V)$, *and* $\mathbf{d} = \mathbf{dim}\, V$. *The following conditions are equivalent*:

(i) $\mathrm{mod}_A(\mathbf{d}) \neq \overline{\mathscr{O}(V)}$.

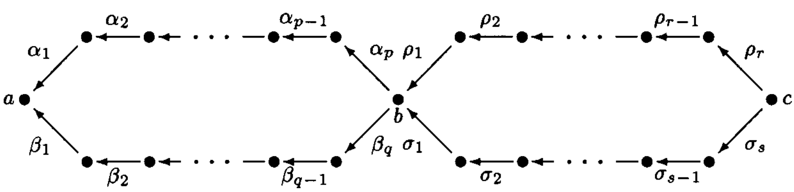
(ii) $\mathrm{mod}_A(\mathbf{d})$ *is not irreducible.*

(iii) $\mathrm{mod}_A(\mathbf{d})$ *is not normal.*

(iv) $\mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$, *where* $\mathbf{h}_1$ *and* $\mathbf{h}_2$ *are connected positive vectors in* $K_0(A)$ *with* $\chi_A(\mathbf{h}_1) = 0$, $\chi_A(\mathbf{h}_2) = 0$, $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_A = 1$, $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle_A = 0$.

(v) *V is a projective-injective B-module and B is one of the 2-parametric tilted algebras* $A(p, q, r, s)$, $p, q, r, s \geq 1$, $F(p, q, r, s)$, $p, r \geq 1$, $q, s \geq 2$, $p + r \geq 3$, $1/(p + r - 1) + 1/q + 1/s > 1$, $D(p, r)$, $p \geq 2$, $r \geq 1$, $E'(p, r)$ *or* $E''(p, r)$, $p \geq 2$, $r \geq 1$, $4 \leq p + r \leq 6$ (*described below*) *or their opposite algebras.*

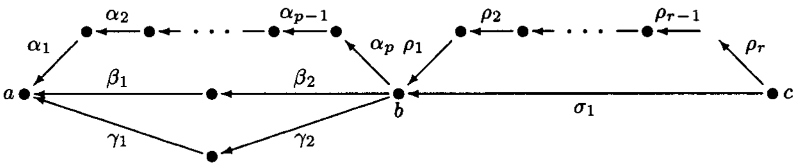
For positive integers $p, q, r, s \geq 1$, $A(p, q, r, s)$ denotes the bound quiver algebra $K\Delta(p, q, r, s)/I(p, q, r, s)$, where $\Delta(p, q, r, s)$ is the quiver



and the ideal $I(p, q, r, s)$ is generated by the relations $\alpha_p \rho_1$, $\beta_q \sigma_1$, $\alpha_1 \cdots \alpha_p \sigma_1 \cdots \sigma_s - \beta_1 \cdots \beta_q \rho_1 \cdots \rho_r$.

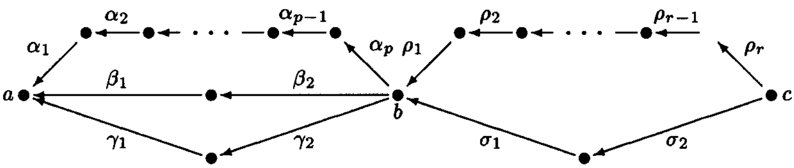
For integers $p, r \geq 1$, $q, s \geq 2$, $p + r \geq 3$, such that $1/(p + r - 1) + 1/q + 1/s > 1$, $F(p, q, r, s)$ denotes the bound quiver algebra $K\Delta(p, q, r, s)/J(p, q, r, s)$, where $\Delta(p, q, r, s)$ is the quiver as above and the ideal $J(p, q, r, s)$ is generated by the relations $\alpha_p \rho_1$, $\beta_q \rho_1 \cdots \rho_r - \beta_q \sigma_1 \cdots \sigma_s$, $\alpha_1 \cdots \alpha_p \sigma_1 - \beta_1 \cdots \beta_q \sigma_1$.

For integers $p \geq 2$, $r \geq 1$, $D(p, r)$ denotes the bound quiver algebra $K\Delta(p, r)/I(p, r)$, where $\Delta(p, r)$ is the quiver

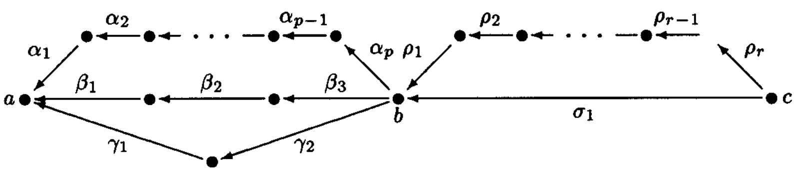


and the ideal $I(p, r)$ is generated by the relations $\alpha_p \rho_1, \gamma_2 \sigma_1, \beta_2 \rho_1 \cdots \rho_r - \beta_2 \sigma_1, \alpha_1 \cdots \alpha_p + \beta_1 \beta_2 + \gamma_1 \gamma_2$.

For integers $p \geq 2$, $r \geq 1$, with $4 \leq p + r \leq 6$, $E'(p, r)$ denotes the bound quiver algebra $K\Delta'(p, r)/I'(p, r)$, where $\Delta'(p, r)$ is the quiver



and the ideal $I'(p, r)$ is generated by the relations $\alpha_p \rho_1, \gamma_2 \sigma_1, \beta_2 \rho_1 \cdots \rho_r - \beta_2 \sigma_1 \sigma_2, \alpha_1 \cdots \alpha_p + \beta_1 \beta_2 + \gamma_1 \gamma_2$. Finally $E''(p, r)$ denotes the bound quiver algebra $K\Delta''(p, r)/I''(p, r)$, where $\Delta''(p, r)$ is the quiver



and the ideal $I''(p,r)$ is generated by the relations $\alpha_p \rho_1, \gamma_2 \sigma_1, \beta_3 \rho_1 \cdots \rho_r - \beta_3 \sigma_1, \alpha_1 \cdots \alpha_p + \beta_1 \beta_2 \beta_3 + \gamma_1 \gamma_2$.

According to [18] the algebras $A(p,q,r,s)$ are all 2-parametric tilted algebras of type $\tilde{\tilde{\mathbb{A}}}_m$ with sincere projective-injective indecomposable modules. In addition we will show in [2] that the algebras $F(p,q,r,s)$, $D(p,r), E'(p,r), E''(p,r)$ form the family of all 2-parametric tilted algebras of types $\tilde{\tilde{\mathbb{D}}}_n$, $\tilde{\tilde{\mathbb{E}}}_6$, $\tilde{\tilde{\mathbb{E}}}_7$, $\tilde{\tilde{\mathbb{E}}}_8$ having sincere projective-injective indecomposable modules V such that $\operatorname{rad} V/\operatorname{soc} V$ is indecomposable.

The following facts are immediate consequences of Theorem 2.

COROLLARY 3. *Let A be a tame algebra without a convex hereditary subcategory of type $\tilde{\mathbb{A}}_m$. Then for any directing A-module V,* $\operatorname{mod}_A(\mathbf{dim}\, V) = \overline{\mathscr{O}(V)}$, *is normal and a complete intersection.*

COROLLARY 4. *Let A be tame algebra and V be a directing A-module which is not projective-injective over* $\operatorname{supp}(V)$. *Then* $\operatorname{mod}_A(\mathbf{dim}\, V) = \overline{\mathscr{O}(V)}$, *is normal and a complete intersection.*

Finally, we note that if the support of an A-module V is not hereditary then $\operatorname{mod}_A(\mathbf{dim}\, V)$ is not smooth (see [5, Proposition 1]).

2. GEOMETRIC PRELIMINARIES

In this section we shall recall and prove some facts applied in our investigations of module varieties over tame tilted algebras. For basic background we refer to [7, 12, 14, 15, 21].

2.1

Let $A = KQ/I$ be a triangular algebra (hence gl.dim $A < \infty$) and $\mathbf{d} \in K_0(A) = \mathbb{Z}^{Q_0}$. Given a module $M \in \operatorname{mod}_A(\mathbf{d})$ we denote by $T_M(\operatorname{mod}_A(\mathbf{d}))$ the tangent space to $\operatorname{mod}_A(\mathbf{d})$ at M and by $T_M(\mathscr{O}(M))$ the tangent space to $\mathscr{O}(M)$ at M. Then there is a canonical monomorphism of K-vector spaces

$$T_M(\operatorname{mod}_A(\mathbf{d}))/T_M(\mathscr{O}(M)) \hookrightarrow \operatorname{Ext}^1_A(M,M)$$

(see [14, (2.7)]). In particular, if $\operatorname{Ext}^1_A(M,M) = 0$ then $\overline{\mathscr{O}(M)}$ is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$ and $\mathscr{O}(M)$ is an open subset of $\operatorname{mod}_A(\mathbf{d})$. The local dimension $\dim_M \operatorname{mod}_A(\mathbf{d})$ is the maximal dimension of the irreducible components of $\operatorname{mod}_A(\mathbf{d})$ containing M. Then the following inequality $\dim_K T_M(\operatorname{mod}_A(\mathbf{d})) \geq \dim_M \operatorname{mod}_A(\mathbf{d})$ holds. Further, $M \in \operatorname{mod}_A(\mathbf{d})$ is said to be a nonsingular point if $\dim_M \operatorname{mod}_A(\mathbf{d}) = \dim_K T_M(\operatorname{mod}_A(\mathbf{d}))$. If M is a nonsingular point of $\operatorname{mod}_A(\mathbf{d})$ then M belongs to exactly one irreducible component of $\operatorname{mod}_A(\mathbf{d})$ [21, (II.2.6)]. Further, the nonsingular points in $\operatorname{mod}_A(\mathbf{d})$ form an open nonempty subset. Clearly, $\mathcal{O}(M)$ is irreducible, M is a nonsingular point of $\mathcal{O}(M)$, and hence $\dim \mathcal{O}(M) = \dim_K T_M(\mathcal{O}(M))$. Moreover, we have $\dim \mathcal{O}(M) = \dim G(\mathbf{d}) - \dim_K \operatorname{End}_A(M)$ (see [15]). It is also known that $M \in \operatorname{mod}_A(\mathbf{d})$ is nonsingular provided $\operatorname{Ext}_A^2(M, M) = 0$. The standard proof of this fact involves schemes and a result by Voigt (see [8, 19, 25]). For triangular algebras of global dimension at most 2 we shall present an elementary proof of this fact below. Finally, $\operatorname{mod}_A(\mathbf{d})$ is said to be a complete intersection provided the vanishing ideal of $\operatorname{mod}_A(\mathbf{d})$ in the coordinate ring $K[\mathbb{A}(\mathbf{d})]$ of the affine space $\mathbb{A}(\mathbf{d}) = \prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ is generated by $\dim \mathbb{A}(\mathbf{d}) - \dim \operatorname{mod}_A(\mathbf{d})$ polynomials. Observe that it is the case when $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$. We note also that the irreducible components of a complete intersection have the same dimension.

2.2

We shall need the following fact.

PROPOSITION. *Let A be a triangular algebra of global dimension at most 2, M an A-module with $\operatorname{Ext}_A^2(M, M) = 0$, and $\mathbf{d} = \mathbf{dim}\, M$. Then M is a nonsingular point of $\operatorname{mod}_A(\mathbf{d})$, $\dim_M \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$, and $\dim \mathcal{O}(M) = a(\mathbf{d}) - \dim_K \operatorname{Ext}_A^1(M, M)$.*

Proof. For any $M \in \operatorname{mod}_A(\mathbf{d})$ we have

$$\begin{aligned} a(\mathbf{d}) &= \dim G(\mathbf{d}) - q_A(\mathbf{d}) = \dim G(\mathbf{d}) - \chi_A(\mathbf{d}) \\ &= \big(\dim G(\mathbf{d}) - \dim_K \operatorname{End}_A(M)\big) \\ &\qquad + \dim_K \operatorname{Ext}_A^1(M, M) - \dim_K \operatorname{Ext}_A^2(M, M) \\ &= \dim \mathcal{O}(M) + \dim_K \operatorname{Ext}_A^1(M, M) - \dim_K \operatorname{Ext}_A^2(M, M) \\ &= \big(\dim_K T_M(\mathcal{O}(M)) + \dim_K \operatorname{Ext}_A^1(M, M)\big) - \dim_K \operatorname{Ext}_A^2(M, M) \\ &\geq \dim_K T_M\big(\operatorname{mod}_A(\mathbf{d})\big) - \dim_K \operatorname{Ext}_A^2(M, M). \end{aligned}$$

Hence, applying Krull's generalized principal ideal theorem, we get the inequalities

$$\begin{aligned} \dim_M \operatorname{mod}_A(\mathbf{d}) \geq a(\mathbf{d}) &\geq \dim_K T_M\big(\operatorname{mod}_A(\mathbf{d})\big) - \dim_K \operatorname{Ext}_A^2(M, M) \\ &\geq \dim_M \operatorname{mod}_A(\mathbf{d}) - \dim_K \operatorname{Ext}_A^2(M, M). \end{aligned}$$

Therefore, if $\operatorname{Ext}_A^2(M, M) = 0$, this leads to the equalities $\dim_M \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d}) = \dim_K T_M(\operatorname{mod}_A(\mathbf{d}))$. Moreover, we have also $\dim \mathcal{O}(M) = a(\mathbf{d}) - \dim \operatorname{Ext}_A^1(M, M)$. This shows our claims.

2.3

We get the following consequences of the above proposition.

COROLLARY. *Let A be a triangular algebra with* gl.dim $A \leq 2$, *and* $\mathbf{d} \in K_0(A)$ *be a positive vector. Assume that for any maximal* $G(\mathbf{d})$*-orbit* $\mathscr{O}(M)$ *in* $\operatorname{mod}_A(\mathbf{d})$ *we have* $\operatorname{Ext}_A^2(M, M) = 0$. *Then* $\operatorname{mod}_A(\mathbf{d})$ *is a complete intersection and* $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$.

Proof. Clearly it is enough to show only that $a(\mathbf{d}) = \dim \operatorname{mod}_A(\mathbf{d})$. Observe that the closure of any $G(\mathbf{d})$-orbit in $\operatorname{mod}_A(\mathbf{d})$ is an irreducible variety. Moreover, any module N from $\operatorname{mod}_A(\mathbf{d})$ belongs to the closure $\overline{\mathscr{O}(M)}$ of a maximal orbit $\mathscr{O}(M)$ of $\operatorname{mod}_A(\mathbf{d})$. Hence, it follows from our assumption that any irreducible component $\mathscr{Z}$ of $\operatorname{mod}_A(\mathbf{d})$ contains a module M with $\operatorname{Ext}_A^2(M, M) = 0$. Applying now (2.2) we conclude that M is a nonsingular point of $\operatorname{mod}_A(\mathbf{d})$, and so $\mathscr{Z}$ is a unique irreducible component of $\operatorname{mod}_A(\mathbf{d})$ containing M. In particular, applying (2.2) again, we have $\dim \mathscr{Z} = \dim_M \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$. Therefore, $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$, and this finishes the proof.

2.4

A module variety $\operatorname{mod}_A(\mathbf{d})$ is said to be normal if the local ring $\mathscr{O}_M$ of any module $M \in \operatorname{mod}_A(\mathbf{d})$ is integrally closed in its total quotient ring. It is well known that if $\operatorname{mod}_A(\mathbf{d})$ is normal then it is nonsingular in codimension one, that is, the set of singular points in $\operatorname{mod}_A(\mathbf{d})$ is of codimension at least two (see [7, Chap. 11]). We shall need the following consequence of Serre's normality criterion.

THEOREM. *Let A be a triangular algebra,* $\mathbf{d} \in K_0(A)$ *a positive vector, and assume that* $\operatorname{mod}_A(\mathbf{d})$ *is a complete intersection. Then* $\operatorname{mod}_A(\mathbf{d})$ *is normal if and only if* $\operatorname{mod}_A(\mathbf{d})$ *is nonsingular in codimension one.*

Proof. See [12, (II.8.2.3)].

