Iteration method in linear elasticity of random structure composites containing heterogeneities of noncanonical shape

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\textbf{A B S T R A C T}

We consider a linearly elastic composite medium, which consists of a homogeneous matrix containing a statistically inhomogeneous random set of heterogeneities of arbitrary shape. The general integral equations connecting the stress and strain fields in the point being considered with the stress and strain fields in the surrounding points are obtained for the random fields of heterogeneities. The method is based on a recently developed centering procedure where the notion of a perturbator is introduced and statistical averages are obtained without any auxiliary assumptions such as, e.g., effective field hypothesis implicitly exploited in the known centering methods. Effective elastic moduli and the first statistical moments of stresses in the heterogeneities are estimated for statistically homogeneous composites with the general case of both the shape and inhomogeneity of the heterogeneities moduli. The explicit new representations of the effective moduli and stress concentration factors are built by the iteration method in the framework of the quasicristallite approximation but without basic hypotheses of classical micromechanics such as both the EFH and “ellipsoidal symmetry” assumption. Numerical results are obtained for some model statistically homogeneous composites reinforced by aligned identical homogeneous heterogeneities of noncanonical shape. Some new effects are detected that are impossible in the framework of a classical background of micromechanics.

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1. Introduction

With the enhancement in available computer hardware and software, numerical techniques offer a powerful tool for modeling the mechanical behavior of composite materials (CMs). A considerable number of methods are known in the linear theory of composites that yield the effective elastic constants and stress field averages in the components. Appropriate, but by no means exhaustive, references are provided by the reviews Willis (1981), Mura (1987), Nemat-Nasser and Hori (1993), Torquato (2002), Milton (2002), Buryachenko (2007) and Li and Wang (2008). It appears to-
of micromechanics such as, e.g., the effective medium method and their modifications, differential scheme, Mori–Tanaka method, and, needless to say, the multiparticle effective field method (MEFM, see Buryachenko, 2007).

In most micromechanical studies heterogeneities are reduced to ellipsoidal shape allowing one to implement the analytical Eshelby (1957) (see the references related with generalization of this solution to non-ellipsoidal inclusion in Zhou et al., 2010) solution in one or the other micromechanical scheme. However, research shows that composite mechanical properties greatly depend on the fiber of nonellipsoidal shape (Antretter and Fisher, 1997; Zhou et al., 2005). To obtain a better load transfer mechanism and better stress distribution, many different fiber geometries have been experimented and analyzed. Kozaczek et al. (1995) studied a single non-ellipsoidal inclusion in an infinite medium, which can be considered as a limiting case of a dilute concentration of inclusions. They demonstrated that the shape of the inclusion plays a role in the stress distribution in the grain boundary region; sharp corners raise stress more effectively than rounded edges of oblong-shaped precipitates. CM reinforced by shaped head fibers provide additional mechanical locking in comparison with straight fibers. Zhou (1994) was likely the first to introduce this concept and showed that matrix composites with dumbbell-shaped steel wires have higher strength than those reinforced by straight wires. Tsai et al. (2005) analyzed stress profiles induced during pullout of two chosen shaped head families using a finite element analysis (FEA). Bagwell and Wetherhold (2005) (see also Wetherhold et al., 2007) investigated shaped fiber ends produced by end-impacting and knotting fibers to facilitate anchoring, while Parthasarathy et al. (2007) analyzed improving the strength and fracture toughness of CM by introducing special interface elements.

2. Preliminaries

2.1. Basic equations

Let a linear elastic body occupy an open simply connected bounded domain \( \Omega \subset \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \) and with an indicator function \( \chi \) and space dimensionality \( d = 2 \) or \( d = 3 \) for 2-D and 3-D problems, respectively. The domain \( \Omega \) contains a homogeneous matrix \( \nu^{(0)} \) and, in general, a statistically homogeneous set \( X = \{ \nu \} \) of heterogeneities with indicator functions \( \chi \) and bounded by the closed smooth surfaces \( \Gamma_i, (i = 1, 2, \ldots, N) \). It is assumed that the heterogeneities can be grouped into components (phases) \( \nu^{(i)} \) with identical mechanical and geometrical properties (such as the size, shape, orientation, and microstructure of heterogeneities). For the sake of definiteness, in the 2-D case we will consider a plane-strain problem. At first no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the heterogeneities.\(^1\)

We will consider the local basic equations of elastostatics of composites

\[
\nabla \nu\mathbf{x} = \mathbf{0},
\]

\[
\sigma\mathbf{x} = L\mathbf{x}\varepsilon\mathbf{x}, \quad \text{or} \quad \varepsilon\mathbf{x} = M\mathbf{x}\sigma\mathbf{x},
\]

\[
\varepsilon\mathbf{x} = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2, \quad \nabla \times \varepsilon\mathbf{x} \times \nabla = \mathbf{0},
\]

where \((\cdot)^T\) denotes matrix transposition and \(\mathbf{x}\) is the vector product. The local stiffness and compliance tensors \(L\mathbf{x}\) and \(M\mathbf{x} \equiv L^{-1}\mathbf{x}\) are the known phase stiffness and compliance fourth-order tensors, and the common notation for contracted products has been employed: \[L_{ij}^g = L_{ijkl} \chi \delta_{ij} \theta_k \] (i,j,k,l = 1,...,d). In particular, for isotropic constituents the local stiffness tensor \(L\mathbf{x}\) is given in terms of the local bulk modulus \(k\mathbf{x}\) and the local shear modulus \(\mu\mathbf{x}\):

\[
L\mathbf{x} = (\delta \mathbf{x}, 2 \mu) = \delta \mathbf{x} + 2 \mu \mathbf{x} \mathbf{n}_1 + 2 \mu \mathbf{x} \mathbf{n}_2, \quad \beta\mathbf{x} = \beta_0\mathbf{x} \delta\mathbf{x},
\]

\[
\mathbf{N}_1 = \delta \mathbf{x} \delta\mathbf{x} / d, \quad \mathbf{N}_2 = \mathbf{I} - \mathbf{N}_1, \quad \delta\mathbf{x} = \mu\mathbf{x} / (1 - 2 \mu) \theta d, \quad \mathbf{I} = \text{unit second-order and fourth-order tensors},
\]

and \(\delta\mathbf{x}\) denotes tensor product. For the fiber composites it is the plane-strain bulk modulus \(k_{33}\mathbf{x}\) instead of the 3-D bulk modulus \(k_{33}\mathbf{x}\) – which plays the significant role: \(k_{33}^g = k_{33}^g + \mu_{33}^g / 3, \mu_{33}^g = \mu_{33}^g\). All tensors \(\mathbf{g}\mathbf{x} = \mathbf{g}(\mathbf{x}) \mathbf{M}\mathbf{x}\) of material properties are decomposed as \(\mathbf{g} \equiv g^{(0)} \mathbf{g}(\mathbf{x}) = g^{(0)} + g^{(m)}(\mathbf{x}) \) at \(\mathbf{x} \in \nu^{(m)}\). The introduction of jumps of material properties allows one to define the stress tensor and strain \(\eta\mathbf{x}\) polarization tensors \((\mathbf{x} \in \nu^{(m)}\).

\(^1\) It is known that for 2-D problems the plane-strain state is only possible for material symmetry no lower than orthotropic (see e.g. Lekhnitskii, 1963) that will be assumed hereafter in 2-D case.
which are simply a notational convenience and vanish inside the matrix \( \tau(x) = L(x) \xi(x) \) and \( \eta(x) = M(x) \sigma(x) \),

\[
(2.5)
\]

respectively, equals 1 at the interface boundaries, i.e., \( \mu^e \big|_{\Gamma_{\text{int}}} = 0 \) and \( \mu^e \big|_{\partial \Omega} = 0 \) on the interface boundary \( \Gamma_{\text{int}} \) and \( \partial \Omega \), respectively. The interface function \( \mu^e \) is defined as the jump of the traction \( \tau(x) \) acting on any plane normal to \( \Gamma_{\text{int}} \) and \( \Gamma_{\partial \Omega} \).

