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Some convolution properties of analytic functions

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Abstract

An interesting criterion was given by Ponnusamy and Singh [S. Ponnusamy, V. Singh, Convolution properties of some classes of analytic functions, Internal Report, SPIC Science Foundation, 1990.] for convolution properties of functions in the class P_α. In this paper we shall get a new criterion for convolution properties of functions in this class.

Keywords: Hadamard product; Starlike functions; Convex functions

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic in the unit disc $U = \{z : |z| < 1\}$. And let S denote the subclass of A consisting of univalent functions f in U. Let S^* and K be the usual subclasses of S consisting of starlike and convex functions, respectively; that is,

$$S^* = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\} \qquad (z \in U) \tag{1.2}$$

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and

$$K = \left\{ f \in A : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\} \qquad (z \in U). \tag{1.3}$$

It is well known that

$$f(z) \in K \qquad \text{if and only if } zf'(z) \in S^*. \tag{1.4}$$

Let $(f * g)(z)$ denote the Hadamard product (convolution) of two functions f and g; that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.5}$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.6}$$

We say that a function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ belongs to the class P_α, if $p(z)$ satisfies the following condition:

$$\mathrm{Re}\, p(z) \geq \alpha, \qquad \alpha \leq 1 \quad \text{and} \quad (z \in U). \tag{1.7}$$

In this paper we shall get a new criterion for convolution properties of functions in the class P_α.

2. Main results

In order to show our main results we have to recall here the following lemmas.

Lemma 1 (*See [2]*). *Suppose that the function* $\Psi : C^2 \times U \to C$ *satisfies the condition* $\mathrm{Re}\,\Psi(\mathrm{i}x, y; z) \leq \delta$ *for all real* $x, y \leq -\frac{(1+x^2)}{2}$ *and all* $z \in U$. *If* $p(z) = 1 + p_1 z + \cdots$ *is analytic in* U *and*

$$\mathrm{Re}\,\Psi(p(z), zp'(z), z) > \delta, \qquad \text{for } z \in U, \tag{2.1}$$

then $\mathrm{Re}\, p(z) > 0$ *in* U.

Lemma 2 (*See [3]*). *Let* $\beta < 1$. *If the function* p *is analytic in* U, *with* $p(0) = 1$, *and*

$$\mathrm{Re}\,[p(z) + zp'(z)] > \beta, \qquad (z \in U) \tag{2.2}$$

then $\mathrm{Re}\, p(z) > (2\beta - 1) + 2(1 - \beta)\ln 2$, $z \in U$. *The result is sharp.*

Lemma 3 (*See [1]*). *For* $\alpha \leq 1$ *and* $\beta \leq 1$,

$$P_\alpha * P_\beta \subset P_\delta, \qquad \delta = 1 - 2(1 - \alpha)(1 - \beta). \tag{2.3}$$

The result is sharp.

With the help of the above lemmas, we derive

Theorem 1. *Let* $f, g \in A$, $\alpha, \beta < 1$. *If* $f' \in P_\alpha$, $g' \in P_\beta$ *and* $\Phi(z) = (f * g)(z)$ *then* $\Phi \in S^*$ *provided*

$$(1 - \alpha)(1 - \beta) < \frac{3}{8(\ln 2 - 1)^2 + 4}. \tag{2.4}$$

Proof. By hypotheses on f and g and Lemma 3 we obtain

$$\text{Re}\left((f' * g')(z)\right) = \text{Re}\,[\Phi' + z\Phi''] > 1 - 2(1-\alpha)(1-\beta). \tag{2.5}$$

By using Lemma 2, from (2.5) we have

$$\text{Re}\,\Phi'(z) > 1 + 4(1-\alpha)(1-\beta)(\ln 2 - 1), \qquad (z \in U). \tag{2.6}$$

From (2.6) and Lemma 2, we have

$$\text{Re}\,\frac{\Phi(z)}{z} > 1 - 8(1-\alpha)(1-\beta)(\ln 2 - 1)^2. \tag{2.7}$$

Now, we let $p(z) = \frac{z\Phi'(z)}{\Phi(z)}$ and $\lambda(z) = \frac{\Phi(z)}{z}$; then $p(z)$ is analytic in U and $p(0) = 1$,

$$\text{Re}\,\lambda(z) > 1 - 8(1-\alpha)(1-\beta)(\ln 2 - 1)^2. \tag{2.8}$$

A simple computation show that

$$\Phi' + z\Phi'' = \lambda(z)[p^2(z) + zp'(z)] = \Psi\left(p(z), zp'(z), z\right), \tag{2.9}$$

where $\Psi(u, v; z) = \lambda(z)(u^2 + v)$. Using (2.5) we have

$$\text{Re}\left[\Psi\left(p(z), zp'(z), z\right)\right] > 1 - 2(1-\alpha)(1-\beta),$$

for $z \in U$. Now for real $x, y \le -\frac{1}{2}(1 + x^2)$, we have

$$\text{Re}\,\{\Psi(\text{i}x, y, z)\} = (y - x^2)\text{Re}\,\lambda(z) \le -\frac{1}{2}(1 + 3x^2)\text{Re}\,\lambda(z) \le -\frac{1}{2}\text{Re}\,\lambda(z) \tag{2.10}$$

for $z \in U$. From (2.8) and (2.10) we get

$$\text{Re}\,\{\Psi(\text{i}x, y, z)\} \le 1 - 2(1-\alpha)(1-\beta),$$

for all $z \in U$. Thus by Lemma 1, Re $p(z) > 0$, in U; that is, $\Phi(z) \in S^*$. □

Corollary. *Let* $f, g \in A$, $\alpha, \beta < 1$. *If* $f' \in P_\alpha$, $g' \in P_\beta$ *and*

$$\xi(z) = \int_0^z \frac{(f * g)(t)}{t}\text{d}t,$$

then $\xi \in K$ *provided*

$$(1-\alpha)(1-\beta) < \frac{3}{8(\ln 2 - 1)^2 + 4}.$$

The proof is now immediate because $z\xi'(z) = \Phi(z)$.

Theorem 2. *Let* $f, g, h \in A$, $\alpha, \beta, \gamma < 1$. *If* $f' \in P_\alpha$, $g' \in P_\beta$, $h' \in P_\gamma$ *and* $\zeta(z) = (f * g * h)(z)$ *then* $\zeta \in K$ *provided*

$$(1-\alpha)(1-\beta)(1-\gamma) < \frac{3}{16(\ln 2 - 1)^2 + 8}. \tag{2.11}$$

Proof. It is sufficient to show that $\eta = z\zeta' \in S^*$. By hypotheses on f, g, h, and Lemma 3, we obtain

$$\text{Re}\left((f' * g' * h')(z)\right) = \text{Re}\,[\eta' + z\eta''] > 1 - 4(1-\alpha)(1-\beta)(1-\gamma), \tag{2.12}$$

and the proof is completed similarly to Theorem 1. □

Theorem 3. *Let $f, g \in A$. If*

$$\text{Re}\left((f * g)'(z)\right) > 1 - \frac{3}{4(\ln 2 - 1)^2 + 2} \tag{2.13}$$

then

$$F(z) = \int_0^z \frac{(f * g)(t)}{t} \mathrm{d}t \in S^*.$$

Proof. From the definition of F we see that

$$\text{Re}\left((f * g)'(z)\right) = \text{Re}\left(F'(z) + zF''(z)\right) > 1 - \frac{3}{4(\ln 2 - 1)^2 + 2},$$

and the proof is completed similarly to Theorem 1. □

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