2.5

The following theorem shows that the degenerations of finite dimensional modules over tame tilted algebras are given by short exact sequences.

THEOREM. *Let A be a tame tilted algebra,* $\mathbf{d} \in K_0(A)$ *a positive vector, and M, N two modules in* $\operatorname{mod}_A(\mathbf{d})$. *Then* $M \in \overline{\mathscr{O}(N)}$ *if and only if there exist A-modules N_i, U_i, V_i and short exact sequences* $0 \to U_i \to N_i \to V_i \to 0$ *in* mod_A *such that* $N_1 = N$, $N_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, $M = N_{s+1}$ *for some natural number s.*

Proof. It is a direct consequence of [24, Theorem 3; 26, Corollary 3] and the well-known fact (see, for example, [6, (1.1)]) that any short exact sequence $0 \to U \to W \to V \to 0$ of modules gives a degeneration $U \oplus V \in \overline{\mathcal{O}(W)}$.

2.6

The following characterization of maximal orbits in the module varieties of tame tilted algebras will be crucial in our investigations.

PROPOSITION. *Let A be a tame tilted algebra, $\mathbf{d} \in K_0(A)$ a positive vector, and M a module in* $\operatorname{mod}_A(\mathbf{d})$. *Then $\mathcal{O}(M)$ is a maximal $G(\mathbf{d})$-orbit in* $\operatorname{mod}_A(d)$ *if and only if* $\operatorname{Ext}^1_A(M', M'') = 0$ *for any decomposition $M = M' \oplus M''$ of M.*

Proof. Suppose $M = M' \oplus M''$ and there exists a nonsplittable short exact sequence

$$0 \to M' \to E \to M'' \to 0.$$

Then $M = M' \oplus M''$ is a proper degeneration of E, and hence the inclusion $\mathcal{O}(M) \subset \overline{\mathcal{O}(E)} \setminus \mathcal{O}(E)$ holds. Therefore, $\mathcal{O}(M)$ is not maximal. Assume now that $\operatorname{Ext}^1_A(M', M'') = 0$ for any decomposition $M = M' \oplus M''$ of M. Let $\mathcal{O}(M) \subset \overline{\mathcal{O}(N)}$ for some module N in $\operatorname{mod}_A(\mathbf{d})$. Applying Theorem 3.5 we conclude that there are A-modules N_i, U_i, V_i and short exact sequences

$$0 \to U_i \to N_i \to V_i \to 0$$

in mod_A such that $N_1 = N$, $N_{i+1} = U_i \oplus V_i$, $1 \le i \le s$, $M = N_{s+1}$ for some natural number s. Invoking now our assumption we infer that the above exact sequences are splittable, and consequently $M = N_{s+1} \simeq N_s \simeq \cdots \simeq N_1 = N$. This shows that $\mathcal{O}(M)$ is a maximal orbit in $\operatorname{mod}_A(\mathbf{d})$.

3. MODULE CATEGORIES OF TILTED ALGEBRAS

In this section we shall introduce some notation and recall some facts on tame tilted algebras and their module categories, needed in the proofs of our main results. For details we refer to [13, 17, 20].

3.1

For an algebra A, we denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^- the Auslander–Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, respectively. We shall not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \ge 1$, is said to be a stable tube of rank r. A stable tube of rank one is said to be homogeneous. Further, a component Γ of Γ_A is called preprojective (respectively, preinjective) if Γ contains no oriented cycle and each module in Γ belongs to the τ_A-orbit of a projective (respectively, injective) module. An indecomposable A-module lying in a homogeneous tube (respectively, preprojective component, preinjective component) of Γ_A is said to be homogeneous (respectively, preprojective, preinjective).

3.2

Let Δ be a finite connected quiver without oriented cycles, $\overline{\Delta}$ the underlying graph of Δ, and $H = K\Delta$ the path algebra of Δ. By a tilting H-module we mean a direct sum T of $n = |\Delta_0|$ pairwise nonisomorphic indecomposable H-modules such that $\mathrm{Ext}^1_H(T, T) = 0$. Then $B = \mathrm{End}_H(T)$ is called a tilted algebra of type $\overline{\Delta}$. The module T determines a torsion theory $(\mathscr{F}(T), \mathscr{G}(T))$ in mod_H, where $\mathscr{F}(T) = \{X \in \mathrm{mod}_H; \mathrm{Hom}_H(T, X) = 0\}$, $\mathscr{G}(T) = \{X \in \mathrm{mod}_H;\ \mathrm{Ext}^1_H(T, X) = 0\}$, and a splitting torsion theory $(\mathscr{Y}(T), \mathscr{X}(T))$ in mod_B, where $\mathscr{Y}(T) = \{Y \in \mathrm{mod}_B;\ \mathrm{Tor}_1^B(T, Y) = 0\}$, $\mathscr{X}(T) = \{Y \in \mathrm{mod}_B;\ T \otimes_B Y = 0\}$. By the Brenner–Bulter theorem the functor $F = \mathrm{Hom}_H(T, -)$ induces an equivalence of $\mathscr{G}(T)$ and $\mathscr{Y}(T)$, and the functor $F' = \mathrm{Ext}^1_H(T, -)$ induces an equivalence of $\mathscr{F}(T)$ and $\mathscr{X}(T)$. The images $F(I)$ of all indecomposable injective H-modules I via F form a faithful section $\Sigma = \Delta^{\mathrm{op}}$ in one component $\mathscr{C}_T$ of Γ_B, called the connecting component of Γ_B determined by T. This component connects the torsion-free part $\mathscr{Y}(T)$ with the torsion part $\mathscr{X}(T)$. Recall that a connected full subquiver Σ of a component $\mathscr{C}$ is called a section if: (S1) Σ has no oriented cycles; (S2) Each τ_A-orbit in $\mathscr{C}$ intersects Σ exactly once; (S3) Each path in $\mathscr{C}$ with source and target in Σ lies entirely in Σ. Finally, the section Σ is called faithful if the direct sum of modules lying on Σ is a faithful B-module. The following practical criterion for an algebra to be tilted has been established independently by S. Liu [16] and the second-named author [23] (see also [20]).

THEOREM. *An algebra B is tilted if and only if Γ_B admits a component $\mathscr{C}$ with a faithful section Σ such that $\mathrm{Hom}_B(X, \tau_B Y) = 0$ for all modules X and Y from Δ. Moreover, in this case, $\mathscr{C}$ is the connecting component of Γ_B determined by a tilting module T over the path algebra $H = K\Delta$ of $\Delta = \Sigma^{\mathrm{op}}$ and $B = \mathrm{End}_H(T)$.*

3.3

It is known that a tilted algebra B is representation-finite if and only if Γ_B is a finite preprojective and preinjective component. In particular, it is the case for all tilted algebras of Dynkin types. The tilted algebras of

Euclidean type (extended Dynkin type) are tame and we refer to [20, (4.9)] for the structure of their module categories. An important class of tame tilted algebras is formed by the tame concealed algebras, that is, the tilted algebras $\operatorname{End}_H(T)$ of Euclidean type with the tilting H-module being a direct sum of preprojective (equivalently, preinjective) modules. Namely, it was shown in [11] that a tame algebra C whose Auslander–Reiten quiver has a preprojective component is minimal representation-infinite if and only if C is a tame concealed algebra. We refer to [4, 11] for a complete classification of tame concealed algebras by quivers and relations. A general structure of the module category of an arbitrary tame tilted algebra (including those of wild type) has been described by O. Kerner [13]. Generally speaking, the Auslander–Reiten quiver of a tame tilted algebra B consists of a connecting component, a finite number of preprojective components and preinjective components, and a finite number of $\mathbb{P}_1(K)$-families of ray tubes (obtained from stable tubes by ray insertions) and coray tubes (obtained from stable tubes by coray insertions). Moreover, we have the following characterization of tame tilted algebras established in [13, Theorem 6.2].

THEOREM. *Let A be a tilted algebra. The following conditions are equivalent*:

(i) *A is tame.*

(ii) *χ_A is weakly nonnegative.*

(iii) $\dim_K \operatorname{Ext}_A^1(X, X) \leq \dim_K \operatorname{End}_A(X)$ *for any indecomposable A-module X.*

Recall that χ_A is called weakly nonnegative if $\chi_A(\mathbf{x}) \geq 0$ for any nonnegative vector $\mathbf{x} \in K_0(A)$. We also note that any tilted algebra A is of global dimension at most 2 [20, (4.2)], and hence $\chi_A = q_A$.

3.4

Moreover, we have the following fact (see [13, Sect. 4, (6.2); 17, (2.1); 20, (2.4)(8), (49)]).

PROPOSITION. *Let A be a tilted algebra. Then A is tame if and only if χ_A controls the category* mod_A, *that is, satisfies the following properties*:

(i) *For any indecomposable A-module X, $\chi_A(\mathbf{dim}\, X) \in \{0, 1\}$.*

(ii) *For any connected positive vector $\mathbf{d} \in K_0(A)$ with $\chi_A(\mathbf{d}) = 1$ there is a unique* (*up to isomorphism*) *indecomposable A-module X with $\mathbf{dim}\, X = \mathbf{d}$.*

(iii) *For any connected positive vector $\mathbf{d} \in K_0(A)$ with $\chi_A(\mathbf{d}) = 0$, there is an infinite family $(X_\lambda)_{\lambda \in \Lambda}$ of pairwise nonisomorphisc indecomposable A-modules with $\mathbf{dim}\, X_\lambda = \mathbf{d}$ for any $\lambda \in \Lambda$.*

3.5

We have also the following consequence of Theorem 3.3 [13, Sect. 4; 17; 20, (4.9), p. 375] and the fact that the support of any directing module is convex [3, (3.2)].

COROLLARY. *Let A be a tame tilted algebra and M an indecomposable A-module. Then there is a convex subcategory B of A such that M is a B-module and one of the following holds*:

(i) *B is representation-infinite tilted of Euclidean type, and M is a nondirecting module from a tube of Γ_A consisting entirely of B-modules.*

(ii) *B is a tame tilted algebra containing at most two different concealed convex subcategories, and M is a directing module lying in a connecting component of Γ_B.*

3.6

In particular, we get also the following

COROLLARY. *Let A be a tame tilted algebra. Then*

(i) *The support of any stable tube of Γ_A is a convex tame concealed subcategory of A.*

(ii) *If M is an indecomposable A-module with $\chi_A(\mathbf{dim}\, M) = 0$ then the support of M is a tame concealed subcategory C of A and M lies in a stable tube of Γ_C.*

3.7

An indecomposable module M over a tame tilted algebra such that $\chi_A(\mathbf{dim}\, M) = 0$ is said to be isotropic. We shall use the following well-known properties of isotropic modules over tame concealed algebras (see [20, (3.1), (4.3)]).

LEMMA. *Let C be a tame concealed algebra and M an isotropic C-module. Then*

(i) $\mathrm{Hom}_A(X, M) \neq 0$ *for any preprojective C-module X.*

(ii) $\mathrm{Hom}_A(M, Y) \neq 0$ *for any preinjective C-module Y.*

3.8

Let A be a tame tilted algebra and $\mathbf{d} \in K_0(A)$ be a connected positive vector. An important consequence of Kerner's work [13] is the fact that all but a finite number of components in Γ_A are homogeneous tubes. Hence, there exists only a finite number (up to isomorphism) of indecomposable

A-modules X which are nonhomogeneous and $\mathbf{dim}\, X \le \mathbf{d}$. We define $\mathscr{M}(\mathbf{d})$ to be a set of (isomorphism classes) and A-modules M without homogeneous direct summands such that $M \oplus N \in \operatorname{mod}_A(\mathbf{d})$ for some A-module N being a direct sum of homogeneous modules. For each $M \in \mathscr{M}(\mathbf{d})$ consider the set $\mathscr{W}_M(\mathbf{d})$ consisting of all modules $L \oplus N \in \operatorname{mod}_A(\mathbf{d})$ such that $L \simeq M$ and N is a direct sum of homogeneous modules. It was shown in [1, (4.6), (4.12)] that $\mathscr{W}_M(\mathbf{d})$ is irreducible. Clearly, $\mathscr{W}_M(\mathbf{d})$ is $G(\mathbf{d})$-invariant. Therefore, we get a decomposition of $\operatorname{mod}_A(\mathbf{d})$ into the disjoint union of finitely many irreducible $G(\mathbf{d})$-invariant subsets $\mathscr{W}_M(\mathbf{d})$, $M \in \mathscr{M}(\mathbf{d})$. Finally, for an A-module Z, denote by $\mu(Z)$ the number of homogeneous direct summands (including the multiplicities) of Z. Then $\dim \mathscr{W}_M(\mathbf{d})$ is the maximum of $\dim \mathscr{O}(Z) + \mu(Z)$ for all modules $Z \in \mathbf{W}_M(\mathbf{d})$.

4. GEOMETRY OF DIRECTING MODULES

Let A be a tame algebra and $\mathbf{d}$ the dimension-vector of a directing (indecomposable) A-module. We shall describe the maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$, the irreducible components of $\operatorname{mod}_A(\mathbf{d})$, discuss the normality of $\operatorname{mod}_A(\mathbf{d})$, and prove that $\operatorname{mod}_A(\mathbf{d})$ is a complete intersection. In particular, we prove Theorems 1 and 2.

4.1

It is known that the support of any directing A-module is a convex subcategory of A [3, (2.3)], and moreover is tilted [20, p. 376]. Therefore, we may assume that A is a tame tilted algebra and $\mathbf{d} = \mathbf{dim}\, V$ for a (uniquely determined) sincere directing A-module V. Then $\operatorname{pd}_A V \le 1$, $\operatorname{id}_A V \le 1$, and $\operatorname{Ext}^1_A(V, V) = 0$ (see [20, (2.4)]). In particular, $\mathscr{O}(V)$ is a maximal $G(\mathbf{d})$-orbit in $\operatorname{mod}_A(\mathbf{d})$ and $\overline{\mathscr{O}(V)}$ is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$ of dimension $a(\mathbf{d})$, since $\operatorname{Ext}^2_A(V, V) = 0$. We shall abbreviate $\langle -, - \rangle = \langle -, - \rangle_A$ and $\chi = \chi_A$.

4.2

We start with the following lemma.

LEMMA. *Let V_1, V_2 be two indecomposable A-modules such that there exists an exact sequence.*

$$0 \to V_1 \to V \to V_2 \to 0$$

and $\dim_K \operatorname{End}_A(V_1 \oplus V_2) = 2$. *Then* $\chi(\mathbf{dim}\, V_1) = 1$, $\chi(\mathbf{dim}\, V_2) = 1$, *and* $\operatorname{Ext}^2_A(V_1 \oplus V_2, V_1 \oplus V_2) = 0$.