We assume that the phases are perfectly bonded, so that the displacements and the traction components are continuous across the interface boundaries, i.e., \( \mu^e \big|_{\Gamma_{\text{int}}} = 0 \) and \( \mu^e \big|_{\Gamma_{\partial \Omega}} = 0 \) on the interface boundary \( \Gamma_{\text{int}} \) and \( \Gamma_{\partial \Omega} \), respectively. The interface function \( \mu^e \) is defined as the jump of the traction \( \tau(x) \) acting on any plane normal to \( \Gamma_{\text{int}} \) and \( \Gamma_{\partial \Omega} \).

The boundary conditions at the interface boundaries will be considered together with the boundary conditions on \( \Gamma_0 \) with the unit outward normal \( n(x) \). Without loss of generality, the homogeneous traction boundary conditions (\( \Gamma_0 \)) are considered

\[
t^i(x) = \sigma^i(x)n(x), \quad \sigma^i = \text{const.}, \quad x \in \Gamma_0,
\]

where \( \sigma^i \) are the given constant symmetric tensors of the macroscopic stress. We will consider the interior problem when the body occupies the interior domain with respect to \( \Gamma_0 \).

2.2. Statistical description of the composite microstructure

It is assumed that the representative macrodomain \( w \) contains a statistically large number of realizations \( x \) (providing validity of the standard probability technique) of heterogeneities \( \nu_k \in \nu^e \) of the component \( \nu^e \) (i.e., \( \nu^e = (1, 2, \ldots, k = 1, 2, \ldots, N) \). A random parameter \( x \) belongs to a sample space \( A \), over which a probability density \( p(x) \) is defined (see, e.g., Willis, 1981). For any given \( x \), any random function \( g(x, x) \) (e.g., \( g = V, V^e, \sigma, \epsilon \)) is defined explicitly as one particular member, with label \( x \), of an ensemble realization. Then, the mean, or ensemble average is defined by the angle brackets enclosing the quantity \( g \)

\[
\langle g \rangle(x) = \int_A g(x, x) p(x) dx.
\]

(2.7)

No confusion will arise below in notation of the random quantity \( g(x) \) if the label \( x \) is removed. One treats two material length scales (see, e.g., Torquato, 2002): the macroscopic scale \( L \) characterizing the extent of \( w \), and the microscopic scale \( a \), related with the heterogeneities \( \nu_k \). Moreover, one supposes that the applied field varies on a characteristic length scale \( \Lambda \). The limit of our interests for both the macroscopic and field one

\[
L \gg \Lambda \gg a.
\]

(2.8)

All the random quantities under discussion are described statistically homogenous random fields. For the alternative description of the random structure of a composite material let us introduce a conditional probability density \( \phi(x|v_i, \xi_x|v_i, \xi_x) \), which is a probability density to find the \( i \)-th heterogeneity with the center \( \xi_x \) in the domain \( v_i \) with fixed heterogeneity \( \nu_k \) with the centers \( \xi_x \). The notation \( \phi(v_i, \xi_x|v_i, \xi_x) \) denotes the case \( \xi_x \neq \xi_x \). To prevent overlapping of different inclinations \( \phi(v_i, \xi_x|v_i, \xi_x) = 0 \) for values of \( \xi_x \) lying inside the "excluded volumes" \( \cup_{v_m} p_{v_m} \), where \( p_{v_m} \) is the indicator function of the "excluded volumes" of \( \nu \), with respect to \( v_m \); it is usually assumed that \( p_{v_m} \big|_{\cup_{v_m} w_{v_m}} = 1 \), and \( \phi(v_i, \xi_x|v_i, \xi_x) \), as \( \xi_x \rightarrow \xi_x \rightarrow \infty \) (since no long-range order is assumed), \( \phi(v_i, \xi_x) \) is a number density, \( n(x) = n \) of component \( \nu \) at the point \( x \) and \( c^{(i)} \) is the concentration, i.e., volume fraction, of the component \( \nu_i \in \nu : c^{(i)} = (\nu) = \nu_i = \text{mes} \nu_i \) (i = 1, 2, \ldots). Averages \( \langle \rangle \) and \( \langle \rangle \) will be used for the average and for the conditional average taken for the ensemble of a statistically homogeneous field \( X = (v) \) at the point \( x \) on the condition that there are no overlapings at the points \( x \). The notation \( \langle \rangle \big|_{\nu_i} \) indicates the average volume over an ensemble realization of surrounding inclusions at the fixed \( \nu_i \) whereas \( \langle \rangle \big|_{\nu_i} \) indicates the volume average over an inclusion \( \nu_i \) in a single realization and \( \langle \rangle \big|_{\nu_i} \equiv (\langle \rangle)_{\nu_i} \).

3. General integral equation

3.1. Stress distributions for one heterogeneity inside macrodomain \( w \)

At first we consider a homogeneous domain \( w \) subjected to the boundary conditions (2.6) which generate a stress distribution inside domain \( w \) (see, e.g., Brebbia et al., 1984; Ballas et al., 1989)

\[
\sigma_0(x) = \int_{\Gamma_0} [L^{(0)} \nabla \sigma^0(x-s)u_0(s)-L^{(0)} \nabla \sigma^0(x-s)u_0(s)]ds,
\]

(3.1)

where \( \sigma_0 = L^{(0)} u_0, \sigma_0 = [V u_0 + (V u_0)]/2, G \) is the infinite body Green's function of the Navier equation with homogeneous elastic modulus tensor \( L^{(0)} \), defined by

\[
\nabla \left( \frac{1}{2} \nabla \sigma^0 + (\nabla \sigma^0)^T \right) = -\delta(x),
\]

(3.2)

of order \( O(1/|x|) \) as \( |x| \rightarrow \infty, \delta(x) \) is the Dirac delta function, and the tensor of the "fundamental traction" \( T \) on \( \Gamma_0 \) (also called the Kupradze tensor) associated with the tensor of "fundamental displacement" \( G \) is given by

\[
T_{u}(x) = L_{ii} \delta_{ij} (\frac{\partial G_{u_j}(x-s)}{\partial x_i} - \frac{\partial G_{u_i}(x-s)}{\partial x_j}),
\]

(3.3)

Let us assume that the domain \( w \) contains one heterogeneity \( \nu_k \subset w \), we define the stress perturbator \( \sigma_0^k(x-x_k, \eta) \) and displacement one \( u_0^k(x-x_k, \eta) \) as the perturbations introduced by the heterogeneity \( \nu_k \)

\[
\sigma_0^k(x-x_k, \eta) = \int_{w} G(x-y, \eta) \nu(y) V_k(y) dy,
\]

(3.6)

\[
\sigma_0^k(x-x_k, \eta) = \int_{w} \nabla G(x-y, \eta) \nu_k(y) V_k(y) dy.
\]

(3.7)

\[
\Gamma_0(x-x_k, \tau) = \int_{w} \nabla G(x-y, \eta) \nu_k(y) V_k(y) dy.
\]

\[
\Gamma_0(x-x_k, \tau) = \int_{w} \nabla G(x-y, \eta) \nu_k(y) V_k(y) dy.
\]

Hereafter

\[
\Gamma_0(x-x_k, \tau) = -\int_{\Gamma_0} \delta_0(x-y) \nabla \nabla G(x-y, \eta) L^{(0)}
\]

are the Green tensors for the strains and stresses, respectively.