Proof. Let $\mathbf{d}_1 = \mathbf{dim}\, V_1$ and $\mathbf{d}_2 = \mathbf{dim}\, V_2$. The existence of the above short exact sequence and the fact that V is directing imply $\mathrm{Ext}_A^1(V_1, V_2) = 0$ and $\mathrm{Ext}_A^2(V_1, V_2) = 0$. Moreover, $\mathrm{Hom}_A(V_1, V_2) = 0$ because $\dim_K \mathrm{End}_A(V_1 \oplus V_2) = 2$. Hence we get

$$\langle \mathbf{d}_1, \mathbf{d}_2 \rangle = \dim \mathrm{Hom}_A(V_1, V_2) - \dim_K \mathrm{Ext}_A^1(V_1, V_2) + \dim_K \mathrm{Ext}_A^2(V_1, V_2) = 0.$$

Similarly, we have $\mathrm{Ext}_A^1(V_1, V) = 0$, $\mathrm{Ext}_A^2(V_1, V) = 0$, and so

$$\langle \mathbf{d}_1, \mathbf{d} \rangle = \langle \mathbf{dim}\, V_1, \mathbf{dim}\, V \rangle = \dim_K \mathrm{Hom}_A(V_1, V) > 0.$$

On the other hand, $\langle \mathbf{d}_1, \mathbf{d} \rangle = \langle \mathbf{d}_1, \mathbf{d}_1 \rangle + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle = \langle \mathbf{d}_1, \mathbf{d}_1 \rangle$ and V_1 is indecomposable. Thus $\chi(\mathbf{d}_1) = 1$. Similarly, we prove that $\chi(\mathbf{d}_2) = 1$. Then

$$1 = \chi(\mathbf{d}) = \chi(\mathbf{d}_1) + \chi(\mathbf{d}_2) + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle + \langle \mathbf{d}_2, \mathbf{d}_1 \rangle = 2 + \langle \mathbf{d}_2, \mathbf{d}_1 \rangle$$

implies $\langle \mathbf{d}_2, \mathbf{d}_1 \rangle = -1$. Since $\dim_K \mathrm{End}_A(V_1 \oplus V_2) = 2$, for any nonsplittable exact sequence

$$0 \to V_1 \to W \to V_2 \to 0,$$

$\mathrm{End}_A(W) = K$ holds, and hence W is indecomposable. Consequently, $W \simeq V$, because $\mathbf{dim}\, W = \mathbf{dim} + V_1 + \mathbf{dim}\, V_2 = \mathbf{d} = \mathbf{dim} + V$, and V is uniquely determined (up to isomorphism) by $\mathbf{d}$. Moreover, $\dim_K \mathrm{Hom}_A(V_1, V) = \langle \mathbf{d}_1, \mathbf{d} \rangle = \langle \mathbf{d}_1, \mathbf{d}_1 \rangle = 1$. This implies that $\dim_K \mathrm{Ext}_A^1(V_2, V_1) = 1$. Hence

$$\dim_K \mathrm{Ext}_A^2(V_2, V_1) = \langle \mathbf{d}_2, \mathbf{d}_1 \rangle - \dim_K \mathrm{Hom}_A(V_2, V_1) + \dim_K \mathrm{Ext}_A^1(V_2, V_1) = 0.$$

Further, since V_1, V_2 are indecomposable modules over a tilted algebra, we have also $\mathrm{Ext}_A^2(V_1, V_1) = 0$, $\mathrm{Ext}_A^2(V_2, V_2) = 0$. Combining the above facts we get $\mathrm{Ext}_A^2(V_1 \oplus V_2, V_1 \oplus V_2) = 0$.

4.3

Our next aim is to prove the following fact.

PROPOSITION. *Let M be a module in $\mathrm{mod}_A(\mathbf{d})$ such that $\mathcal{O}(M)$ is a maximal orbit of $\mathrm{mod}_A(\mathbf{d})$ but different from $\mathcal{O}(V)$. Then there exist indecomposable direct summands U_1 and U_2 of M such that $\mathrm{Hom}_A(U_1, V) \neq 0$, $\mathrm{Hom}_A(V, U_2) \neq 0$, $\mathrm{Ext}_A^1(U_1, U_1) \neq 0$, and $\mathrm{Ext}_A^1(U_2, U_2) \neq 0$. Moreover, $U_1 \not\simeq U_2$.*

Proof. We have the equalities

$$\begin{aligned}1 &= \langle \mathbf{d}, \mathbf{d} \rangle = \langle \mathbf{dim}\, M, \mathbf{dim}\, V \rangle \\ &= \dim_K \mathrm{Hom}_A(M, V) - \dim_K \mathrm{Ext}^1_A(M, V) + \dim_K \mathrm{Ext}^2_A(M, V) \\ &= \dim_K \mathrm{Hom}_A(M, V) - \dim_K \mathrm{Ext}^1_A(M, V),\end{aligned}$$

since $\mathrm{id}_A V \le 1$. Hence $\mathrm{Hom}_A(M, V) \ne 0$, and so there exists an indecomposable direct summand U_1 of M such that $\mathrm{Hom}_A(U_1, V) \ne 0$. We claim that also $\mathrm{Ext}^1_A(U_1, U_1) \ne 0$. Put $\mathbf{d}_1 = \mathbf{dim}\, U_1$. Observe that $U_1 \not\simeq V$ because $M \not\simeq V$ and U_1 is a submodule of M. Since V is directing and $\mathrm{Hom}_A(U_1, V) \ne 0$, we get $\mathrm{Hom}_A(V, U_1) = 0$. Also $\mathrm{Ext}^2_A(V, U_1) = 0$ because $\mathrm{pd}_A V \le 1$. Therefore, we have

$$\langle \mathbf{d}, \mathbf{d}_1 \rangle = \langle \mathbf{dim}\, V, \mathbf{dim}\, U_1 \rangle = -\dim_K \mathrm{Ext}^1_A(V, U_1) \le 0.$$

On the other hand, we have

$$\begin{aligned}\langle \mathbf{d}, \mathbf{d}_1 \rangle &= \langle \mathbf{dim}\, M, \mathbf{dim}\, U_1 \rangle \\ &= \dim_K \mathrm{Hom}_A(M, U_1) - \dim_K \mathrm{Ext}^1_A(M, U_1) + \dim_K \mathrm{Ext}^2_A(M, U_1).\end{aligned}$$

Clearly, $\mathrm{Hom}_A(M, U_1) \ne 0$ because U_1 is a direct summand of M. Hence, $\langle \mathbf{d}, \mathbf{d}_1 \rangle \le 0$ implies $\mathrm{Ext}^1_A(M, U_1) \ne 0$. Now, if $M = U_1 \oplus N$, then we get $\mathrm{Ext}^1_A(N, U_1) = 0$ because $\mathcal{O}(M)$ is a maximal $G(\mathbf{d})$-orbit in $\mathrm{mod}_A(\mathbf{d})$ (see (2.6)). Consequently $\mathrm{Ext}^1_A(U_1, U_1) \ne 0$. Similarly, we prove that there exists an indecomposable direct summand U_2 of M such that $\mathrm{Hom}_A(V, U_2) \ne 0$ and $\mathrm{Ext}^1_A(U_2, U_2) \ne 0$. Since $V \not\simeq U_1$, $V \not\simeq U_2$, and V is directing, we conclude also that $U_1 \not\simeq U_2$.

4.4

It is known that if V is preprojective (respectively, preinjective) and U is an indecomposable A-module such that $\mathrm{Hom}_A(U, V) \ne 0$ (respectively, $\mathrm{Hom}_A(V, U) \ne 0$) then U is also preprojective (respectively, preinjective), and hence $\mathrm{Ext}^1_A(U, U) = 0$. We then get the following consequence of the above proposition and (2.1)–(2.3).

PROPOSITION. *Assume that V is preprojective (respectively, preinjective). Then* $\mathrm{mod}_A(\mathbf{d}) = \overline{\mathcal{O}(V)}$. *In particular,* $\mathrm{mod}_A(\mathbf{d})$ *is irreducible, a complete intersection, and has dimension $a(\mathbf{d})$.*

4.5

We shall prove now the following

PROPOSITION. *Assume that V is preprojective* (*respectively, preinjective*). *Then the variety* $\mathrm{mod}_A(\mathbf{d})$ *is normal.*

Proof. Assume first that A is representation-finite. Then $\mathrm{mod}_A(\mathbf{d}) = \mathscr{O}(V)$ has only a finite number of $G(\mathbf{d})$-orbits. Moreover, by (2.5), for any $G(\mathbf{d})$-orbit $\mathscr{O}(M)$ in $\mathrm{mod}_A(\mathbf{d})$ of codimension one there exists a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with $M \simeq V_1 \oplus V_2$, $\dim_K \mathrm{End}_A(M) = 2$, and V_1, V_2 indecomposable. Applying (4.2) we get $\mathrm{Ext}_A^2(M, M) = 0$, and hence $\mathscr{O}(M)$ consists of nonsingular points. This shows that $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one, and so is normal, by (2.4). Therefore, we may assume that A is representation-infinite and, by symmetry, that V is a (sincere) preprojective module. Then Γ_A is of the form

$$\Gamma_A = \mathscr{P} \vee \mathscr{T} \vee \mathscr{Q},$$

where $\mathscr{P}$ is a preprojective component containing V, $\mathscr{Q}$ is a preinjective component, and $\mathscr{T}$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal coray tubes separating $\mathscr{P}$ from $\mathscr{Q}$. Moreover, all indecomposable projective A-modules are predecessors of V in $\mathscr{P}$, and so all indecomposable A-modules of injective dimension 2 are predecessors of V in $\mathscr{P}$. On the other hand, any indecomposable A-module of projective dimension 2 lies either in a tube of $\mathscr{T}$, containing an injective module, or in $\mathscr{Q}$. Since $\mathrm{mod}_A(\mathbf{d}) = \overline{\mathscr{O}(V)}$ is a complete intersection, we have to show that $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one. We use notation introduced in (3.8). Observe first that $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d}) = \dim G(\mathbf{d}) - \chi(\mathbf{d}) = \dim G(\mathbf{d}) - 1$. We shall prove that if $M \in \mathscr{M}(\mathbf{d})$ and $\mathscr{W}_M(\mathbf{d})$ contains a singular module, then $\dim \mathscr{W}_M(\mathbf{d}) \leq \dim G(\mathbf{d}) - 3$. Clearly, this will imply that $\mathrm{mod}_A(\mathbf{d})$ is normal. Assume $M \in \mathscr{M}(\mathbf{d})$ and $L \oplus N \in \mathscr{W}_M(\mathbf{d})$ with $L \simeq M$ is a singular point of $\mathrm{mod}_A(\mathbf{d})$. Then $\mathrm{Ext}_A^2(L, L) = \mathrm{Ext}_A^2(L \oplus N, L \oplus N) \neq 0$, because $\mathrm{pd}_A N = \mathrm{id}_A N = 1$. Suppose now that $L \oplus N = Z_1 \oplus Z_2 \oplus W$ for some indecomposable direct summands Z_1 and Z_2 with $\mathrm{Hom}_A(Z_1, Z_2) \neq 0$ or $\dim_K \mathrm{End}_A(L) \geq 3$. Then we have

$$\begin{aligned} \dim \mathscr{O}(L \oplus N) &= \dim G(\mathbf{d}) - \dim_K \mathrm{End}_A(L \oplus N) \\ &\leq \dim G(\mathbf{d}) - 3 - \mu(N) \leq \dim \mathrm{mod}_A(\mathbf{d}) - 2 - \mu(N), \end{aligned}$$

and hence $\dim \mathscr{W}_M(\mathbf{d}) \leq \dim \mathrm{mod}_A(\mathbf{d}) - 2$. Therefore, we may assume that $L \oplus N$ is a direct sum of pairwise orthogonal indecomposable modules, and $L = L_1 \oplus L_2$ with $\dim_K \mathrm{End}_A(L) = 2$. Suppose $N = 0$. Then $\mathscr{W}_M(\mathbf{d}) = \mathscr{O}(M)$ and $\dim_K \mathrm{End}_A(M) = 2$. Hence M is a minimal degeneration of V and there is an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with $V_1 \oplus V_2 \simeq$

$L_1 \oplus L_2 \simeq M$ (see (2.5)), and so $\mathrm{Ext}_A^2(L_2, L_1) = 0$ by (4.2), a contradiction. Thus $N \neq 0$. Since $\mathscr{T}$ separates $\mathscr{P}$ from $\mathscr{Q}$, invoking (3.7), we conclude that $\mathrm{Hom}_A(N, X) \neq 0$ for any indecomposable A-module X lying in $\mathscr{Q}$. But then $\mathrm{pd}_A L_2 = 2$ implies that L_2 lies in a coray tube of $\mathscr{T}$. Further, $\mathrm{id}_A L_1 = 2$, and so L_1 lies in $\mathscr{P}$. Take now an indecomposable A-module H lying on the mouth of a homogeneous tube of $\mathscr{T}$ which does not contain direct summands of N, and put $\mathbf{h} = \mathbf{dim}\, H$. Since L_1 and N are orthogonal, $\langle \mathbf{dim}\, L_1, \mathbf{dim}\, N \rangle = \dim_K \mathrm{Hom}_A(L_1, N) = 0$, and so $\langle \mathbf{dim}\, L_1, \mathbf{h} \rangle = 0$. On the other hand, $\langle \mathbf{dim}\, L_1, \mathbf{h} \rangle = \dim_K \mathrm{Hom}_A(L_1, H)$ and consequently $\mathrm{Ext}_A^1(H, L_1) \simeq D\overline{\mathrm{Hom}}_A(L_1, \tau_A H) = D\overline{\mathrm{Hom}}_A(L_1, H) = 0$. Then

$$\begin{aligned}\langle \mathbf{h}, \mathbf{d} \rangle &= \langle \mathbf{dim}\, H, \mathbf{dim}(L_1 \oplus L_2 \oplus N) \rangle \\ &= \dim_K \mathrm{Hom}_A(H, L_1 \oplus L_2 \oplus N) - \dim_K \mathrm{Ext}_A^1(H, L_1 \oplus L_2 \oplus N) \\ &\quad + \dim_K \mathrm{Ext}_A^2(H, L_1 \oplus L_2 \oplus N) = 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle \mathbf{h}, \mathbf{d} \rangle &= \langle \mathbf{dim}\, H, \mathbf{dim}\, V \rangle \\ &= \dim_K \mathrm{Hom}_A(H, V) - \dim_K \mathrm{Ext}_A^1(H, V) + \dim_K \mathrm{Ext}_A^1(H, V) \\ &= -\dim_K D\overline{\mathrm{Hom}}_A(V, \tau_A H) = -\dim_K D\,\mathrm{Hom}_A(V, H) < 0\end{aligned}$$

because $\tau_A H = H$, $\mathrm{pd}_A H = 1$, V is a sincere module lying in $\mathscr{P}$, $\mathscr{Q}$ contains injective modules, and $\mathscr{T}$ separates $\mathscr{P}$ from $\mathscr{Q}$. The obtained contradiction completes the proof.

4.6

In our further considerations we assume that V is an internal directing A-module, that is, is neither preprojective nor preinjective. Then, since A is a tame tilted algebra and V is a sincere internal directing A-module, we know (see [17, (2.4)]) that there exist convex subcategories C_1, B_1, C_2, B_2 of A such that

(1) C_1 is tame concealed and B_1 is tilted of Euclidean type obtained from C_1 by a tubular extension.

(2) C_2 is tame concealed and B_2 is tilted of Euclidean type obtained from C_2 by a tubular coextension.

(3) The Auslander–Reiten quiver Γ_A of A has the form

$$\Gamma_A = \mathscr{P}_1 \vee \mathscr{T}_1 \vee \mathscr{C} \vee \mathscr{T}_2 \vee \mathscr{Q}_2,$$

where $\mathscr{P}_1$ is a preprojective component consisting of C_1-modules, $\mathscr{T}_1$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal ray tubes, obtained from the unique $\mathbb{P}_1(K)$-family $\mathscr{T}_1'$ of stable tubes in Γ_{C_1} by ray insertions, $\mathscr{Q}_2$ is a preinjective component consisting of C_2-modules, $\mathscr{T}_2$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal coray tubes, obtained from the unique $\mathbb{P}_1(K)$-family $\mathscr{T}_2'$ of stable tubes in Γ_{C_2} by coray insertions, and $\mathscr{C}$ is a directing connecting component.

(4) The connecting component $\mathscr{C}$ has a decomposition

$$\mathscr{C} = \mathscr{D}_1 \vee \mathscr{E} \vee \mathscr{D}_2,$$

where $\mathscr{D}_1$ is a left stable full translation subquiver of $\mathscr{C}$ consisting of preinjective B_1-modules and closed under predecessors, $\mathscr{D}_2$ is a right stable full translation subquiver of $\mathscr{C}$ consisting of preprojective B_2-modules and closed under successors, and $\mathscr{E}$ is a finite full translation subquiver containing all indecomposable sincere A-modules (hence the module V).

(5) For any indecomposable A-module X in $\mathscr{C}$, its restriction to C_1 (respectively, to C_2) is either zero or a direct sum of indecomposable preinjective C_1-modules (respectively, is either zero or direct sum of indecomposable preprojective C_2-modules).

The ordering of the families $\mathscr{P}_1, \mathscr{T}_1, \mathscr{C}, \mathscr{T}_2, \mathscr{Q}_2$ in the decomposition (3), from left to right, indicates that there are nonzero maps only from these families to itself or to the families to its right. We set $\mathscr{P} = \mathscr{P}_1 \vee \mathscr{T}_1$ and $\mathscr{Q} = \mathscr{T}_2 \vee \mathscr{Q}_2$. Further, we denote by $\mathbf{h}_1$ the dimension-vector of a module H_1 lying on the mouth of a homogeneous tube of $\mathscr{T}_2$ and by $\mathbf{h}_2$ the dimension-vector of a module H_2 lying on the mouth of a homogeneous tube of $\mathscr{T}_2$. Then $\mathbf{h}_1$ generates the radical of χ_{C_1} (and χ_{B_1}) and $\mathbf{h}_2$ generates the radical of χ_{C_2} (and χ_{B_2}) (see [20, (4.9)]).

4.7

The internal directing module V is said to be exceptional if its dimension-vector $\mathbf{d}$ has a decomposition $\mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$ wit h $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$. If V is not exceptional, V will be called an ordinary internal directing module. Our next aim is to prove that if V is an ordinary directing internal module then $\operatorname{mod}_A(\mathbf{d})$ is irreducible, normal, and a complete intersection.

4.8

We start with the following

PROPOSITION. *Assume V is an ordinary internal directing module. Then* $\operatorname{mod}_A(\mathbf{d}) = \overline{\mathscr{O}(V)}$.

Proof. Let M be a module in $\operatorname{mod}_A(\mathbf{d})$ and assume that $\mathscr{O}(M)$ is maximal but different from $\mathscr{O}(V)$. Consider a decomposition $M = P \oplus W \oplus Q$ with $P \in \operatorname{add} \mathscr{P}$, $W \in \operatorname{add} \mathscr{C}$, $Q \in \operatorname{add} \mathscr{Q}$, and put $\mathbf{p} = \operatorname{\mathbf{dim}} P$, $\mathbf{w} = \operatorname{\mathbf{dim}} W$, $\mathbf{q} = \operatorname{\mathbf{dim}} Q$. Applying (4.3) we get $\operatorname{Hom}_A(P, V) \neq 0$ and $\operatorname{Hom}_A(V, Q) \neq 0$. Since $V \in \mathscr{C}$, we have also $\operatorname{Hom}_A(V, P) = 0$ and $\operatorname{Hom}_A(Q, V) = 0$. Hence, invoking $\operatorname{pd}_A V \leq 1$ and $\operatorname{id}_A V \leq 1$, we obtain

$$\langle \mathbf{d}, \mathbf{p} \rangle = \langle \operatorname{\mathbf{dim}} V, \operatorname{\mathbf{dim}} P \rangle = -\dim_K \operatorname{Ext}_A^1(V, P) \leq 0.$$

On the other hand, we have also

$$\begin{aligned}\langle \mathbf{d}, \mathbf{p} \rangle &= \langle \operatorname{\mathbf{dim}} M, \operatorname{\mathbf{dim}} P \rangle \\ &= \dim_K \operatorname{Hom}_A(M, P) - \dim_K \operatorname{Ext}_A^1(M, P) + \dim_K \operatorname{Ext}_A^2(M, P).\end{aligned}$$

Note that $\operatorname{Hom}_A(M, P) = \operatorname{End}_A(P)$. Further, since $\mathscr{O}(M)$ is maximal and $M = P \oplus W \oplus Q$, we get $\operatorname{Ext}_A^1(M, P) = \bigoplus_{i=1}^r \operatorname{Ext}_A^1(P_i, P_i)$ for a decomposition $P = P_1 \oplus \cdots \oplus P_r$ into a direct sum of indecomposable A-modules. Therefore,

$$\begin{aligned}\langle \mathbf{d}, \mathbf{p} \rangle &= \sum_{i=1}^r \chi(\operatorname{\mathbf{dim}} P_i) + \sum_{i \neq j} \dim_K \operatorname{Hom}_A(P_i, P_j) + \dim_K \operatorname{Ext}_A^2(M, P) \\ &\geq 0.\end{aligned}$$

Combining the above inequalities we infer that $\langle \mathbf{d}, \mathbf{p} \rangle = 0$, and hence also $\langle \operatorname{\mathbf{dim}} P_i, \operatorname{\mathbf{dim}} P_j \rangle = 0$ for all $1 \leq i, j \leq r$. Similarly, if $Q = Q_1 \oplus \cdots \oplus Q_s$ is a decomposition of Q into a direct sum of indecomposable A-modules, we conclude that $\langle \mathbf{q}, \mathbf{d} \rangle = 0$ and $\langle \operatorname{\mathbf{dim}} Q_i, \operatorname{\mathbf{dim}} Q_j \rangle = 0$ for all $1 \leq i, j \leq s$. Since $P \in \operatorname{add} \mathscr{P}$ and $V \in \mathscr{C}$ we have $\operatorname{Ext}_A^1(P, V) = D\,\overline{\operatorname{Hom}}_A(V, \tau_A P) = 0$, and consequently

$$\langle \mathbf{p}, \mathbf{d} \rangle = \langle \operatorname{\mathbf{dim}} P, \operatorname{\mathbf{dim}} V \rangle = \dim_K \operatorname{Hom}_A(P, V) > 0.$$

This implies the inequality

$$\langle \mathbf{w}, \mathbf{d} \rangle = \langle \mathbf{d}, \mathbf{d} \rangle - \langle \mathbf{p}, \mathbf{d} \rangle - \langle \mathbf{q}, \mathbf{d} \rangle \leq 0,$$

because $\langle \mathbf{d}, \mathbf{d} \rangle = 1$ and $\langle \mathbf{q}, \mathbf{d} \rangle = 0$. Next observe that the maximality of $\mathscr{O}(M)$ implies $\operatorname{Ext}_A^1(W, P) = 0$, $\operatorname{Ext}_A^1(W, Q) = 0$, and hence $\langle \mathbf{w}, \mathbf{p} \rangle = \langle \operatorname{\mathbf{dim}} W, \operatorname{\mathbf{dim}} P \rangle = \dim_K \operatorname{Ext}_A^2(W, P) \geq 0$ and $\langle \mathbf{w}, \mathbf{q} \rangle = \langle \operatorname{\mathbf{dim}} W, \operatorname{\mathbf{dim}} Q \rangle = \dim_K \operatorname{Hom}_A(W, Q) \geq 0$. Hence we get

$$\langle \mathbf{w}, \mathbf{d} \rangle = \langle \mathbf{w}, \mathbf{p} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{q} \rangle \geq 0,$$

and consequently $\langle \mathbf{w}, \mathbf{d} \rangle = 0$. In particular, $\langle \mathbf{w}, \mathbf{w} \rangle = 0$. We claim that $\mathbf{w} = 0$. Indeed, if $W \neq 0$ and $W = W_1 \oplus \cdots \oplus W_t$ is a decomposition of W

into a direct sum of indecomposable A-modules, then $W_1, \dots, W_t$ are directing modules from $\mathscr{C}$, and hence $\chi(\mathbf{dim}\, W_i) = 1$ for any $1 \le i \le t$. Moreover, since W is a direct summand of M and $\mathscr{O}(M)$ is maximal, we have $\mathrm{Ext}^1_A(W_i, W_j) = 0$ for all $i \ne j$ from $\{1, \dots, t\}$. Then we get

$$\begin{aligned}\langle \mathbf{w}, \mathbf{w} \rangle &= \sum_{i=1}^{t} \chi(\mathbf{dim}\, W_i) + \sum_{i \ne j} \dim_K \mathrm{Hom}_A(W_i, W_j) + \dim_K \mathrm{Ext}^2_A(W, W) \\ &> 0,\end{aligned}$$

a contradiction with $\langle \mathbf{w}, \mathbf{w} \rangle = 0$. Therefore, $W = 0$ and $M = P \oplus Q$. From the above considerations we have also

$$\langle \mathbf{p}, \mathbf{d} \rangle = \langle \mathbf{d}, \mathbf{d} \rangle - \langle \mathbf{w}, \mathbf{d} \rangle - \langle \mathbf{q}, \mathbf{d} \rangle = \langle \mathbf{d}, \mathbf{d} \rangle = 1,$$

and hence $\dim_K \mathrm{Hom}_A(P, V) = \langle \mathbf{p}, \mathbf{d} \rangle = 1$. On the other hand, for each $1 \le i \le r$, P_i is an indecomposable A-module from $\mathscr{P}$ with $\chi(\mathbf{dim}\, P_i) = 0$, and so is an isotropic C_1-module. But then $\mathrm{Hom}_A(P_i, V) \ne 0$ because $V|_{C_1}$ is a direct sum of indecomposable preinjective C_1-modules. Therefore, $\dim_K \mathrm{Hom}_A(P, V) = 1$ implies $P = P_1$. Similarly, we prove that $Q = Q_1$. In particular, $p = \alpha \mathbf{h}_1$ and $q = \beta \mathbf{h}_2$ for some positive integers α, β. Finally, observe that

$$\alpha\beta \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = \langle \mathbf{p}, \mathbf{q} \rangle = \langle \mathbf{p}, \mathbf{d} \rangle - \langle \mathbf{p}, \mathbf{p} \rangle = 1 - 0 = 1,$$

and so $\alpha = 1 = \beta$ and $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$. Thus $\mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$ with $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$, and this contradicts our assumption that V is an ordinary internal directing module. Therefore, we proved that $\mathscr{O}(V)$ is a unique maximal $G(\mathbf{d})$-orbit in $\mathrm{mod}_A(\mathbf{d})$, and hence $\mathrm{mod}_A(\mathbf{d}) = \overline{\mathscr{O}(V)}$.

4.9

As a direct consequence of the above proposition and (2.3) we get the following

COROLLARY. *Assume V is an ordinary internal directing A-module. Then* $\mathrm{mod}_A(\mathbf{d})$ *is a complete intersection, irreducible, and* $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$.

4.10

We complete the above considerations by the following

PROPOSITION. *Assume V is an ordinary internal directing module. Then* $\mathrm{mod}_A(\mathbf{d})$ *is normal.*

Proof. Since $\mathrm{mod}_A(\mathbf{d})$ is a complete intersection we have to prove that $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one. Since $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d}) =$

$\dim G(\mathbf{d}) - \chi(\mathbf{d}) = \dim G(\mathbf{d}) - 1$, it is enough to show that if $M \in \mathscr{M}(\mathbf{d})$ (in the notation of (3.8)) and $\mathscr{W}_M(\mathbf{d})$ contains a singular module, then $\dim \mathscr{W}_M(\mathbf{d}) \leq \dim G(\mathbf{d}) - 3$. Assume that $M \in \mathscr{M}(\mathbf{d})$ and $R = L \oplus N$, for $L \simeq M$ and a direct sum N of modules from homogeneous tubes is a singular point of $\operatorname{mod}_A(\mathbf{d})$. In contrast to our previous considerations, we cannot assume here that $\operatorname{Ext}_A^2(L, L) \neq 0$, because $\operatorname{pd}_A N \leq 1$ and $\operatorname{id}_A N \leq 1$ usually does not hold. Hence, we must consider also the case $M = 0$. On the other hand, if $\dim \operatorname{End}_A(M) \geq 3$ then $\dim \mathscr{W}_M(\mathbf{d}) \leq \dim G(\mathbf{d}) - 3 = \dim \operatorname{mod}_A(\mathbf{d}) - 2$. Hence we may assume that $\dim_K \operatorname{End}_A(M) \leq 2$. Assume that $N = 0$. If $\dim_K \operatorname{End}_A(M) = 1$ then M is an indecomposable A-module such that $\mathbf{dim}\, M = \mathbf{d} = \mathbf{dim}\, V$, and so $M \simeq V$ is nonsingular. If $\dim_K \operatorname{End}_A(M) = 2$ then M is a minimal degeneration of V, because $\operatorname{mod}_A(\mathbf{d}) = \overline{\mathscr{O}(V)}$ and $\dim_K \operatorname{End}_A(V) = 1$. Then there exists an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with $V_1 \oplus V_2 = M$, V_1, V_2 indecomposable, and again M is nonsingular, by (4.2). Therefore, we may assume $N \neq 0$. Let $N = X_1 \oplus \cdots . \oplus X_r \oplus Y_1 \oplus \cdots \oplus Y_s$, for $r, s \geq 0$, be a decomposition of N such that $X_1, \ldots, X_r$ are modules from homogeneous tubes of $\mathscr{T}_1$ and $Y_1, \ldots, Y_s$ are modules from homogeneous tubes of $\mathscr{T}_2$. By our assumption we have $r + s \geq 1$. We shall use also the following observation. Let U be an indecomposable isotropic direct summand of R lying in $\mathscr{T}_1$ (so $\mathbf{dim}\, U = m\mathbf{h}_1$ for some $m \geq 1$). Then

$$\dim_K \operatorname{Hom}_A(U, V) = \langle \mathbf{dim}\, U, \mathbf{dim}\, V \rangle = \langle m\mathbf{h}_1, \mathbf{d} \rangle > 0$$

because V is sincere and $\mathscr{T}_1$ separates $\mathscr{P}$ from $\mathscr{C} \vee \mathscr{T}_2 \vee \mathscr{Q}_2$. On the other hand, we have

$$\begin{aligned} \langle \mathbf{dim}\, U, \mathbf{d} \rangle &= \langle \mathbf{dim}\, U, \mathbf{dim}\, R \rangle \\ &= \dim_K \operatorname{Hom}_A(U, R) - \dim_K \operatorname{Ext}_A^1(U, R) + \dim_K \operatorname{Ext}_A^2(U, R) \\ &= \dim_K \operatorname{Hom}_A(U, R) - \dim_K \operatorname{Ext}_A^1(U, R) \end{aligned}$$

because $\operatorname{pd}_A U \leq 1$. Hence, there exists an indecomposable direct summand U' of R such that $\langle \mathbf{dim}\, U, \mathbf{dim}\, U' \rangle$ is positive, and so U' is not an isotropic module from $\mathscr{T}_1$. Dually, for any indecomposable isotropic direct summand Z of R lying in $\mathscr{T}_2$ there exists an indecomposable direct summand Z' of R which is not an isotropic module lying in $\mathscr{T}_2$.

Assume now $M = 0$. It follows from the above remarks that we have $\operatorname{Hom}_A(X_i, Y_j) \neq 0$ for some i, j. In particular, $r \geq 1$, $s \geq 1$, and $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle > 0$. Let $\mathbf{dim}\, X_i = p_i \mathbf{h}_1$, for $1 \leq i \leq r$, and $\mathbf{dim}\, Y_j = q_j \mathbf{h}_2$, for $1 \leq j \leq S$. Then we get

$$\dim_K \operatorname{Hom}_A\big(X_i, Y_j\big) = \langle \mathbf{dim}\, X_i, \mathbf{dim}\, Y_j \rangle = p_i q_j \langle \mathbf{h}_1, \mathbf{h}_2 \rangle \geq p_i q_j.$$

Therefore, we obtain the inequality

$$\dim_K \operatorname{End}_A(N) \geq \sum_{i=1}^{r} p_i + \sum_{i=1}^{r}\sum_{j=1}^{s} p_i q_j + \sum_{j=1}^{s} q_j.$$

Observe now that if $\dim_K \operatorname{End}_A(N) \geq r+s+3$, then $\dim \mathscr{O}(N) \leq \dim G(\mathbf{d}) - 2 - (r+s)$, and so $\dim \mathscr{W}_M(\mathbf{d}) = \dim \mathscr{W}_0(\mathbf{d}) \leq \dim \operatorname{mod}_A(\mathbf{d}) - 2$, as required. Thus it remains to consider (up to symmetry) only the following two cases:

(α) $r = s = 1$, $p_1 = q_1 = 1$

(β) $r = 1$, $s = 2$, $p_1 = q_1 = 1$.