The Cauchy data \( \{u_0(x), \sigma_0(x)\} \) at the smooth surface \( \Gamma_0 \) can be found, e.g., from the conventional BIE taking the limit \( x \rightarrow \Gamma_0 \)

\[
\frac{1}{2} u_0(x) = \int_{\Gamma_0} [G(s-x, \xi_0(s) \xi_0)|T - |s, \xi_0(s)|u_0(s)|ds + \Sigma(x-s-x_k)]
\]

(3.9)
Without loss of generality, the traction boundary conditions \( (\Gamma_1 = \Gamma_0) \) (2.6) and isotropic matrix are considered.

### 3.2. General integral equations

Let us consider an arbitrary random realization \( x \) of inclusions in the domain \( w \) described by an analog of Eq. (3.4) generalized to any number of inhomogeneities. Then the centering method (see for details Buryachenko and Brun, 2012b) subtracting from both sides of the mentioned equation their statistical averages leads to

\[
\sigma(x, x) = \langle \sigma \rangle(x) + \int [L_{r}^{iL}(x - x_{0}, \eta)V_{e}^{0}(x_{0}, x) - \langle L_{r}^{iL}(x - x_{0}, \eta) \rangle(x_{0})] \, dx_{0}.
\tag{3.10}
\]

Eq. (3.10) is only obtained at the internal points \( x \in \Gamma \) of the macromodel \( w \) at sufficient distance from the boundary

\[
a \ll |x - s|, \quad \forall s \in \Gamma.
\tag{3.11}
\]

This Eq. (3.10) is not valid in a “boundary layer” region close to the surface \( s \in \Gamma \) where boundary data \( (u(o(s), t_{e}(s)) \) not prescribed by the boundary conditions depend on perturbations introduced by all inhomogeneities (see (3.4)), and, therefore \( \sigma_{w}(x) = \sigma_{0}(x, x). \)

The volume integral in (3.10) converges absolutely because the integrand in the square brackets in Eq. (3.10) is of order \( O(|x - y|^{-d}) \) as \( |x - y| \to \infty \). For no long-range order assumed, the function \( \phi(v_{r}, x, \eta) \) decays at infinity sufficiently rapidly and guarantees an absolute convergence of the integral involved. Therefore, for \( x \in w \) far enough from the boundary \( \Gamma_0 \), the right-hand side integral in (3.10) does not depend on the shape and size of the domain \( w \), and it can be replaced by the integrals over the whole space \( \mathbb{R}^d \) (the domain integration \( \mathbb{R}^d \) will be omitted hereafter for simplicity of notation).

Let the inclusions \( v_{1}, \ldots, v_{n} \) be fixed and let us consider some conditional statistical averages of the general integral Eq. (3.10) leading to an infinite system of new integral equations. The first equation of this system \( (n = 1) \) can be rearranged as \( (x \in v_{1}) \)

\[
\langle \sigma \rangle|_{v_{1}}, x_{1})(x) = \langle \sigma \rangle|_{v_{1}}, x_{1})(x) + \langle L_{r}^{iL}(x - x_{1}, \eta) \rangle|_{v_{1}, x_{1}}.
\tag{3.12}
\]

\[
\langle \sigma \rangle|_{v_{1}}, x_{1})(x) = \langle \sigma \rangle(x) + \int \langle \langle L_{r}^{iL}(x - x_{1}, \eta) \rangle \, dx_{1} \rangle|_{v_{1}, x_{1}}.
\tag{3.13}
\]

The definitions of the effective field \( \sigma(x) \), as well as its statistical average \( \langle \sigma \rangle(x) \) are nothing more than a notation convention for different terms of the corresponding infinite systems Eqs. (3.12) and (3.13).

If the perturbator \( L_{r}^{iL}(x - x_{1}, \eta) \) is expressed in term of the volume integral (3.6), then Eq. (3.10) is reduced to the known equation obtained in Buryachenko (2010b), namely

\[
\sigma(x, x) = \langle \sigma \rangle(x) + \int [\Gamma(x - y)|\eta|y, x) - \langle \Gamma(x - y)|\eta \rangle(y)] \, dy.
\tag{3.14}
\]

Buryachenko (2010b,c) demonstrated both the qualitative and quantitative advantages of Eq. (3.12) with respect to the classical ones (see for references and details Buryachenko, 2007)

\[
\sigma(x, x) = \langle \sigma \rangle(x) + \int \Gamma(x - y)|\eta|y, x) - \langle \eta \rangle|y \rangle \, dy.
\tag{3.15}
\]

The new exact Eqs. (3.10) and (3.14) forming a new background of micromechanics (see for details Buryachenko, 2010a,b,c, 2011a,b; Buryachenko and Brun, 2011, 2012a,b) yield the known approximate one (3.15) only with some additional assumptions

\[
\begin{align*}
\mathbf{U}(x - y)|\tau(y) = \mathbf{U}(x - y)|\tau(y), \quad \langle \Gamma(x - y)|\eta \rangle(y) = \Gamma(x - y)|\eta \rangle(y),
\end{align*}
\tag{3.16}
\]

which are fulfilled at \( |x - y| \to \infty \). The advantages of Eq. (3.10) with respect to (3.15) will be considered in Section 6.

The effective compliance \( M \) in the governing equation \( (\tau) = M'(\sigma) \) is expressed through the stress concentrator factor \( B'\mathbf{x}) \):

\[
M' = M^{(0)} + \{M^i\mathbf{B}_i\}, \quad \langle \sigma \rangle_i(x) = B^i(x)|\sigma\rangle.
\tag{3.17}
\]

### 4. Some classical hypotheses and approaches

#### 4.1. Effective field hypothesis

In order to approximately solve the exact system we now apply the so-called effective field hypothesis (EFH) which is the main approximate hypothesis of many micromechanical methods:

**Hypothesis 1a, H1a.** Each heterogeneity \( v_i \) is located in the field \( \sigma_i(y) \equiv \langle \sigma \rangle_i(x) \quad (y \in v_i) \),

\[
\tag{4.1}
\]

which is homogeneous over the heterogeneity \( v_i \).

In some methods (such as, e.g., the MEFM) this basic hypothesis H1a is complimented by a satellite hypothesis presented in the form of the perturbar rather than the Green’s function:

**Hypothesis 1b, H1b.** The operator \( L_{r}^{i}(x - x_{i}, \eta) \) of perturbation generated by the heterogeneities \( v_i \) at the point \( x \neq v_i \) is reduced to the decoupled tensorial multiplications

\[
L_{r}^{iL}(x - x_{i}, \eta) = L_{r}^{i}(x - x_{i}, \eta)|_{(\eta)_{99}}.
\tag{4.2}
\]

For the perturbator \( L_{r}^{i}(x - x_{i}, \eta) \), (3.6) expressed through the Green’s functions, the assumptions (4.2) is reduced to the known ones (see e.g. Buryachenko, 2007) with the perturbator factors \( (x \neq v_i) \), \( y \in v_i \)

\[
L_{r}^{iL}(x - x_{i}) = \{\Gamma(x - y)|_{(\eta)_{99}} \} \equiv T_{r}^{iL}(x - x_{i}).
\tag{4.3}
\]

where the tensor \( T_{r}^{iL}(x - x_{i}) \) is written in terms of both the internal \( S_i(x) \) and external \( S_e(x) \) Eshelby (1957) tensors (see also for references Buryachenko, 2007) \( (x \in \mathbb{R}^d) \)

\[
T_{r}^{iL}(x - x_{i}) = \begin{cases}
\rho_v^{'L}L^{(0)}S_v(x), & \text{if } x \neq v_i \\
-\rho_eS_v & \text{if } x \in v_i.
\end{cases}
\tag{4.4}
\]

and \( Q_v = L^{(0)}(1 - S_v) \). The tensors \( T_{r}^{iL}(x - x_{i}) \), and \( T_{r}^{iL}(x - x_{i}) \equiv (T_{r}^{iL}(x - x_{i}))_{99} \quad (x \in v_i) \) are known and have an analytical representations for the spherical inclusions (in both 2D and 3D cases) in an isotropic matrix (see for references Buryachenko, 2007) regardless of whether the inclusions are coated or uncoated; the case of ellipsoidal inclusions of different sizes and orientations is analyzed by Franciosi and Lebail (2004). The representations (4.2) are only exact for both the homogeneous ellipsoidal inclusions and homogeneous loading \( \tau(x), \sigma(x) \equiv \text{const., } x \in v_i \) (4.1), otherwise the representations (4.2) are just the approximations which are asymptotically fulfilled at \( |x - x_i| \to \infty \).