Assume (α). Then $\mathscr{W} = \mathscr{W}_0(\mathbf{d}) = \mathscr{W}_M(\mathbf{d})$ is the (irreducible) subset of all modules $R = X \oplus Y$ in $\operatorname{mod}_A(\mathbf{d})$ such that X and Y are indecomposable A-modules with $\mathbf{dim}\, X = \mathbf{h}_1$ and $\mathbf{dim}\, Y = \mathbf{h}_2$. It follows from our assumption that $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \geq 2$. If $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \geq 3$ then $\dim_K \operatorname{End}_A(R) \geq 5$ and hence $\dim \mathscr{W} \leq (\dim G(\mathbf{d}) - 5) + 2 = \dim G(\mathbf{d}) - 3 = \operatorname{mod}_A(\mathbf{d}) - 2$. Hence we may assume $\dim_K \operatorname{End}_A(R) = 4$, and equivalently $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = \dim \operatorname{Hom}_A(X, Y) = 2$. Observe now that if $\mathscr{O}(Z)$ kis an orbit in $\operatorname{mod}_A(\mathbf{d})$ of codimension at most 2, that is, $\dim_K \operatorname{End}_A(Z) \leq 3$, then Z does not contain an indecomposable direct summand of dimension-vector $\mathbf{h}_1$ or $\mathbf{h}_2$. Thus $\operatorname{mod}_A(\mathbf{d})$ contains only a finite number of orbits of codimension 1 and 2, and let Ω be the union of their closures. Then it follows from (4.2) that the set of all singular points in the intersection $\Omega \cap \mathscr{W}$ is of codimension at least 2. The modules $R = X \oplus Y$ from $\mathscr{W} \setminus \Omega$ are minimal degenerations of V, and so we have short exact sequences $0 \to X \to V \to Y \to 0$. Furthermore, if $R = Y \oplus Y \in \mathscr{W}$ and $\dim_K \operatorname{End}_A(R) = 4$, then

$$\begin{aligned} 1 = \langle \mathbf{d}, \mathbf{d} \rangle &= \dim_K \operatorname{Hom}_A(X, Y) - \dim_K \operatorname{Ext}_A^1(Y, X) + \dim_K \operatorname{Ext}_A^2(Y, X) \\ &= 2 - \dim_K \operatorname{Ext}_A^1(Y, X) + \dim_K \operatorname{Ext}_A^2(Y, X). \end{aligned}$$

Hence, $\dim_K \operatorname{Ext}_A^2(Y, X) \neq 0$ implies $\dim_K \operatorname{Ext}_A^1(Y, X) \geq 2$. Observe also that $\dim_K \operatorname{Hom}_A(X, V) = \langle \mathbf{h}_1, \mathbf{d} \rangle = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 2$. Therefore, if $R = X \oplus Y \in \mathscr{W} \setminus \Omega$ and $\operatorname{Ext}_A^2(R, R) \neq 0$ then $X \oplus Y$ is (up to isomorphism) a unique module in $\mathscr{W} \setminus \Omega$ of the form $X \oplus Y'$, with Y' indecomposable of dimension-vector $\mathbf{h}_2$ such that $\operatorname{Ext}_A^2(X \oplus Y', X \oplus Y') \neq 0$. This shows that the set of all singular points in $\mathscr{W} \setminus \Omega$ is of codimension at least 2, and consequently the set of all singular points of $\mathscr{W}$ is of codimension at least 2.

Assume now (β). Then we get the equalities

$$\langle \mathbf{d}, \mathbf{d} \rangle = \langle \mathbf{h}_1 + 2\mathbf{h}_2, \mathbf{h}_1 + 2\mathbf{h}_2 \rangle = 2\big(\langle \mathbf{h}_1, \mathbf{h}_2 \rangle + \langle \mathbf{h}_2, \mathbf{h}_1 \rangle\big)$$

because $\langle \mathbf{h}_1, \mathbf{h}_1 \rangle = 0 = \langle \mathbf{h}_2, \mathbf{h}_2 \rangle$, which contradicts $1 = \langle \mathbf{d}, \mathbf{d} \rangle$.

Assume $M \neq 0$. If M is a direct sum of two indecomposable modules then $\dim_K \mathrm{End}_A(R) = \dim_K \mathrm{End}_A(M \oplus N) \geq r + s + 3$, because $\dim_K \mathrm{End}_A(M) \geq 2$, and $r + s \geq 1$ implies $\mathrm{Hom}_A(M \oplus X_1 \oplus \cdots \oplus X_r, Y_1 \oplus \cdots \oplus Y_s) \neq 0$ (if $s \geq 1$) or $\mathrm{Hom}_A(X_1 \oplus \cdots \oplus X_r, M \oplus Y_1 \oplus \cdots \oplus Y_s) \neq 0$ (if $r \geq 1$). In such a case, $\dim \mathscr{W}_M(\mathbf{d}) \leq \dim \mathrm{mod}_A(\mathbf{d}) - 2$. Finally, assume that M is an indecomposable module and put $\mathbf{e} = \mathbf{dim}\, M$. Observe that if $\sum_{i=1}^r p_i \geq 2$ or $\sum_{j=1}^s q_j \geq 2$ then again $\dim_K \mathrm{End}_A(R) \geq r + s + 3$, and our claim $\dim \mathscr{W}_M(\mathbf{d}) \leq \dim \mathrm{mod}_A(\mathbf{d}) - 2$ follows. Let $r = s = 1$ and $p_1 = q_1 = 1$. By symmetry, we may assume that M does not lie in $\mathscr{T}_2$. In addition we have $\mathrm{Hom}_A(X_1 \oplus M, Y_1) \neq 0$ and $\mathrm{Hom}_A(X_1, M \oplus Y_1) \neq 0$, and so $\dim_K \mathrm{End}_A(R) \geq 4$. If $\dim_K \mathrm{End}_A(R) \geq 5 = r + s + 3$ then the required claim also follows. Thus assume $\dim_K \mathrm{End}_A(R) = 4$. Then we have $\mathrm{End}_A(M) = K$, $\mathrm{End}_A(X_1) = K$, $\mathrm{End}_A(Y_1) = K$, $\mathrm{Hom}_A(Y_1, M \oplus X_1) = 0$, $\mathrm{Hom}_A(M, Y_1) = 0$, $\mathrm{Hom}_A(M, X_1) = 0$, $\mathrm{Hom}_A(X_1, M) = 0$, and $\mathrm{Hom}_A(X_1, Y_1) = K$. Since $\mathrm{Hom}_A(M, Y_1) = 0$ we conclude that $\chi(\mathbf{e}) = \chi(\mathbf{dim}\, M) = 1$, and also $M|_{C_2} = 0$. Moreover, we have

$$\langle \mathbf{h}_1, \mathbf{e} \rangle = \dim_K \mathrm{Hom}_A(X_1, M) - \dim_K \mathrm{Ext}_A^1(X_1, M) + \dim_K \mathrm{Ext}_A^2(X_1, M) = 0,$$

because $\mathrm{Ext}_A^1(X_1, M) \simeq D\, \overline{\mathrm{Hom}}_A(M, X_1) = 0$ and $\mathrm{pd}_A X_1 \leq 1$,

$$\langle \mathbf{e}, \mathbf{h}_1 \rangle = \dim_K \mathrm{Hom}_A(M, X_1) - \dim_K \mathrm{Ext}_A^1(M, X_1) + \dim_K \mathrm{Ext}_A^2(M, X_1) \geq 0,$$

because $\mathrm{Ext}_A^1(M, X_1) \simeq D\, \underline{\mathrm{Hom}}_A(X_1, M) = 0$,

$$\langle \mathbf{e}, \mathbf{h}_2 \rangle = \dim_K \mathrm{Hom}_A(M, Y_1) - \dim_K \mathrm{Ext}_A^1(M, Y_1) + \dim_K \mathrm{Ext}_A^2(M, Y_2) = 0,$$

because $\mathrm{Ext}_A^1(M, Y_1) = D\, \underline{\mathrm{Hom}}(Y_1, M) = 0$ and $\mathrm{id}_A Y_1 \leq 1$,

$$\langle \mathbf{h}_2, \mathbf{e} \rangle = \dim_K \mathrm{Hom}_A(Y_1, M) - \dim_K \mathrm{Ext}_A^1(Y_1, M) + \dim_K \mathrm{Ext}_A^2(Y_1, M) \geq 0,$$

because $\mathrm{Ext}_A^1(Y_1, M) \simeq D\, \overline{\mathrm{Hom}}_A(M, Y_1) = 0$,

$$\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = \dim_K \mathrm{Hom}_A(X_1, Y_1) - \dim_K \mathrm{Ext}_A^1(X_1, Y_1) + \dim_K \mathrm{Ext}_A^2(X_1, Y_1) = 1,$$

because $X_1 \in \mathscr{T}_1$ and $Y_1 \in \mathscr{T}_2$. Hence we get

$$\begin{aligned} 1 = \langle \mathbf{d}, \mathbf{d} \rangle &= \langle \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{e}, \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{e} \rangle \\ &= 2 + \langle \mathbf{h}_2, \mathbf{h}_1 \rangle + \langle \mathbf{e}, \mathbf{h}_1 \rangle + \langle \mathbf{h}_2, \mathbf{e} \rangle \geq 2 + \langle \mathbf{h}_2, \mathbf{h}_1 \rangle, \end{aligned}$$

and so $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle \leq -1$. On the other hand, we have

$$\begin{aligned} \langle \mathbf{h}_2, \mathbf{h}_1 \rangle &= \dim_K \operatorname{Hom}_A(Y_1, X_1) - \dim_K \operatorname{Ext}_A^1(Y_1, X_1) \\ &\quad + \dim_K \operatorname{Ext}_A^2(Y_1, X_1) \\ &= -\dim_K \mathrm{D}\, \overline{\operatorname{Hom}}_A(X_1, Y_1) + \dim_K \operatorname{Ext}_A^2(Y_1, X_1) \geq -1. \end{aligned}$$

Consequently, $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle = -1$. But this implies $\dim_K \operatorname{Ext}_A(Y_1, X_1) = 0$, $\dim_K \operatorname{Ext}_A^2(M, X_1) = \langle \mathbf{e}, \mathbf{h}_1 \rangle = 0$ and $\dim_K \operatorname{Ext}_A^2(Y_1, M) = \langle \mathbf{h}_2, \mathbf{e} \rangle = 0$. Hence, $\operatorname{Ext}_A^2(R, R) = \operatorname{Ext}_A^2(X_1 \oplus Y_1 \oplus M, X_1 \oplus Y_1 \oplus M) = 0$, since $\operatorname{pd}_A X_1 \leq 1$, $\operatorname{id}_A Y_1 \leq 1$, and $\operatorname{pd}_A M \leq 1$ or $\operatorname{id}_A M \leq 1$, and so R is nonsingular. Finally, assume (by symmetry) that $r = 1$, $s = 0$, and $p_1 = 1$. Thus $\mathbf{d} = \mathbf{e} + \mathbf{h}_1$. Since X_1 is an isotropic module from $\mathscr{T}_1$, hence we know that M is not an isotropic module from $\mathscr{T}_1$, and $\operatorname{Hom}_A(X_1, M) \neq 0$. Clearly, then $\operatorname{Hom}_A(M, X_1) = 0$. If $\dim_K \operatorname{End}_A(R) = \dim_K \operatorname{End}_A(M \oplus X_1) \geq 4 = r + 3 = r + s + 3$ then our claim follows. Hence, assume $\dim_K \operatorname{End}_A(R) = 3$. Then $\operatorname{End}_A(M) = K$ and $\operatorname{Hom}_A(X_1, M) = K$. Observe that if M is an isotropic module then $\mathbf{e} = \mathbf{h}_2$ and $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$, and we get contradiction with our assumption on V. Thus $\langle \mathbf{e}, \mathbf{e} \rangle = 1$. Moreover, $\langle \mathbf{h}_1, \mathbf{e} \rangle = \dim_K \operatorname{Hom}_A(X_1, M) = 1$. Hence we get

$$1 = \langle \mathbf{d}, \mathbf{d} \rangle = \langle \mathbf{e} + \mathbf{h}_1, \mathbf{e} + \mathbf{h}_1 \rangle = 2 + \langle \mathbf{e}, \mathbf{h}_1 \rangle,$$

so $\langle \mathbf{e}, \mathbf{h}_1 \rangle = -1$. On the other hand,

$$\begin{aligned} \langle \mathbf{e}, \mathbf{h}_1 \rangle &= \dim_K \operatorname{Hom}_A(M, X_1) - \dim_K \operatorname{Ext}_A^1(M, X_1) \\ &\quad + \dim_K \operatorname{Ext}_A^2(M, X_1) \\ &= -\dim_K \mathrm{D}\, \underline{\operatorname{Hom}}_A(X_1, M) + \dim_K \operatorname{Ext}_A^2(M, X_1) \\ &\geq -1 + \dim_K \operatorname{Ext}_A^2(M, X_1). \end{aligned}$$

But then $\langle \mathbf{e}, \mathbf{h}_1 \rangle = -1$ implies $\operatorname{Ext}_A^2(M, X_1) = 0$. Therefore, $\operatorname{Ext}_A^2(R, R) = \operatorname{Ext}_A^2(M \oplus X_1, M \oplus X_1) = 0$, because $\operatorname{pd}_A X_1 \leq 1$ and either $\operatorname{pd}_A M \leq 1$ or $\operatorname{id}_A M \leq 1$. This finishes our proof that $\operatorname{mod}_A(\mathbf{d})$ is nonsingular in codimension one, and so is normal.

4.11

Our next aim is to prove the following fact.

PROPOSITION. *Assume V is an exceptional internal directing A-module. Then the following statements hold.*

(i) $\operatorname{mod}_A(\mathbf{d})$ *is a complete intersection and* $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$.

(ii) $\operatorname{mod}_A(\mathbf{d})$ *has two irreducible components.*

(iii) $\operatorname{mod}_A(\mathbf{d})$ *is not normal.*

(iv) $\mathscr{O}(V)$ *is the open sheet of* $\operatorname{mod}_A(\mathbf{d})$.

(v) *All $G(\mathbf{d})$-orbits in* $\operatorname{mod}_A(\mathbf{d})$ *of codimension one are contained in* $\overline{\mathscr{O}(V)}$.