It should be mentioned that the popular formulation of the EFH (hypothesis H1) is a combination of the hypotheses H1a and H1b.

#### 4.2. A single inhomogeneity in an infinite matrix

According to hypothesis H1a and in view of the linearity of the problem, there exist the fourth-rank tensors \( B_i(x) \) and \( R_i(x) \) defined in a full space \( x \in \mathbb{R}^d \) rather than in the domain \( v_i \)

\[
\sigma(x) = B_i(x)|\tau(x)|, \quad \varepsilon_i(x) = R_i(x)|\sigma(x)|,
\tag{4.4}
\]

where \( R_i(x) = v_iM^{(i)}|B_i(x)|, \) and the operator \( \mathcal{L}^i(x - x, \tau) \) is decomposed
\[ \mathcal{L}_i^m(x-x, \sigma) = \mathcal{L}_i^m(x-x, \sigma) \mathcal{B} = \mathcal{L}_i^m(x-x) = \mathcal{B}(x) - I. \] (4.5)

According to Eshelby’s (1957) theorem there is the following relation between the averaged tensors (4.5) \( \mathbf{R} = \mathbb{P} \mathbf{Q}^{-1}(1 - \mathbf{B}_i) \), where \( \mathbf{g} = \mathbf{g}(x) \) \( \mathbf{g} \) stands for \( \mathbf{B}, \mathbf{R} \). For example, for the homogeneous ellipsoidal domain with the semi-axes \( a = (d_1, \ldots, d_3) \),

\[ \mathbf{M}_i^m(x) = M_i^m(x) \equiv \text{const}, \quad x \in \nu_i; \quad \nu_i = \{x: |(a_1^2 - x_1^2)^{-1} < 1\}, \] (4.6)

we obtain \( \mathbf{B}_i = (1 + \mathbf{Q}_i \mathbf{M}_i^m)^{-1} \). In the general case of coated inclusions \( \nu_i \), the tensors \( \mathbf{B}_i(x) \) can be found by the transformation method by Dvorak and Benveniste (1992) (see for references and details Buryachenko, 2007).

It should be mentioned that the operator \( \mathcal{L}_i^m(x-x, \sigma) \) has the physical interpretation of perturbation introduced by a single heterogeneity \( \nu_i \) in the infinite homogeneous matrix subjected by the effective field \( \mathbf{\sigma}_i(x) \); where at no restrictions are imposed on the inhomogeneities of effective fields. The mentioned perturbation can be found by any available numerical method, such as e.g. the volume integral equation (VIE), boundary element method (BEM), FEA, hybrid FEA–BEM, multipole expansion method, complex potential method among others (see for references Buryachenko, 2007). Each method has its advantages and disadvantages and it is crucial for the analyst to be aware of their range of applications. In particular, the VIE method enables one to restrict discretization to the inclusions only (in contrast to the FEA), and an inhomogeneous structure of inclusions (see, e.g., Chen et al., 1990; Jayaraman and Reifsneider, 1992; You et al., 2006) presents no problem in the framework of the same numerical scheme (compared to the standard BIE method). The first method used for solution of the counterpart of Eqs. (3.12) and (3.13) was the VIE method (see Buryachenko, 2010b) which has well developed routines for the solution of integral equations (such as, e.g., the iteration method and the quadrature schemes) and allows to analyze arbitrary inhomogeneous effective fields. On the contrary, the VIE method is quite time-consuming and no optimized commercial software exists for its application. Because of this, we at first will make use of the FEA which is supported by well developed commercial softwares and gives strong advantages in term of CPU-time.

The FEA is very effective for estimating the perturbation factor \( \mathcal{L}_i^m(x-x, \sigma) \) [or that is equivalent, the stress concentrator \( \mathbf{B}_i(x) \) (4.4)] at the constant effective fields considered now. Indeed, let the inclusion \( \nu_i \) be subjected to homogeneous remote stress \( \mathbf{\sigma} = \text{const} \) with a single nonzero component \( \sigma_i = 1; \) otherwise \( \sigma_k = 0 (j,k = 1, \ldots, 3d - 3; k \neq j) \). We assume that the stress field \( \mathbf{\sigma}(x) \) \( x \in \mathbb{R}^3 \) is estimated by the FEA. Then the tensor \( \mathbf{B}_i(x) \) is represented explicitly over the known stress field \( \mathbf{\sigma}(x) \) \( x \in \mathbb{R}^3 \):

\[ B_{ijm}(x) = \sigma_i \text{ for } \sigma_j = 1, \quad \sigma_k = 0 (j \neq k) \] (4.7)

where \( j, k, m = 1, \ldots, 3d - 3 \).

However, the FEA is only very effective for estimations of the perturber factor at the constant effective field considered above when a prescription of a homogeneous loading at the boundary of a large sample \( w \) is obvious. At the same time analysis of inhomogeneous effective fields is not so straightforward in the FEA. In such a case, the VIE method using a prescription of the effective field \( \mathbf{\sigma}(x) \) only inside the domain \( x \in \nu_i \) is found to be more effective. A situation is complicated by the fact that the kernel of the operator \( \mathcal{L}_i^m(x-x, \sigma) \) is singular at \( x = x \). The mentioned difficulty can be eliminated in the framework of a subtraction technique transforming Eqs. (3.4) and (3.6) in the following manner. Namely, let the inclusions \( \nu_i \) be fixed and loaded by the inhomogeneous effective field \( \mathbf{\sigma}(x) \):

\[ \mathbf{\sigma}(x) = \mathbf{\sigma}_i(x) + \int \Gamma(x-y) V_i(y) \mathbf{\eta}(y) dy. \] (4.8)