(vi) *The maximal $G(\mathbf{d})$-orbits in* $\operatorname{mod}_A(\mathbf{d})$ *consist of nonsingular modules.*

Proof. Clearly, $\overline{\mathscr{O}(V)}$ is an irreducible component and $\dim_V \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$. Denote by $\mathscr{W}$ the set of all modules in $\operatorname{mod}_A(\mathbf{d})$ of the form $X \oplus Y$, where X and Y are indecomposable A-modules with $\operatorname{\mathbf{dim}} X = \mathbf{h}_1$ and $\operatorname{\mathbf{dim}} Y = \mathbf{h}_2$. We know that $\mathscr{W}$ is irreducible. Moreover, if $X \oplus Y \in \mathscr{W}$ then $\dim_K \operatorname{End}_A(X) = 1$, $\dim_K \operatorname{End}_A(Y) = 1$, $\dim_K \operatorname{Hom}_A(X, Y) = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$, $\operatorname{Hom}_A(Y, X) = 0$, and hence $\dim \mathscr{O}(X \oplus Y) = \dim G(\mathbf{d}) - \dim \operatorname{End}_A(X \oplus Y) = \dim G(\mathbf{d}) - 3 = \dim \operatorname{mod}_A(\mathbf{d}) - 2$. Therefore $\overline{\mathscr{W}}$ is an irreducible closed subset of $\operatorname{mod}_A(\mathbf{d})$ of dimension $a(\mathbf{d})$, and so is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$. Further, it follows from the proof of Proposition 4.8 that if $\mathscr{O}(M)$ is a maximal orbit in $\operatorname{mod}_A(\mathbf{d})$ different from $\mathscr{O}(V)$ then $\mathscr{O}(M)$ is contained in $\mathscr{W}$. This shows that $\operatorname{mod}_A(\mathbf{d})$ is a union of $\overline{\mathscr{O}(V)}$ and $\overline{\mathscr{W}}$. Observe that $\overline{\mathscr{O}(V)} \neq \overline{\mathscr{W}}$. Indeed, if $\overline{\mathscr{W}} \subset \overline{\mathscr{O}(V)}$ then $\mathscr{W} \subset \overline{\mathscr{O}(V)} \setminus \mathscr{O}(V)$, and hence $\dim \overline{\mathscr{W}} \leq \dim(\overline{\mathscr{O}(V)} \setminus \mathscr{O}(V)) < \dim \overline{\mathscr{O}(V)}$, a contradiction. Therefore, $\operatorname{mod}_A(\mathbf{d})$ has exactly two irreducible components: $\overline{\mathscr{O}(V)}$ and $\overline{\mathscr{W}}$. Clearly, $\operatorname{mod}_A(\mathbf{d})$ is a complete intersection and $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$. Moreover, $\mathscr{O}(V)$ is a unique $G(\mathbf{d})$-orbit in $\operatorname{mod}_A(\mathbf{d})$ of (maximal) dimension $a(\mathbf{d})$, and hence is the open sheet of $\operatorname{mod}_A(\mathbf{d})$. Further, all $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ of codimension one are contained in $\overline{\mathscr{O}(V)}$. Our next aim is to show that $\operatorname{mod}_A(\mathbf{d})$ is not normal. Observe that the intersection $\overline{\mathscr{O}(V)} \cap \overline{\mathscr{W}}$ contains the set $\mathscr{Z}$ of all modules $X \oplus Z$ with X and Z indecomposable A-modules such that $\operatorname{\mathbf{dim}} X = \mathbf{h}_1$, $\operatorname{\mathbf{dim}} Z = \mathbf{h}_2$, and there exists an exact sequence of the form $0 \to X \to V \to Z \to 0$. Clearly $\mathscr{Z}$ consists of singular points of $\operatorname{mod}_A(\mathbf{d})$, as a subset of the intersection of two irreducible compondents of $\operatorname{mod}_A(\mathbf{d})$. Hence in order to prove that $\operatorname{mod}_A(\mathbf{d})$ is not normal, it is enough to show that $\dim \overline{\mathscr{Z}} = a(\mathbf{d}) - 1 = \dim \operatorname{mod}_A(\mathbf{d}) - 1$.

First observe that $1 = \langle \mathbf{d}, \mathbf{d} \rangle = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle + \langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 1 + \langle \mathbf{h}_2, \mathbf{h}_1 \rangle$ implies $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0$. Let X and Y be two indecomposable modules with $\mathbf{dim}\, X = \mathbf{h}_1$ and $\mathbf{dim}\, Y = \mathbf{h}_2$. Then we have

$$\begin{aligned} 0 &= \langle \mathbf{h}_2, \mathbf{h}_1 \rangle \\ &= \dim_K \operatorname{Hom}_A(Y, X) - \dim_K \operatorname{Ext}_A^1(Y, X) + \dim_K \operatorname{Ext}_A^2(Y, X) \\ &= -\dim_K \operatorname{Ext}_A^1(Y, X) + \dim_K \operatorname{Ext}_A^2(Y, X), \end{aligned}$$

and so $\dim_K \operatorname{Ext}_A^1(Y, X) = \dim_K \operatorname{Ext}_A^2(Y, X)$. Further,

$$\dim_K \operatorname{Hom}_{C_1}\big(X, V|_{C_1}\big) = \dim_K \operatorname{Hom}_A(X, V) = \langle \mathbf{h}_1, \mathbf{d} \rangle = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1,$$

and $\dim_K \operatorname{Hom}_{C_1}(X, Y|_{C_1}) = \dim_K \operatorname{Hom}_A(X, Y) = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$. Since $V|_{C_1}$ and $Y|_{C_1}$ are direct sums of indecomposable preinjective C_1-modules, we conclude that $V|_{C_1}$ and $Y|_{C_1}$ are indecomposable C_1-modules. Let $\mathbf{d}' = \mathbf{d}|_{C_1}$. Since $V|_{C_1}$ is an indecomposable preinjective C_1-module, $\mathscr{O}(V|_{C_1})$ is a unique maximal $G(\mathbf{d}')$-orbit in the variety $\operatorname{mod}_{C_1}(\mathbf{d}')$, and consequently $\operatorname{Ext}_{C_1}^1(Y|_{C_1}, X) \neq 0$. Moreover, $\dim_K \operatorname{End}_{C_1}(X \oplus Y|_{C_1}) = 3$ and $\chi_{C_1}(\mathbf{d}') = \chi_{C_1}(\mathbf{dim}\, V|_{C_1}) = 1$, and hence $\mathscr{O}(X \oplus Y|_{C_1})$ is a $G(\mathbf{d}')$-orbit of $\operatorname{mod}_{C_1}(\mathbf{d}')$ of codimension two. Let $\mathscr{W}'$ be the set of all modules in $\operatorname{mod}_{C_1}(\mathbf{d}')$ which are the restrictions of modules from $\mathscr{W}$ to C_1. Clearly, $\dim \mathscr{W}' = a(\mathbf{d}') - 1$. Further, it follows from (4.2) that $\operatorname{mod}_{C_1}(\mathbf{d}')$ has only finitely many $G(\mathbf{d}')$-orbits of codimension one. This implies that the subset $\mathscr{W}''$ of $\mathscr{W}'$ consisting of all modules which are not minimal degenerations of $V|_{C_1}$ has codimension 2. Therefore, for all but finitely many indecomposable A-modules X with $\mathbf{dim}\, X = \mathbf{h}_1$ there exists an exact sequence

$$0 \to X \to V|_{C_1} \to Y|_{C_1} \to 0$$

for some indecomposable A-module Y with $\mathbf{dim}\, Y = \mathbf{h}_2$. Since we have $\operatorname{Hom}_{C_1}(X, V|_{C_1}) = \operatorname{Hom}_A(X, V)$ and $\operatorname{Hom}_A(X, V)$ is of dimension one, any nonzero map from X to V is a monomorphism, and consequently there exists an exact sequence

$$0 \to X \to V \to Z \to 0,$$

where Z is an A-module with $\mathbf{dim}\, Z = \mathbf{h}_2$, uniquely determined by X. Observe that such a module Z is indecomposable, because V is directing and then $\dim_K \operatorname{Hom}_A(V, Z) = \langle \mathbf{d}, \mathbf{h}_2 \rangle = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$. Therefore, we have proved that $\dim \overline{\mathscr{Z}} = a(\mathbf{d}) - 1$, and so $\operatorname{mod}_A(\mathbf{d})$ is not normal. Finally, observe that any maximal $G(\mathbf{d})$-orbit in $\operatorname{mod}_A(\mathbf{d})$ different from $\mathscr{O}(V)$ is of the form $\mathscr{O}(X \oplus Y)$ for indecomposable A-modules X and Y with $\mathbf{dim}\, X = \mathbf{h}_1$, $\mathbf{dim}\, Y = \mathbf{h}_2$, and $\operatorname{Ext}_A^1(Y, X) = 0$. But then we have the following equalities $\dim_K \operatorname{Ext}_A^2(Y, X) = \dim_K \operatorname{Ext}_A^1(Y, X) = 0$ and consequently

$\operatorname{Ext}^2_A(X \oplus Y, X \oplus Y) = 0$. Hence all maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ consist of nonsingular modules.

4.12

We may now complete the proof of Theorem 1.

PROPOSITION. *Let A be a tame algebra and V a directing A-module and $\mathbf{d} = \mathbf{dim}\, V$. Then all but a finite number of $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ have codimension at least two.*

Proof. We may assume that A is the support algebra of V, and so A is a tilted algebra. It follows from (4.4), (4.8), and (4.11) that $\mathscr{O}(V)$ is a unique $G(\mathbf{d})$-orbit in $\operatorname{mod}_A(\mathbf{d})$ of codimension 0, and if $\mathscr{O}(M)$ is a $G(\mathbf{d})$-orbit in $\operatorname{mod}_A(\mathbf{d})$ of codimension 1 then M is a minimal degeneration of V, and so there exists a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$, where V_1 and V_2 are indecomposable A-modules such that $M = V_1 \oplus V_2$, and clearly $\dim_K \operatorname{End}_A(M) = 2$. Moreover, then $\chi_A(\mathbf{dim}\, V_1) = 1$ and $\chi_A(\mathbf{dim}\, V_2) = 1$, by (4.2). We know from (3.4) that there are only finitely many choices (up to isomorphism) for $M = V_1 \oplus V_2$ satisfying the above conditions. Therefore, the number of $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ of codimension 1 is finite. This finishes the proof.

4.13

Our final aim in this section is to complete the proof of Theorem 2, by showing the equivalence of its statements (iv) and (v). We start with the following fact.

PROPOSITION. *Assume V is a sincere exceptional internal directing A-module. Then V is projective-injective.*

Proof. We use notation introduced in (4.6). Hence $\mathbf{dim}\, V = \mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$, where $\mathbf{h}_1$, $\mathbf{h}_2$ are generators of the radicals of χ_{C_1} and χ_{C_2}, and $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$, $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0$. Then $\langle \mathbf{d}, \mathbf{h}_1 \rangle = \langle \mathbf{h}_1, \mathbf{h}_1 \rangle + \langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0$ and $\langle \mathbf{h}_2, \mathbf{d} \rangle = \langle \mathbf{h}_2, \mathbf{h}_1 \rangle + \langle \mathbf{h}_2, \mathbf{h}_2 \rangle = 0$. Put $\mathbf{d}_1 = \mathbf{dim}\, \tau_A V$ and $\mathbf{d}_2 = \mathbf{dim}\, \tau_A^{-1} V$. We observe first that $\operatorname{supp} \tau_A V$ and C_1 (respectively, $\operatorname{supp} \tau_A^{-} V$ and C_2) have no common objects. Let H_1 and H_2 be indecomposable modules lying on the mouth of homogeneous tubes in $\mathscr{T}_1$ and $\mathscr{T}_2$, respectively. We then get

$$\begin{aligned} \dim_K \operatorname{Hom}_A(H_1, \tau_A V) &= \dim_K D\, \overline{\operatorname{Hom}}_A(H_1, \tau_A V) = \dim_K \operatorname{Ext}^1_A(V, H_1) \\ &= -\langle \mathbf{dim}\, V, \mathbf{dim}\, H_1 \rangle = -\langle \mathbf{d}, \mathbf{h}_1 \rangle = 0 \end{aligned}$$

and

$$\dim_K \operatorname{Hom}_A(\tau_A^- V, H_2) = \dim_K D\,\underline{\operatorname{Hom}}_A(\tau_A^- V, H_2) = \dim_K \operatorname{Ext}_A^1(H_2, V)$$
$$= -\langle \mathbf{dim}\, H_2, \mathbf{dim}\, V\rangle = -\langle \mathbf{h}_2, \mathbf{d}\rangle = 0.$$

Since $\operatorname{supp} H_1 = C_1$, $\mathscr{P}_1$ contains all indecomposable projective C_1-modules and $\mathscr{T}_1$ separates $\mathscr{P}_1$ from $\mathscr{C} \vee \mathscr{T}_2 \vee \mathscr{Q}_2$, we conclude that $\operatorname{supp} \tau_A V$ and C_1 have no common objects. Similarly, $\operatorname{supp} H_2 = C_2$, $\mathscr{Q}_2$ contains all indecomposable injective C_2-modules and $\mathscr{T}_2$ separates $\mathscr{P}_1 \vee \mathscr{T}_1 \vee \mathscr{C}$ from $\mathscr{Q}_2$, and so $\operatorname{supp} \tau_A^- V$ and C_2 have no common objects. We claim now that $\operatorname{Ext}_A^1(\tau_A^- V, \tau_A V) = 0$. Suppose we have an exact sequence

$$e\colon 0 \to \tau_A V \to W \to \tau_A^- V \to 0.$$

Let $x \xrightarrow{\alpha} y$ be an arrow in Q_A with x in Q_{C_2} and y in Q_{C_1}. Then $(\tau_A V)_y = 0$, $(\tau_A^- V)_x = 0$, and hence we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\tau_A V)_x & \longrightarrow & W_x & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle W_\alpha} & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & W_y & \longrightarrow & (\tau_A^- V)_y & \longrightarrow & 0.
\end{array}$$

Thus $W_\alpha = 0$. Moreover, Q_A has no arrows with source in Q_{C_1} and target in Q_{C_2}, and clearly $W_z = 0$ for any common object of C_1 and C_2. All together this implies $W = M \oplus N$, where M is the restriction of W to C_1 and N is the restriction of W to C_2. Considering now the restrictions of the above exact sequence e to C_1 and C_2, we conclude that $M \simeq \tau_A V$, $N \simeq \tau_A^{-1} V$, and so the sequence e is splittable. Therefore, $\operatorname{Ext}_A^1(\tau_A^- V, \tau_A V) = 0$. We claim now that $\operatorname{Hom}_A(\tau_A V, V) = 0$. If V is not injective, then it has no injective predecessors in mod_A, and we have

$$\dim_K \operatorname{Hom}_A(\tau_A V, V) = \dim_K D\,\overline{\operatorname{Hom}}_A(\tau_A V, V)$$
$$= \dim_K \operatorname{Ext}_A^1(\tau_A^- V, \tau_A V) = 0.$$

On the other hand, if V is injective then $V = I(a)$ for some vertex a of Q_{C_1}, and $\operatorname{Hom}_A(\tau_A V, V) = 0$ because $\operatorname{supp} \tau_A V$ and C_1 have no common objects. Suppose now that the module V ius not projective. Then $\tau_A V$ is a directing module, and hence $\mathbf{d}_1 = \mathbf{dim}\, \tau_A V$ is a positive connected vector with $\langle \mathbf{d}_1, \mathbf{d}_1\rangle = 1$. Moreover, we have

$$\langle \mathbf{d}, \mathbf{d}_1\rangle = \langle \mathbf{dim}\, V, \mathbf{dim}\, \tau_A V\rangle = -\dim_K \operatorname{Ext}_A^1(V, \tau_A V)$$
$$= -\dim_K D\,\overline{\operatorname{Hom}}_A(\tau_A V, \tau_A V) = -\dim_K \operatorname{End}_A(\tau_A V) = -1$$

and

$$\langle \mathbf{d}_1, \mathbf{d} \rangle = \langle \mathbf{dim}\, \tau_A V, \mathbf{dim}\, V \rangle = \dim_K \operatorname{Hom}_A(\tau_A V, V) = 0.$$

Since $\operatorname{supp} \tau_A V$ is contained in C_2, we have also

$$\begin{aligned}\langle \mathbf{d}_1, \mathbf{h}_2 \rangle &= \langle \mathbf{dim}\, \tau_A V, \mathbf{dim}\, H_2 \rangle = \dim_K \operatorname{Hom}_{C_2}(\tau_A V, H_2) \\ &= \dim_K \operatorname{Hom}_{C_2}(\tau_A V, \tau_A H_2) = \dim_K D\, \overline{\operatorname{Hom}}_{C_2}(\tau_A V, \tau_C H_2) \\ &= \dim_K \operatorname{Ext}^1_{C_2}(H_2, \tau_A V) = \dim_K \operatorname{Ext}^1_A(H_2, \tau_A V) \\ &= -\langle \mathbf{dim}\, H_2, \mathbf{dim}\, \tau_A V \rangle = -\langle \mathbf{h}_2, \mathbf{d}_1 \rangle.\end{aligned}$$

Therefore, we get

$$\begin{aligned}0 &\le \langle \mathbf{h}_1 + \mathbf{d}_1, \mathbf{h}_1 + \mathbf{d}_1 \rangle \\ &= \langle \mathbf{d} + \mathbf{d}_1, \mathbf{d} + \mathbf{d}_1 \rangle - \langle \mathbf{h}_1 + \mathbf{d}_1, \mathbf{h}_2 \rangle - \langle \mathbf{h}_2, \mathbf{h}_1 + \mathbf{d}_1 \rangle - \langle \mathbf{h}_2, \mathbf{h}_2 \rangle \\ &= 1 - (\langle \mathbf{d}_1, \mathbf{h}_2 \rangle + \langle \mathbf{h}_2, \mathbf{d}_1 \rangle) - \langle \mathbf{h}_1, \mathbf{h}_2 \rangle - \langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0,\end{aligned}$$

and so $\langle \mathbf{h}_1 + \mathbf{d}_1, \mathbf{h}_1 + \mathbf{d}_1 \rangle = 0$. Further, $0 = \langle \mathbf{d}_1, \mathbf{d} \rangle = \langle \mathbf{d}_1, \mathbf{h}_1 \rangle + \langle \mathbf{d}_1, \mathbf{h}_2 \rangle$ implies $\langle \mathbf{d}_1, \mathbf{h}_1 \rangle \neq 0$, because $\tau_A V$ is a nonzero C_2-module and then $\langle \mathbf{d}_1, \mathbf{h}_2 \rangle = \dim_K \operatorname{Hom}_A(\tau_A V, H_2) \neq 0$. Thus $\mathbf{d}_1 + \mathbf{h}_1$ is a positive connected vector of $K_0(A)$ with $\chi_A(\mathbf{d}_1 + \mathbf{h}_1) = 0$. Applying now (3.4) we conclude that $\mathbf{d}_1 + \mathbf{h}_1 = \mathbf{dim}\, X$ for an indecomposable A-module lying in a homogeneous tube of Γ_A. On the other hand, $\mathscr{T}_1$ and $\mathscr{T}_2$ are unique families of tubes of Γ_A, and clearly $\mathbf{d}_1 + \mathbf{h}_1$ is neither a multiple of $\mathbf{h}_1$ nor a multiple of $\mathbf{h}_2$. This contradiction shows that in fact V is projective. Clearly, then $V = P(x)$ for some vertex x of Q_{C_2}. Suppose now that V is not injective. Then

$$\dim_K \operatorname{Hom}_A(V, \tau_A^- V) = \dim_K \operatorname{Hom}_A(P(x), \tau_A^- V) = 0$$

because $\operatorname{supp} \tau_A^- V$ and C_2 have no common objects. Moreover, $\mathbf{d}_2 = \mathbf{dim}\, \tau_A^- V$ is a positive connected vector with $\langle \mathbf{d}_2, \mathbf{d}_2 \rangle = 1$, because τ_A^-V is directing. Further, we have

$$\begin{aligned}\langle \mathbf{d}_2, \mathbf{d} \rangle &= \langle \mathbf{dim}\, \tau_A^- V, \mathbf{dim}\, V \rangle = -\dim_K \operatorname{Ext}^1_A(\tau_A^- V, V) \\ &= -\dim_K D\, \overline{\operatorname{Hom}}_A(\tau_A^- V, \tau_A^- V) = -\dim_K \operatorname{End}_A(\tau_A^- V) = -1\end{aligned}$$

and

$$\langle \mathbf{d}, \mathbf{d}_2 \rangle = \langle \mathbf{dim}\, V, \mathbf{dim}\, \tau_A^- V \rangle = \dim_K \operatorname{Hom}_A(V, \tau_A^- V) = 0,$$

by the above remark, and consequently

$$\langle \mathbf{d} + \mathbf{d}_2, \mathbf{d} + \mathbf{d}_2 \rangle = \langle \mathbf{d}, \mathbf{d} \rangle + \langle \mathbf{d}, \mathbf{d}_2 \rangle + \langle \mathbf{d}_2, \mathbf{d} \rangle + \langle \mathbf{d}_2, \mathbf{d}_2 \rangle = 1.$$

since supp $\tau_A^- V$ is contained in C_1, we have also

$$\begin{aligned}\langle \mathbf{h}_1, \mathbf{d}_2 \rangle &= \langle \mathbf{dim}\, H_1, \mathbf{dim}\, \tau_A^- V) = \dim_K \operatorname{Hom}_A(H_1, \tau_A^- V) \\ &= \dim_K \operatorname{Hom}_{C_1}\left(\tau_{C_1}^- H_1, \tau_A^- V\right) = \dim_K D\, \underline{\operatorname{Hom}}_{C_1}\left(\tau_{C_1}^- H_1, \tau_A^- V\right) \\ &= \dim_K \operatorname{Ext}^1_{C_1}(\tau_A^- V, H_1) = \dim_K \operatorname{Ext}^1_A(\tau_A^- V, H_1) \\ &= -\langle \mathbf{dim}\, \tau_A^- V, \dim H_1 = -\langle \mathbf{d}_2, \mathbf{h}_1 \rangle.\end{aligned}$$

Then we get

$$\begin{aligned}0 &\le \langle \mathbf{h}_2 + \mathbf{d}_2, \mathbf{h}_2 + \mathbf{d}_2 \rangle \\ &= \langle \mathbf{d} + \mathbf{d}_2, \mathbf{d} + \mathbf{d}_2 \rangle - \langle \mathbf{h}_2 + \mathbf{d}_2, \mathbf{h}_1 \rangle - \langle \mathbf{h}_1, \mathbf{h}_2 + \mathbf{d}_2 \rangle - \langle \mathbf{h}_1, \mathbf{h}_1 \rangle \\ &= 1 - \left(\langle \mathbf{d}_2, \mathbf{h}_1 \rangle + \langle \mathbf{h}_1, \mathbf{d}_2 \rangle\right) - \langle \mathbf{h}_1, \mathbf{h}_2 \rangle - \langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0\end{aligned}$$

and hence $\langle \mathbf{h}_2 + \mathbf{d}_2, \mathbf{h}_2 + \mathbf{d}_2 \rangle = 0$. Further, since $\tau_A^- V$ is an indecomposable C_1-module, we have $\langle \mathbf{h}_1, \mathbf{d}_2 \rangle = \dim_K \operatorname{Hom}_A(H_1, \tau_A^- V) \neq 0$, and then $0 = \langle \mathbf{d}, \mathbf{d}_2 \rangle = \langle \mathbf{h}_1, \mathbf{d}_2 \rangle + \langle \mathbf{h}_2, \mathbf{d}_2 \rangle$ implies $\langle \mathbf{h}_2, \mathbf{d}_2 \rangle \neq 0$. Thus $\mathbf{h}_2 + \mathbf{d}_2$ is a positive connected vector of $K_0(A)$ with $\chi_A(\mathbf{h}_2 + \mathbf{d}_2) = 0$. This again leads to a contradiction with (3.4) because $\mathbf{h}_2 + \mathbf{d}_2$ is neither a multiple of $\mathbf{h}_1$ nor a multiple of $\mathbf{h}_2$. Hence V is also injective. This completes our proof.

4.14

The following final result completes the proof of Theorem 2.

PROPOSITION. *Assume V is a sincere internal directing A-module. Then V is exceptional if and only if A or A^{op} is isomorphic to one of the 2-parametric tilted algebras $A(p,q,r,s)$, $p,q,r,s \ge 1$, $F(p,q,r,s)$, $p,r \ge 1$, $q,s \ge 2$, $p+r \ge 3$, $1/(p+r-1) + 1/q + 1/s > 1$, $D(p,q)$, $p \ge 2$, $r \ge 1$, $E'(p,r)$ or $E''(p,r)$, $p \ge 2$, $r \ge 1$, $4 \le p+r \le 6$, and V is its unique indecomposable projective-injective module.*

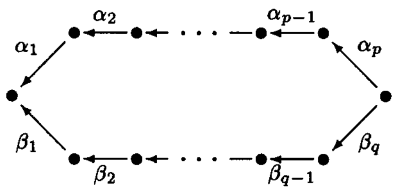
Proof. Assume V is an exceptional internal directing sincere A-module. We use notation introduced in (4.5). Hence $\mathbf{dim}\, V = \mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$ with $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$, and $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0$. Moreover, it follows from (4.13) that V is projective-injective. Hence, there is a unique sink a and a unique source c in Q_A such that $I(a) = V = P(c)$. Further, there is an Auslander−Reiten sequence in mod_A (see [9]) of the form

$$0 \to \operatorname{rad} V \to \operatorname{rad} V/\operatorname{soc} V \oplus V \to V/\operatorname{soc} V \to 0,$$

and hence rad V, $V/\operatorname{soc} V$ are directing modules lying in the connecting component $\mathscr{C}$. Hence, for any convex subcategory D of A containing the sink a (respectively, the source c), the restriction $V|_D$ of V to D is an indecomposable injective (respectively, indecomposable projective) D-module. Let D_1 (respectively, D_2) be the convex subcategory of A given by all objects of A except c (respectively, except a). Then it follows from [17, Proposition 3.3] that D_1 (respectively, D_2) is a one-parameter tame tilted algebra having the sincere indecomposable module rad V (respectively, $V/\operatorname{soc} V$) in the unique preinjective component of Γ_{D_1} (respectively, in the unique preprojective component of Γ_{D_2}). Consider now the restrictions $M = V|_{C_1}$ of V to C_1 and $N = V|_{C_2}$ of V to C_2. It follows from the above remarks that M is a sincere indecomposable injective C_1-module and N is a sincere indecomposable projective C_2-module. Moreover, we have $\mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$. Then a simple inspection of the Bongartz–Happel–Vossieck list of the frames of tame concealed algebras [4, 11] shows that C_1 and C_2 are tame concealed canonical algebras, that is, are of one of the forms:

$\Lambda(p, q)$, $p, q \geq 1$, the path algebra $KQ(p, q)$ of the quiver

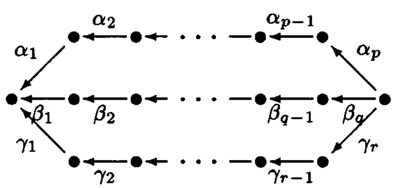
$Q(p,q)$:



or

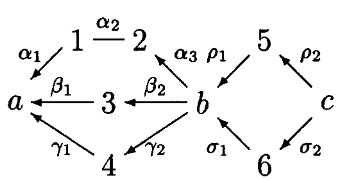
$\Lambda(p, q, r)$, with $(p, q, r) = (3, 3, 2), (4, 3, 2), (5, 3, 2)$ or $(m-2, 2, 2)$, $m \geq 4$, the bound quiver algebra $KQ(p, q, r)/J(p, q, r)$, where

$Q(p,q,r)$:

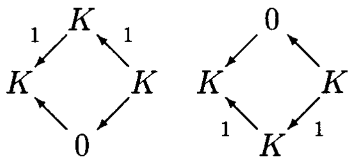


and $J(p,q,r)$ is the ideal in $KQ(p,q,r)$ generated by $\alpha_1 \cdots \alpha_p + \beta_1 \cdots \beta_q + \gamma_1 \cdots \gamma_r$. We know also that the tubular extension B_1 of C_1 (in our notation (4.6)) is a tilted algebra of Euclidean type having the sincere indecomposable injective module $V|_{B_1}$ in the preinjective component of Γ_{B_1}. Similarly, the tubular coextension B_2 of C_2 is a tilted algebra of Euclidean type having the sincere indecomposable projective module $V|_{B_2}$ in the preprojective component of Γ_{B_2}. Now we involve the complete description of all indecomposable modules lying on the mouth of tubes of tame concealed canonical algebras given in [20, (3.7)]. In particular, we know that the mouth of any tube in Γ_{C_1} or Γ_{C_2} contains exactly one module which is not simple. Hence, if B_1 is a proper tubular extension of C_1 (respectively, if B_2 is a proper tubular coextension of C_2) then the presence of an indecomposable sincere injective B_1-module (respectively, of indecomposable sincere projective B_2-module) implies that in this extension (respectively, coextension) process only the nonsimple indecomposable modules from the mouth of tubes of Γ_{C_1} (respectively, Γ_{C_2}) are used. Then a simple analysis shows that $B_1 = D_1$, $B_2 = D_2$, A is the one-point extension $A = B_1[\operatorname{rad} V]$ of B_1 by the indecomposable injective B_1-module $\operatorname{rad} V$, A is the one-point coextension $[V/\operatorname{soc} V]B_2$ of B_2 by the indecomposable projective B_2-module $V/\operatorname{soc} V$, and A or A^{op} is isomorphic to one of the algebras $A(p,q,r,s)$, $p,q,r,s \geq 1$, $F(p,q,r,s)$, $p,r \geq 1$, $q,s \geq 2$, $p+r \geq 3$, $1/(p+r-1) + 1/q + 1/s > 1$, $D(p,r)$, $p \geq 2$, $r \geq 1$, $E'(p,r)$ or $E''(p,r)$, $p \geq 2$, $r \geq 1$, $4 \leq p+r \leq 6$.

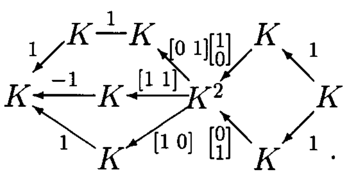
Conversely, if A or A^{op} is one of the above algebras $A(p,q,r,s)$, $F(p,q,r,s)$, $D(p,r)$, $E'(p,r)$, or $E''(p,r)$ then we easily deduce that A is a 2-parametric tilted algebra and its unique projective-injective indecomposable A-module V is an exceptional sincere directing A-module. We shall indicate it in the case $A = E'(3,2)$. In this case, A is the bound quiver algebra $K\Delta'(3,2)/I'(3,2)$, where $\Delta'(3,2)$ is the quiver



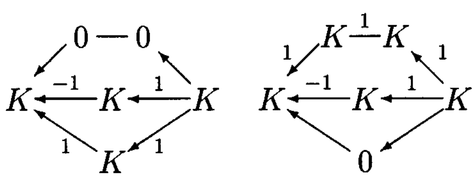
and $I'(3,2)$ is the ideal of $K\Delta'(3,2)$ generated by $\alpha_3\rho_1$, $\gamma_2\sigma_1$, $\beta_2\rho_1\rho_2 - \beta_1\sigma_1\sigma_2$, $\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2 + \gamma_1\gamma_2$. Then the injective envelope $I(a)$ of the simple module $S(a)$ coincides with the projective cover $P(c)$ of the simple module $S(c)$, and this projective-injective A-module $I(a) = V = P(c)$ is given by the representation



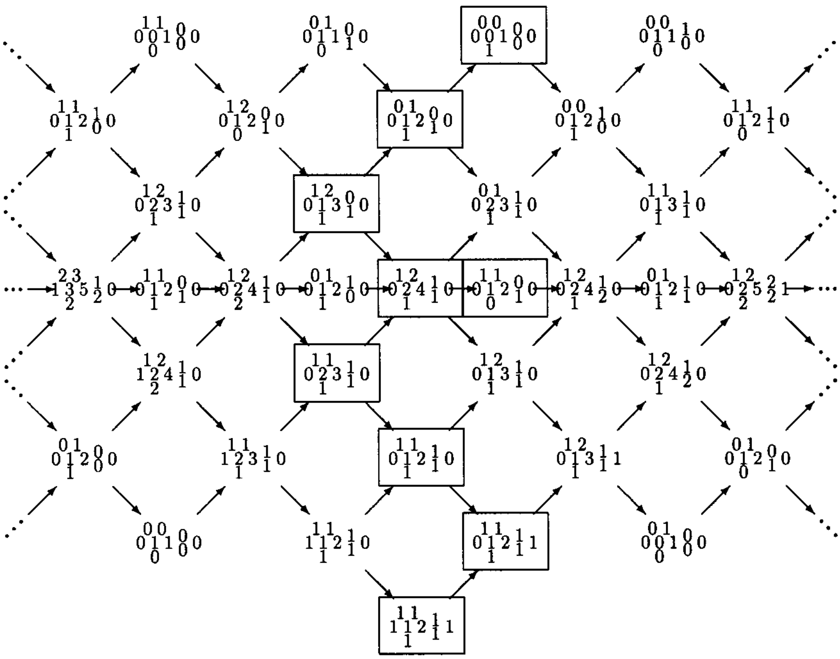
Let C_1 (respectively, B_1) be the convex subcategory of A given by the objects a, 1, 2, 3, 4, b (respectively, a, 1, 2, 3, 4, b, 5, 6) and C_2 (respectively, B_2) the full subcategory of A given by the objects b, 5, 6, c (respectively, 1, 2, 3, 4, b, 5, 6, c). Then C_1 is a tame concealed algebra of type $\tilde{\mathbb{D}}_5$ and B_1 is a tilted algebra of type $\tilde{\mathbb{E}}_7$, obtained from C_1 by two one-point extensions using nonsimple modules



lying respectively on the mouth of the tube of rank 3 in Γ_{C_1} containing the simple modules $S(1)$ and $S(2)$, and the tube of rank 2 in Γ_{C_1} containing the simple module $S(4)$. Similarly, C_2 is a tame concealed algebra of type $\tilde{\mathbb{A}}_4$ and B_2 is a tilted algebra of type $\tilde{\mathbb{E}}_7$, obtained from C_2 by a tubular coextension using the nonsimple modules



lying on the mouth of two tubes in Γ_{C_1} of rank 2 (and one branch of length 2 given by the vertices 1 and 2) and the module



lying on the mouth of a tube of rank 1. Moreover, A is the one-point extension $B_1[\operatorname{rad} V]$ and the one-point coextension $[V/\operatorname{soc} V]B_2$. Further Γ_A has a component $\mathscr{C}$ being the following glueing of the preinjective component of Γ_{B_1} and the preprojective component of Γ_{B_2}:

$$\begin{tikzcd} & K \arrow[dl, "1"'] & \\ K & & K \arrow[ul, "1"'] \arrow[dl, "1"] \\ & K \arrow[ul, "1"] & \end{tikzcd}$$