The difficulties with the troublesome singularities can be avoided if a rearrangement of Eq. (3.1) is performed in the spirit of a subtraction technique used in the modified quadrature method (see, e.g., Delves and Mohamed, 1985)

\[ \eta(x) = \mathbf{\eta}_i(x) + \int K_i(x, y) \|\mathbf{\eta}(y) - \mathbf{\eta}_i(x)\| dy, \quad x \in \nu_i, \] (4.9)

where \( \mathbf{\eta}_i(x) = \mathbf{\eta}_i(x) \Gamma_0(x) \mathbf{\eta}(y) \mathbf{\eta}_i(x) \) is called the effective strain polarization tensor in the inclusion \( \nu_i \) and (no sum on \( i \))

\[ K_i(x, y) = E_i(x) \Gamma_0(x - y) V_i(y), \] (4.10)

where the tensor \( E_i(x) = \mathbf{E}_i(x) \mathbf{\Gamma}_0(x) \mathbf{\eta}_i(x) \) is found by the FEA (see for details Buryachenko, 2007) analogously to Eq. (4.7) by the use of stress estimations inside domain \( \nu_i \) with the elastic moduli \( L^{(i)} \) and the constant fictitious eigenstrains \( \beta_i = 1, \beta_k = 0 (j \neq k) \) where \( j, k = 1, \ldots, 3d - 3 \). We rewrite Eq. (4.9) in symbolic form:

\[ \eta = \mathbf{\eta}_i + \mathbf{K} \eta, \] (4.12)

where \( (\mathbf{K} \eta)(x) = \int K_i(x, y) \mathbf{\eta}(y) dy \) defines the integral operator \( \mathbf{K} \)

\[ \mathbf{\eta} = \mathbf{\eta}_i \] (4.13)

where the inverse operator \( \mathbf{L}_i = (1 - \mathbf{K})^{-1} \) will be constructed by the iteration method based on the recursion formula

\[ \eta^{k+1} = \eta + \mathbf{K} \eta^k \] (4.14)

to construct a sequence of functions \( \eta^k \) that can be treated as an approximation of the solution of Eq. (4.12). We presented the point Jacobi (called also Richardson and point total-step) iterative scheme for ease of calculations. The details of the real iteration method used for the solution of Eq. (4.14) will be presented in Section 6. Usually the driving term of this equation is used as an initial approximation:

\[ \eta^0(x) = \mathbf{\eta}_i(x), \] (4.15)

which is exact for a homogeneous ellipsoidal inclusion subjected to remote homogeneous stress field \( \mathbf{\sigma}(x) \equiv \sigma = \text{const} \). when \( \mathbf{B}_i(x) = \mathbf{E}_i(x) = \text{const.} (x \in \nu_i) \). The sequence \( \eta^k \) (4.14) with arbitrary continuous \( \mathbf{K} \eta^k \) converges to a unique solution \( \eta \) if the norm of the integral operator \( \mathbf{K} \) turns out to be small "enough" (less than 1), and the problem is reduced to the computation of the integrals involved, the density of which is given. The desired for Eq. (3.4) operator \( \mathcal{L}_i(x-x, \sigma) \) can be presented through the found operator \( \mathbf{L}_i \)

\[ \mathcal{L}_i^m(x-x, \sigma) = \int \Gamma(x-y) L_i \Gamma E_i(x) \mathbf{\sigma}(y) V_i(y) dy. \] (4.16)

A singularity in the right-hand side integrals in Eqs. (3.6) and (4.16) can be eliminated by way of a subtraction technique

\[ \mathcal{L}_i^m(x-x, \sigma) = \mathbf{Q}_i(x) \mathbf{g}(x) + \int \Gamma(x-y) \mathbf{g}(y) - \mathbf{g}(x) \Gamma E_i(x) \mathbf{\sigma}(y) V_i(y) dy, \] (4.17)

\[ \mathcal{L}_i^m(x-x, \sigma) = \mathbf{Q}_i(x) \mathbf{g}(x) + \int \Gamma(x-y) \mathbf{g}(y) - \mathbf{g}(x) \Gamma E_i(x) \mathbf{\sigma}(y) V_i(y) dy, \] (4.18)

for \( x \in \nu_i \) and \( x \not \in \nu_i \), respectively; here \( \mathbf{g}(x) = \arg \min \{y - x \mid y \in \nu_i, x \not \in \nu_i\} \), \( \mathbf{Q}_i(x) = \mathbf{Q}_i(x) \Gamma E_i(x) \mathbf{\sigma}(x) \), \( \mathbf{g}(y) = \Gamma E_i(x) \mathbf{\sigma}(y) V_i(y) \).
cability of a standard quadrature rule for the numerical estimation of the integral (4.17). Thus, we constructed the solution (4.8) for a perturbation of the stress field inside and outside the inclusion \( \nu_i \) in the operator form obtained by the VIE for an arbitrary effective field \( \bar{\sigma}(x) \) \( (x \in \nu_i) \).

4.3. Some other classical hypotheses and approaches

For termination of the hierarchy of statistical moment Eqs. (3.12) and (3.13) we use the closing effective field hypothesis called the “quasi-crystalline” approximation by Lax (1952) which in our notations has a form.

Hypothesis 2, “quasi-crystalline” approximation. It is supposed that the mean value of the effective field at a point \( x \in \nu_i \) does not depend on the stress field inside surrounding heterogeneities \( \nu_i \neq \nu_i \):

\[
\bar{\sigma}(x) = \bar{\sigma}(x), \quad x \in \nu_i.
\]

In the framework of the hypothesis H1 (combining the hypotheses H1a and H1b), substitution of the solution (4.5) into the first equations of the systems (3.12) and (3.13) at \( n = 1 \) and at the closing hypothesis H2 leads to the solutions \( (x \in \nu_i) \) for the statistical averages of strains and stresses fields and for the effective properties

\[
\bar{\sigma}(x) = \mathbf{B}(x) \mathbf{R}^{-1} \mathbf{Y} \mathbf{R}(\sigma), \quad \mathbf{M} = \mathbf{M}^{(0)} + n^{(0)} \mathbf{Y} \mathbf{R}.
\]

Here the matrix \( \mathbf{Y} \) determines the actions of the surrounding inclusions on the considered one and has the inverse matrix given by

\[
(\mathbf{Y}^{-1}) = \mathbf{I} - \mathbf{R} \int \mathbf{T}_i(x_i - x_q) \mathbf{\phi}(\nu_i, x_q) \mathbf{T}_i(x_i - x_q) \mathbf{\phi}(\nu_i, x_i) \mathbf{\phi}(\nu_i, x_i) - \mathbf{T}_i(x_i - x_q) n^{(1)} dx_q.
\]

The solution (4.20) and (4.21) is obtained by the so-called method of effective field (MEF). The general case of the closing hypothesis taking \( n \) interacting heterogeneities (defining the MEFM) is considered in Chapter 8 in Buryachenko (2007).

To make further progress, the hypothesis of “ellipsoidal symmetry” for the distribution of inclusions attributed to Willis (1977) is widely used:

Hypothesis 3, H3, “ellipsoidal symmetry”. The conditional probability density function \( \mathbf{\phi}(\nu_i, x_i) \) depends on \( x_i - x_k \) only through the combination \( \rho = ||(a_i^0)^{-1}(x_i - x_k)||\):

\[
\phi(\nu_i, x_i) = \mathbf{h}(\rho).
\]

where the matrix \( (a_i^0)^{-1} \) (which is symmetric in the indexes \( i \) and \( j \) and \( a_i^0 = a_i^0 \)) defines the ellipsoid excluded volume \( \nu_i^0 = \{x: ||a_i^0^{-1}x|| < 1\} \).

For spherical inclusions the relation (4.22) is realized for a statistical isotropy of the composite structure. It is reasonable to assume that \( (a_i^0)^{-1} \) identifies a matrix of affine transformation that transfers the ellipsoid \( \nu_i^0 \) being the “excluded volume” (“correlation hole”) into a unit sphere and, therefore, the representation of the matrices \( \mathbf{Y} \) and \( \mathbf{Y} \) can be simplified:

\[
\mathbf{Y}^{-1} = \mathbf{I} - n^{(1)} \mathbf{R} \mathbf{Q}^0.
\]

where for the sake of simplicity of the subsequent calculation we will usually assume that the shape of “correlation hole” \( \nu_i^0 \) does not depend on the inclusion \( \nu_i : \nu_i^0 = \nu_i^0 \) and \( \mathbf{Q}^0 = \mathbf{Q}^0 \equiv \mathbf{Q}(\nu_i)$.

The essence of the Hypothesis H3 was analyzed by Ponte Castañeda and Willis (1993) (see also Buryachenko, 2007) in the framework of the hypothesis H1. Buryachenko (2010c) and Buryachenko and Brun (2011) demonstrated that the real destination of the Hypothesis H3 is providing the conditions for realizing of the Hypothesis H1a rather than a simplified reduction of the representations (4.22) and (4.23). Abandoning the ellipsoidal symmetry hypothesis (4.33) will necessarily leads to the inhomogeneity of the effective field \( \bar{\sigma} \) (see for details Buryachenko, 2010c; Buryachenko and Brun, 2011) acting on the inclusion \( x \in \nu_i \) that is prohibited for the classical version of the MEFM by Buryachenko (2007) (see also Ponte Castañeda and Willis, 1995).

As pointed out by Benveniste (1987), the essential assumption in the Mori and Tanaka (1973) method (MTM) states that each inclusion \( \nu_i \) behaves as an isolated one in the infinite matrix and subjected to some effective field \( \bar{\sigma} \), coinciding with the average strain (stress) in the matrix

\[
(\mathbf{\sigma})_0 = \langle \mathbf{\sigma} \rangle_0
\]

Using Eq. (4.24) as the closing assumption leads to the following representations for the statistical average stress fields and for the effective properties

\[
\bar{\sigma}(x) = \mathbf{B}(x) \mathbf{C}^{0,1} \mathbf{I} + c^{(0)} \mathbf{B}^{-1}_i
\]

For the identical ellipsoidal inhomogeneous heterogeneities \( \nu_i \) homothetical to \( \nu_i \), equivalences of Eqs. (4.20), (4.20), and (4.25), (4.26), respectively, are demonstrated in, e.g., Buryachenko (2007). However, the representations (4.20) and (4.22) do not coincide with (4.25) and (4.26), respectively, even for the identical aligned isotropic heterogeneities if \( \nu_i \) and \( \nu_i \) are not homothetic (particularly, if \( \nu_i \) is not an ellipsoid, see Buryachenko, 2007).

5. New iteration method for estimation of both the average stress fields and effective elastic moduli

5.1. Initial approximation

In order to simplify the exact systems for stresses (3.12) and (3.13) we accept the hypotheses H1a and H2 while the hypotheses H1b and H3 are not used. This leads to the following representation for the mean of the effective fields in the fixed inhomogeneity \( x \in \nu_i \)

\[
\langle \bar{\sigma}(x) \rangle = \langle \mathbf{\sigma} \rangle + \int \mathbf{L}^0_q(x - x_q) \langle \bar{\sigma}(x) \rangle_q \mathbf{T}_q(x_q - x_i) n^{(1)} dx_q.
\]

which allows one to obtain the explicit solution (called an initial approximation) for identical aligned heterogeneities \( \nu_i = \nu_i = \nu_i \)

\[
\langle \mathbf{\sigma} \rangle_q(x) = \mathbf{\bar{Y}}(\mathbf{\sigma}),
\]

\[
\langle \mathbf{\sigma} \rangle_q(x) = \mathbf{[L}_q^0(x - x_q)] \mathbf{I} + \mathbf{L}_q \mathbf{\bar{Y}}(\mathbf{\sigma}),
\]

\[
\mathbf{M}^{(0)} = \mathbf{M}^{(0)} + n^{(0)} \mathbf{L}_q \mathbf{\bar{Y}},
\]

\[
\mathbf{\bar{Y}}^{-1} = \mathbf{I} - \int \mathbf{T}_q(x_q - x_i) \mathbf{\phi}(\nu_i, x_q) \mathbf{T}_q(x_q - x_i) n^{(1)} dx_q.
\]

It should be mentioned that the domain of the tensor \( \mathbf{L}^0_q(x - x_q) \) also includes a vicinity of the heterogeneity \( x \neq \nu_i \) rather than only open domain \( \nu_i \). Because of this, Eq. (5.3) can also be used for estimation of the stresses \( \langle \mathbf{\sigma} \rangle_q(x) \) in the vicinity of the heterogeneity near the point \( y \in \nu_i \) with the unit vector \( \mathbf{n} \) outward normal to the heterogeneity boundary \( \nu_i \). Eqs. (5.2)–(5.4) obtained for a general case of the perturbator (3.4) are reduced to the corresponding equations proposed for the particular case of the perturbator (3.6) by Buryachenko and Brun (2011) who have demonstrated that even in the referred particular example for the noncanonical and homogeneous heterogeneities, the estimations obtained by the corresponding new Eqs. (5.2)–(5.5), MEF [4.20] and [4.23] and MTM [4.25] and [4.26]] quantitatively differ from one another.
5.2. The next iterations

An abandonment of the effective field hypothesis $H_1a$ gives no way of decomposing of the operator $L_0^y(x - x_i, \mathbf{y}) \ (4.5)$, that leads Eq. (3.13) in the framework of the hypothesis $H_2$ to the following operator equation

$$\mathbf{[\mathbf{y}]}(x) = [\mathbf{y}](x) + \int L_0^y(x - x_i, [\mathbf{y}])(x_i) [\mathbf{y}](v_i, x_i) - n^{(0)} dx_i$$

(5.6)

rather than to the linear algebraic equation (5.1). The first that come to mind is a solution of Eq. (5.6) by the iteration method

$$[\mathbf{y}]^{(i+1)}(x) = [\mathbf{y}](x) + \int L_0^y(x - x_i, [\mathbf{y}]^{(i)})(x_i) [\mathbf{y}](v_i, x_i) - n^{(0)} dx_i$$

(5.7)

with the initial approximation Eq. (5.2). In so doing, the operator $L_0^y(x - x_i, [\mathbf{y}]^{(0)}) \ (4.16)$ is the integral operator with the singular kernel $\Gamma(x - y)$ which is treated with Eq. (4.17). Moreover, the operator $L_0^y(x - x_i, [\mathbf{y}]^{(i)}) \ (4.16)$ in its own is found by the iteration scheme equations (4.13) and (4.14).

Thus, a solution of Eq. (5.6) can be formally presented in the form

$$[\mathbf{y}](x) = \mathbf{B}^i(x) [\mathbf{y}](x)$$

(5.8)

implying the Neumann series forms for the solutions of both Eq. (4.16) and (5.5). A found effective field concentrator factor $\mathbf{B}^i(x)$ (which is inhomogeneous as opposed to the hypothesis $H_1$) makes it possible to estimate the stress concentrator factor

$$[\mathbf{y}](x) = \mathbf{B}^i(x) [\mathbf{y}](x) + \int \Gamma(x - y) L_0^y(y) [\mathbf{B}^i(x)](y) V(y) dy,$$

(5.9)

and effective compliance $\mathbf{M}^*$

$$\mathbf{M}^* = \mathbf{M}^0 + |\mathbf{M}_1(x)\mathbf{B}^i(x) V(x)|,$$

(5.10)

where a singular integral in Eq. (5.9), is estimated analogously to Eq. (4.17), and the index $i$ in $\mathbf{B}^i(x)$ indicated on a representative heterogeneity $v_i$ in Eq. (5.10) is omitted.