The modules in the boxes form a section Σ of type $\overline{\overline{\Sigma}} = \tilde{\tilde{\mathbb{E}}}_7$ in $\mathscr{C}$. Clearly the equality $\operatorname{Hom}_A(X, \tau_A Y) = 0$ holds for any modules X and Y from Σ, and Σ is a faithful section of $\mathscr{C}$, because it contains the projective-injective module V. Then applying (3.2) we conclude that A is a 2-parametric tame tilted algebra of type $\tilde{\tilde{\mathbb{E}}}_7$. Then $\operatorname{gl.dim} A \leq 2$, and consequently the Euler form coincides with the Tits form q_A, and so the bilinear form $\langle -, - \rangle$ is given by

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle = {} & x_a y_a + x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_b y_b + x_5 y_5 + x_6 y_6 + x_c y_c \\ & - x_1 y_a - x_3 y_a - x_4 y_a - x_2 y_1 - x_b y_2 - x_b y_3 - x_b y_4 \\ & - x_5 y_b - x_6 y_b - x_c y_5 - x_c y_6 + x_b y_a + x_5 y_2 + x_6 y_4 + x_c y_3 .\end{aligned}$$

Finally, observe that

$$\mathbf{h}_1 = \begin{smallmatrix} & 1 & & 1 & & & \\ 1 & & 1 & & 1 & \genfrac{}{}{0pt}{}{0}{0} & 0 \\ & & 1 & & & & \end{smallmatrix} \qquad \text{and} \qquad \mathbf{h}_2 = \begin{smallmatrix} & 0 & & 0 & & & \\ 0 & & 0 & & 1 & \genfrac{}{}{0pt}{}{1}{1} & 1 \\ & & 0 & & & & \end{smallmatrix},$$

$\mathbf{dim}\, V = \mathbf{h}_1 + \mathbf{h}_2$ and $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = 1$, $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0$. Therefore, V is an exceptional internal directing sincere (projective-injective) A-module.

5. EXAMPLES

We shall illustrate our considerations by some examples.

5.1

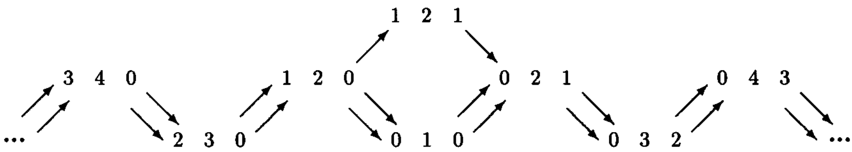
Let $A = KQ/I$ be the bound quiver algebra given by the quiver

$$Q\colon 1 \underset{\gamma}{\overset{\alpha}{\leftleftarrows}} 2 \underset{\sigma}{\overset{\beta}{\leftleftarrows}} 3$$

and the ideal I in KQ generated by $\alpha\beta$, $\gamma\sigma$, and $\gamma\beta - \alpha\sigma$. Then A is a tame tilted algebra obrtained by a glueing of two Kronecker algebras $C_1 = KQ^{(1)}$ and $C_2 = KQ^{(2)}$, where $Q^{(1)}$ (respectively, $Q^{(2)}$) is the full subquiver of A given by vertices 1 and 2 (respectively, 2 and 3). Moreover, Γ_A is of the form

$$\Gamma_A = \mathscr{P}_1 \vee \mathscr{T}_1 \vee \mathscr{C} \vee \mathscr{T}_2 \vee \mathscr{Q}_2,$$

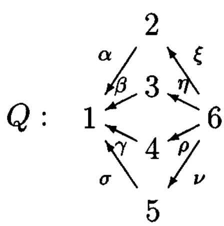
where $\mathscr{P}_1$ is the preprojective component of Γ_{C_1}, $\mathscr{T}_1$ is a $\mathbb{P}_1(K)$-family of stable (homogeneous) tubes of Γ_{C_1}, $\mathscr{T}_2$ is a $\mathbb{P}_1(K)$-family of stable (homogeneous) tubes of Γ_{C_2}, $\mathcal{Q}_2$ is the preinjective component, and $\mathscr{C}$ is the connecting component of the form



obtained by the glueing of the preinjective component of Γ_{C_1} with the preprojective component of Γ_{C_2} using the indecomposable projective-injective A-module P of dimension-vector $\mathbf{d} = (1, 2, 1)$. Observe that $\mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$, where $\mathbf{h}_1 = (1, 1, 0)$ and $\mathbf{h}_2 = (0, 1, 1)$ are generators of the radicals of χ_{C_1} and χ_{C_2}, respectively. Moreover, $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_A = 1$ and $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle_A = 0$, and it follows from Theorem 2 that $\operatorname{mod}_A(\mathbf{d})$ is neither normal nor irreducible. On the other hand, any indecomposable A-module nonisomorphic to P is either a C_1-module or a C_2-module. Since C_1 and C_2 are hereditary algebras we conclude that $\operatorname{mod}_A(\mathbf{e})$ is normal and irreducible for the remaining dimension-vectors $\mathbf{e}$ of indecomposable A-modules.

5.2

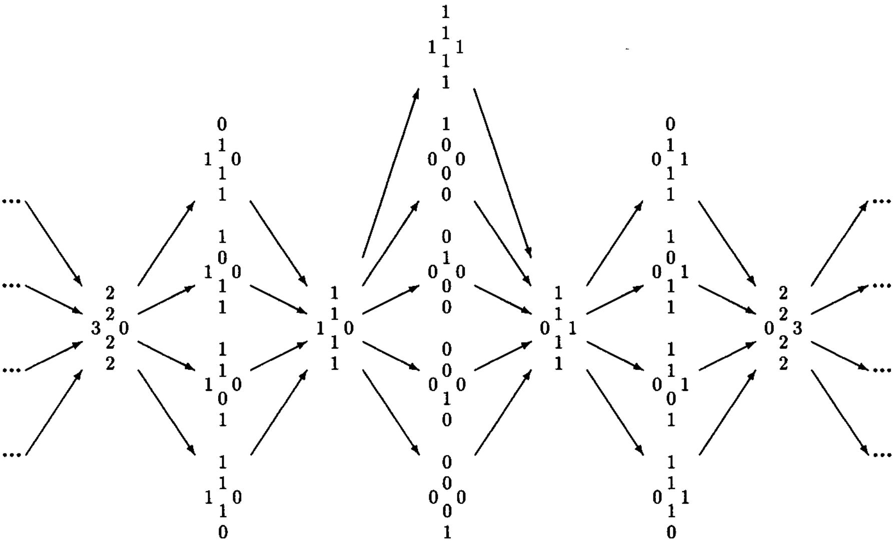
Let $A = KQ/I$ be the found quiver algebra given by the quiver



and the ideal I in KQ generated by $\alpha\xi - \beta\eta$, $\beta\eta - \gamma\rho$, $\gamma\rho - \sigma\nu$. Denote by C_1 (respectively, C_2) the tame hereditary algebra of type $\tilde{\mathbb{D}}_4$ given by the vertices 1, 2, 3, 4, 5 (respectively 2, 3, 4, 5, 6). Then A is a tame tilted algebra whose Auslander–Reiten quiver is of the form

$$\Gamma_A = \mathscr{P}_1 \vee \mathscr{T}_1 \vee \mathscr{C} \vee \mathscr{T}_2 \vee \mathscr{Q}_2,$$

where $\mathscr{P}_1$ is the preprojective component of Γ_{C_1}, $\mathscr{T}_1$ is the $\mathbb{P}_1(K)$-family of stable tubes of Γ_{C_1}, $\mathscr{T}_2$ is the $\mathbb{P}_1(K)$-family of stable tubes of Γ_{C_2}, $\mathscr{Q}_2$ is the preinjective component of Γ_{C_2}, and $\mathscr{C}$ is the connecting component of the form



obtained by the glueing of the preinjective component of Γ_{C_1} with the preprojective component of Γ_{C_2} using the indecomposable projective-injective A-module P of dimension-vector

$$\mathbf{d} = \begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}.$$

Observe that $\mathbf{d} \neq \mathbf{h}_1 + \mathbf{h}_2$, where

$$\mathbf{h}_1 = \begin{smallmatrix} & 1 & \\ & 1 & \\ 2 & & 0 \\ & 1 & \\ & 1 & \end{smallmatrix} \qquad \text{and} \qquad \mathbf{h}_2 = \begin{smallmatrix} & 1 & \\ & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{smallmatrix}$$

are generators of the radicals of χ_{C_1} and χ_{C2}. Hence P is an ordinary internal directing A-module. Moreover, any indecomposable A-module nonisomorphic to P is either a C_1-module or a C_2-module. Since C_1 and C_2 are hereditary algebras, we conclude that the module varieties $\operatorname{mod}_A(\mathbf{e})$

given by the dimension-vectors $\mathbf{e}$ of all indecomposable A-modules are irreducible, complete intersections and normal.

5.3

Finally, let $A = KQ/I$ be the bound quiver algebra given by the quiver

$$Q: \quad \begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & 5 \arrow[dr, "\xi"] & \\ & 3 \arrow[r, "\gamma"] & 4 \arrow[ur, "\sigma"] \arrow[dr, "\delta"'] & & 7 \\ 2 \arrow[ur, "\beta"'] & & & 6 \arrow[ur, "\eta"'] & \end{tikzcd}$$

and the ideal I generated by $\xi\sigma\gamma - \eta\delta\gamma$ and $\xi\sigma\gamma\alpha$. Denote by C_1 the tame hereditary algebra of type $\tilde{\mathbb{A}}_4$ given by the vertices 4, 5, 6, 7, and by C_2 the tame hereditary algebra of type $\tilde{\mathbb{D}}_5$ given by ver tices 1, 2, 3, 4, 5, 6. Moreover, let B_1 be the convex subcategory of A given by all vertices except 1. Then B_1 is a tilted algebra of type $\tilde{\mathbb{D}}_5$ being the tubular extension of C_1 using the simple homogeneous module.

$$\begin{tikzcd} & K \arrow[dr, "1"] & \\ K \arrow[ur, "1"] \arrow[dr, "1"'] & & K \\ & K \arrow[ur, "1"'] & \end{tikzcd}$$

Moreover, A is the one-point extension $B_1[R]$ of B_1 by the indecomposable preinjective B_1-module R of the form

$$\begin{tikzcd} & & & K \arrow[dr] & \\ & K \arrow[r, "1"'] & K \arrow[ur, "1"] \arrow[dr, "1"'] & & 0 \\ 0 \arrow[ur] & & & K \arrow[ur] & \end{tikzcd}$$

Further, the radicals of χ_{C_1} and χ_{C_2} are generated respectively by

$$\mathbf{h}_1 = \begin{smallmatrix} 0 & & & 1 & \\ & 0 & 1 & & 1 \\ 0 & & & 1 & \end{smallmatrix} \qquad \text{and} \qquad \mathbf{h}_1 = \begin{smallmatrix} 1 & & & 1 & \\ & 2 & 2 & & 0 \\ 1 & & & 1 & \end{smallmatrix}.$$

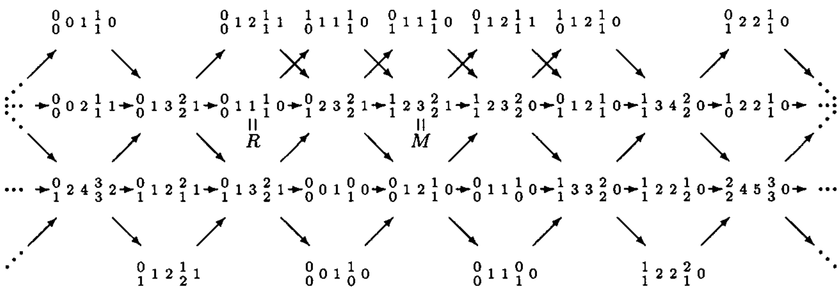
Then it follows (see also [18, (1.5)]) that A is a tame tilted algebra whose Auslander–Reiten quiver is of the form

$$\Gamma_A = \mathscr{P}_1 \vee \mathscr{T}_1 \vee \mathscr{C} \vee \mathscr{T}_2 \vee \mathscr{Q}_2,$$

where $\mathscr{P}_1$ is the preprojective component of Γ_{C_1}, $\mathscr{T}_1$ is the $\mathbb{P}_1(K)$-family of ray tubes of Γ_{B_1}, $\mathscr{T}_2$ is the $\mathbb{P}_1(K)$-family of stable tubes of Γ_{C_2}, $\mathscr{Q}_2$ is the preinjective component of Γ_{C_2}, and the connecting component $\mathscr{C}$ contains a unique sincere indecomposable (directing) module M and

$$\mathbf{dim}\, M = \begin{smallmatrix} 1 & & & 2 & \\ & 2 & 3 & & 1 \\ 1 & & & 2 & \end{smallmatrix} = \begin{smallmatrix} 0 & & & 1 & \\ & 0 & 1 & & 1 \\ 0 & & & 1 & \end{smallmatrix} + \begin{smallmatrix} 1 & & & 1 & \\ & 2 & 2 & & 0 \\ 1 & & & 1 & \end{smallmatrix} = \mathbf{h}_1 + \mathbf{h}_2.$$

In fact, $\mathscr{C}$ is a glueing of the preinjective component of Γ_{B_1} with the preprojective component of Γ_{C_2}, and the neighborhood of M in $\mathscr{C}$ is



Moreover, it is easy to check that $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_A = 2$, and hence M is an ordinary internal directing A-module. Observe that if N is an arbitrary indecomposable directing A-module then N is not an exceptional internal directing A-module, and consequently $\operatorname{mod}_A(\mathbf{dim}\, N)$ is irreducible, normal, and a complete intersection. On the other hand, we note that the tubular family $\mathscr{T}_1$ has one nonstable tube containing indecomposable modules with the dimension-vectors

$$\begin{smallmatrix} 0 & & & n & \\ & 1 & n & & n \\ 0 & & & n & \end{smallmatrix} = \begin{smallmatrix} 0 & & & 0 & \\ & 1 & 0 & & 0 \\ 0 & & & 0 & \end{smallmatrix} + n\mathbf{h}_1, \qquad \begin{smallmatrix} 0 & & & n & \\ & 1 & n & & n \\ 1 & & & n & \end{smallmatrix} = \begin{smallmatrix} 0 & & & 0 & \\ & 1 & 0 & & 0 \\ 1 & & & 0 & \end{smallmatrix} + n\mathbf{h}_1, \qquad n \geq 1,$$

for which the associated modules varieties are, according to [1, Theorem 2], neither irreducible nor normal.

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