6. Numerical results

With the non-essential restriction on space dimensionality $d$ and the shape of inclusions we will consider 2-D plane strain problems for statistically homogeneous composites filled by the aligned infinite fibers with the noncircular section shape schematically presented in the Fig. 1 and described by the curve

$$\begin{align*}
(x - R_1 + r_1)^2 + (y - R_2 + r_2)^2 &= r^2, \\
\text{for } |x| > R_1 - r_1 &\cap |y| > R_2 - r_2, \\
|x + yR_1/R_2| + |x - yR_1/R_2| &= 2R_1, \\
\text{for } |x| < R_1 - r_1 &\cup |y| < R_2 - r_2,
\end{align*}$$

(6.1)

which reduces to a circle and a rectangular in the limiting cases $R_1 = R_2 = r_1 = a$ and $r_1 = 0$, respectively. We will consider the fixed values $R_1 = 1, r_1/R_1 = 0.1$ and the isotropic constituents with the Young’s moduli $E^{(1)} = E^{(2)} = 5$, Poisson ratio $\nu^{(1)} = 0.45$, $\nu^{(0)} = 0.45$. We deliberately consider the same inclusion shape (6.1) as in Buryachenko and Brun (2011) for demonstrating the significant distinction between their numerical results appearing due to abandoning in the current paper of the hypothesis $H_1a$ accepted by Buryachenko and Brun (2011).

Figs. 1 and 2 present not only the geometrical parameters of the inhomogeneities $v_i$ and $v_j$, but also a schematic comparison of both the classical approach (MEF, see Fig. 1) and the new one (see Fig. 2).
to use the perturbation factor $L_i^c(x - x_i)$ Eq.(5.5) estimations for one heterogeneity in a sample in the nodes of just one realization of the mesh (6.2) which is exploited as an “output” mesh for a solution obtained on a standard inhomogeneous mesh $\Omega^{FEM}$ of the FEA (see, e.g., Zienkiewicz and Taylor, 2005; Fish and Belytschko, 2007). Optimality of the square mesh (6.2) and accuracy estimations for the discretizations $\Omega^{A}$ and $\Omega^{FEM}$ (the commercial finite element code ABAQUS, 2001, was used) were analyzed by Buryachenko and Brun (2011, 2012a). From the other side, the singular operator $L_i^c(x - x_i; \sigma^n_{ij})$ is estimated in the next iterations (5.7) in the regular subtraced form Eq.(4.17) that makes it possible to apply a standard quadratic rule.Elimination of singularity at $x = y$ in Eq.(4.17) leads to the necessity of using a complementary polar mesh $\Omega^{P}$ with the center $y = x \in \Omega^{A}$ when the function $g(y)$ Eq. (4.17) in the nodes $y \in \Omega^{P}$ is linearly approximated by the values of the function $g(y)$ previously found in the nearest nodes $y_i \in \Omega^{A}$.

We are coming now to the analysis of the conditional probability density $\rho(v_i; x_i; v_m; x_m)$. This function is well investigated only for identical spherical (3D and 2D) cases) inclusions with a radius $a$ when the pair distribution function $g(x = x_i - x_m) \equiv \rho(v_i; x_i; v_m; x_m)/\pi a^2$ depending only on $|x_i - x_m|$ is called the radial distribution function (RDF). Two alternative RDFs of inclusion will be examined (see Torquato and Lado, 1992; Hansen and McDonald, 1986)

$$g(r/a) \equiv \rho(v_i; x_i; v_m; x_m)/\pi a^2 = H(r/a - 2), \quad (6.3)$$

$$g(r/a) = H(r/a - 2) \left\{ 1 + \frac{4c}{\pi} \left[ 2 - \arcsin \left( \frac{r}{2a} \right) \right] - \frac{r^2}{2a^2} \right\} H(4 - r/a), \quad (6.4)$$

where $H$ denotes the Heaviside step function, $r \equiv |x_i - x_m|$ is the distance between the nonintersecting inclusions $v_i$ and $v_m$, and $c$ is the volume fraction of fibers of radius $a$. The formula (6.3) describes a well-stirred approximation while Eq. (6.4) takes into account a neighboring order in the distribution of the inclusions. Due to the absence of $\rho(v_i; x_i; v_m; x_m)$ for nonspherical inclusions $v_i$, $v_m$, $v_i = (0, 0, 0)$, we will construct it for identical aligned heterogeneities from the known $\rho(n) = \rho(v_i; x_i; v_m; x_m)$ for the noncircular shape (and even for the nonspherical one) of inclusions is absent. This issue merits additional detailed consideration which is beyond the scope of the current study.

We start our estimation from evaluation of the stress perturbation factors $L_i^c(x - x_i)$ for one heterogeneity in an infinite matrix. The infinite dimensions of the matrix were approximated with a length of 40 inclusion diameters: $R = 40r$. Increasing the matrix dimensions further did not significantly change the results (difference is less than 1%, see for details Buryachenko and Brun, 2011). We are expected to get a larger difference of results obtained in the framework of the backgrounds (3.14) and (3.15) for composites with non-ellipsoidal inclusions $x_i$ demonstrating essentially inhomogeneous stress distribution inside inclusions. In more details we will analyze the inclusion shape (6.1) with the different aspect ratios $R_2/R_1 = 0.32, 0.64, 1$, and fixed $r_i/R_1 = 0.1$ (see Figs. 3, 4).

stress concentration factor $B_i(x)$ related with the stress perturbation factors $L_i^c(x - x_i)$ by Eq. (4.5) is estimated by the FEA for a single inclusion in a large matrix sample, and the components $B_{1111}(x|x_i)$ and $B_{2222}(x|x_i)$ in the cross sections $x = (x_1, 0)^T$ and $x = (0, x_2)^T$ are presented in Figs. 3, 4, respectively; the values $B_i(x)$ coincide with $B_i(x|x_i)$ at $c = 0$. As can be seen in Fig. 4, the components $B_{2222}(x|x_i)$ for all considered values $R_2/R_1 = 0.32, 0.64, 1$ exhibit a change of sign along a cross-section $x_1 = 0$ (corresponding values $B_{2222}(x|x_i)$ don’t change signs at $R_2/R_1 = 0.1$). However, the most significant inhomogeneity of the component $B_{1111}(x|x_i)$ in Fig. 3 is displayed for $R_2/R_1 = 0.32$, and, because of this we will consider the composites reinforced by inhomogeneities only with the ratio $R_2/R_1 = 0.32$.

For the case $c = 0.8$ of composite materials reinforced by cylinder inclusions with $R_2/R_1 = 0.32$, the stress concentration factors $B_{1111}(x|x_i)$ and $B_{2222}(x|x_i)$ in the cross sections $x = (x_1, 0)^T$ and $x = (0, x_2)^T$ are presented in Figs. 5 and 6, respectively, for the binary correlation functions $\rho(v_i; x_i|v_m; x_m)$ (6.3) and (6.5).

As can be seen in Fig. 5, $B_{1111}(x|x_i)$ for the initial (5.3) and the 8th iteration (5.9) of the new approach (NA) differ one from another just on 10% while the difference between the estimations obtained by the MEF Eqs. (4.20) and (4.23) (curve 1) and MTM Eqs. (4.25) (curve 4) is more significant (more detailed comparison of residual streses estimations by the MEF, MT, and the initial approximation of the NA was performed by Buryachenko and Brun, 2011, 2012a).

More dramatic difference between the estimations of the stress concentration factors obtained in the different iterations of the NA is observed in Fig. 6 for the components $B_{2222}(x|x_i)$. A fast convergence of the proposed iteration method take place, the eighth iteration differs from the sixth, fourth, first, and initial iterations on 5.5%, 20%, 44%, and 101%, respectively. So much significant difference of the values $B_{2222}(x|x_i)$ of the initial approximation with the next iterations is understandable if we recall that the initial approximation Eq. (5.3) was constructed in the framework of the hypothesis H1a when $\sigma^{(0)}_{ij}(x|x_i) \equiv const. \quad (5.3)$ while in the iterations Eq. (5.9) we estimate $\sigma^{(i)}_{ij}(x|x_i) \equiv const. \quad (x|x_i)$.

The components of the effective stress concentration factors $B_{1111}(x|x_i)$ as the functions of the normalized coordinate $x/R_1$ are presented in Fig. 7 for the iterations $n = 0, 2, 4, 8$. As can be seen a few iterations of Eq. (5.9) produce the inhomogeneous $B_{1111}(x|x_i)$ and this process converges very rapidly. In so doing a convergence of the effective fields $B_{1111}(x|x_i)$ is faster than convergence of $B^0_{ij}(x_i)$ while their inhomogeneity is less than inhomogeneity of stress concentrator factors.

![Fig. 3. B_{1111}(x|x_i) vs. x/R_1 for R_2/R_1 = 0.32 (1), 0.64 (2), and 1 (3).](image-url)
Now we keep track of transformation of the stress concentrator factor $B_{2211}(x_i)$ with variation of the volume fractions $c$ of heterogeneities (see Fig. 8). It is interesting that for small volume fraction $c < 0.3$ $B_{2211}(x_1)$ changes a sign in a cross-section $x_2 = 0$ while $B_{2211}(x_i) > 0$ at $c > 0.4$ for all $x_i \in [-R_2, R_2]$.

The curves of the normalized effective Young’s moduli $E_i/E_0$ (the curves $E_i/E_0$ are similar) are presented in Fig. 9 as the functions of the volume concentration $c$ of inclusions at $R_2/R_1 = 0.32$. Curves 1–4 are estimated by the new approach (5.4) and (5.7)–(5.10). The curves 1–4 are predicted by the MEF and MTM. The initial approximation (5.4) is presented by the curves 1 and 3, 4, respectively. The initial approximation (1, 3) and corresponding eight iterations (2, 4) are negligible.

Thus, stress concentrator factors (Figs. 5 and 6) are significantly more sensitive values to the choice of the approach than effective elastic moduli (Fig. 2). It should be mentioned that all numerical results presented in Figs. 5, 6, and 9 were obtained by the different methods (NA, MEF, and MTM) used the same stress concentrator and the effective moduli (see Figs. 3 and 4). The results of exploiting these data Figs. 3 and 4 in subsequent evaluation of both the effective stress concentrator factors (see Figs. 5 and 6) and the effective moduli (see Fig. 9) can be essentially distinguished for the different methods (either NA, MEF, or MTM). In so doing Buryachenko and Brun (2012a) demonstrated that for limiting case of residual stresses $L(x) = 0$, NA provides the exact estimations of the effective stress concentrator factors $B_i$ (as in Fig. 7) differing from the corresponding evaluations by the MEF and MTM on 40%.

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methods (NA, MEF, and MTM) used the same stress concentrator factors (see Figs. 5 and 6) and the effective moduli (see Fig. 9). It should be mentioned that all numerical results presented in Figs. 5, 6, 9 and obtained by the different methods (NA, MEF, and MTM) used the same stress concentrator factors B_i(x) (x ∈ ν_i) estimated by the FEA for a single heterogeneity ν_i (see Figs. 3 and 4). The results of exploiting these data (Figs. 3 and 4) in subsequent evaluation of both the effective stress concentrator factors (see Figs. 5 and 6) and the effective moduli (see Fig. 9) can be essentially distinguished for the different methods (either NA, MEF, or MTM.) In so doing Buryachenko and Brun (2012a) demonstrated that for limiting case of residual stresses L(x) ≡ 0, NA provides the exact estimations of the effective stress concentrator factors \( \bar{B}_i \) (as in Fig. 7) differing from the corresponding evaluations by the MEF and MTM on 40%.

7. Conclusion

We have proposed the new micromechanical model based on the iteration method for solution of the new integral equation (5.6) presented in terms of perturbators which can be found by any available numerical or semianalytical method. More rich in content is a discussion of the main hypotheses as well as the limitations of the proposed estimations and their possible generalizations.

The current paper is dedicated to development of a new direction in micromechanical modeling initiated by proposed integral equation (3.14) (see Buryachenko, 2010a,b) which was generalized to Eq. (3.10) in this work. A fundamental deficiency of the classical equation (3.15) is the dependence of the renormalizing terms \( \Gamma(x - y)/|y| \) [obtained in the framework of the asymptotic approximation of the hypothesis H1b] only on the statistical average \( \langle y \rangle \) while the renormalizing terms \( \langle c^\alpha_i(x - x, \eta)\rangle \) (3.10) explicitly depend on distributions \( \langle \eta \rangle \), \( x, x_i \rangle (y \in \nu_i) \). What seems to be only a formal trick is in reality a new background of micromechanics yielding to revision of classical background of micromechanics with potential abandonment of many classical concepts of micromechanics used in most popular methods, namely: effective field hypothesis H1, quasi-crystalline approximation H2, the hypothesis of “ellipsoidal symmetry” H3, and Eshelby tensor (see for details Buryachenko, 2010b). Abandonment of a few different combinations of these hypotheses leads to detection of some new effects that are impossible in the framework of a classical background of micromechanics. For example, the hypotheses H1a and H2 were used while the hypotheses H1b and H3 were not accepted by Buryachenko and Brun (2011, 2012b). Buryachenko (2010b) has exploited the hypotheses H2 and H3, while the hypotheses H1a and H1b were not to be accepted for analysis of composites with circle inhomogeneous heterogeneities. Analysis of composites with the circle homogeneous inclusions (when analytical representation for the perturbator factor (4.5) is known) significantly simplifies the problem that enabled Buryachenko (2011a) to abandon the hypotheses H1 and H2 and use only hypothesis H3. Buryachenko (2011b) also considered the case of circle homogeneous inclusions but with nonlocal constitutive law with the use of the hypotheses H2 and H3 and abandonment of the hypothesis H1 (the list of some other problems where one expects to get fundamentally new results in the case of using of the new background of micromechanics (3.10) is presented in Buryachenko, 2010c).

In particular, even in the case of statistically homogeneous media subjected to homogeneous boundary conditions, new effects have been found. So, the final classical representations of the effective properties obtained by both the MEF (4.20), (4.23) and MTM (4.25) depend only on the average stress concentrator factor B, while the effective properties estimated by the new approach (5.10) implicitly depend on the inhomogeneous tensor B_i(x) in both inside and outside inclusion \( \nu_i \) i.e. extension of B_i(x) (\( \nu_i \neq \nu_i \)) is necessary; it allows us to abandon the hypothesis H1b whose accuracy is questionable for inclusions of noncanonical shape. Then the size of the excluded volume \( \nu_i^p \) as well as the binary correlation function \( \phi(|x_i, x|, |\nu|) \) impact on the effective field even in the framework of hypothesis H2. A larger difference between the use of the backgrounds (3.10) and (3.15) was obtained for composites reinforced by non-ellipsoidal inclusions demonstrating essentially inhomogeneous stress distribution inside isolated inclusions even in the framework of the hypothesis H1a. It was quantitatively estimated that the use of the new background (3.10) instead of the old one (3.15) leads to just a small corrections of effective properties (effective moduli, coefficient of thermal expansion, and stored energy) while the estimations of the statistical averages of local stresses can be dramatically different with possible change of the sign of predicted local stresses. The next step in abandonment of basic hypotheses of micromechanics is performed in the current paper. Namely, we abandoned of the hypotheses H1a and H1b (forming the hypothesis H1), and H3 for composites with non-ellipsoidal inclusions while the quasi-crystallite approximation H2 was used; additional abandonment...
the hypothesis H1a was possible due to development of the new iteration method for solution of Eq. (5.7). As was expected, the stress concentrator factors (Figs. 5 and 6) are significantly more sensitive values to the choice of the approach than effective elastic moduli (Fig. 9).

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