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Bounding the coefficients of the characteristic polynomials of simple binary matroids

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ABSTRACT

We give an upper bound and a class of lower bounds on the coefficients of the characteristic polynomial of a simple binary matroid. This generalizes the corresponding bounds for graphic matroids of Li and Tian (1978) [3], as well as a matroid lower bound of Björner (1980) [1] for simple binary matroids. As the flow polynomial of a graph G is the characteristic polynomial of the dual matroid $M^*(G)$, the bound applies to flow polynomials.

1. Introduction

Let G be a graph with p vertices and x be a natural number, the value $P_G(x)$ is the number of proper colorings $f : V(G) \to [x]$. $P_G(x)$ is a degree p polynomial, called the *chromatic polynomial* of G. It is well known that this polynomial has degree $|V(G)|$ with integer coefficients alternating in sign. Let G be a simple connected graph on p vertices and q edges. For the rest of the paper, we write that $P_G(x) = \sum_{k=0}^{p}(-1)^{p-k}a_k x^k$.

The *characteristic polynomial* of a rank-r matroid M with ground set E is defined as

$$\chi(M, x) = \sum_{X \subseteq E}(-1)^{|X|}x^{r(M)-r(X)}.$$

Clearly, this polynomial has degree $r(M)$. It has integer coefficients alternating in sign [6].

The characteristic polynomial of a graphic matroid $M(G)$ is related to the graph chromatic polynomial of G by the equation

$$P_G(x) = x^{\omega(G)}\chi(M(G), x),$$

where $\omega(G)$ is the number of components of G. For notation, we generally follow Oxley [4]. Throughout the paper assume that M is a rank-r matroid on n elements, c_k is the number of k-element circuits of M, and $d_k(e)$ is the number of k-element circuits of M containing the specified element e of $E(M)$. A k-element circuit will also be called a k-circuit. The matroid obtained by deleting or contracting of an element e from M is denoted by $M \backslash e$ and M/e, respectively. The *girth* of a graph or a matroid is the length of a shortest circuit of the graph or matroid.

The next two theorems, due to Li and Tian, give an upper bound and a class of lower bounds on coefficients of the chromatic polynomial of a graph.

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Theorem 1.1 (*[3]*). *Let G be a simple connected graph on p vertices and q edges with girth g and c_g cycles of size g. Then*

$$a_k \leq \binom{q}{p-k} - \binom{q-g+2}{p-k-g+2} + \binom{q-c_g-g+2}{p-k-g+2}.$$

Theorem 1.2 (*[3]*). *Let G be a simple connected graph with p vertices and q edges. Suppose that G has c_i cycles of size i, for $i = 3, 4, \ldots, l$ where l is an integer and $3 \leq l \leq p$. Then for $0 \leq k \leq p$,*

$$a_k \geq \sum_{i=0}^{l-1} \binom{q-p+i}{i} \binom{p-1-i}{k-1} - \sum_{j=3}^{l} c_j \sum_{i=0}^{l-j} \binom{q-p+i}{i} \binom{p-j-i}{k-1}.$$

Let M be a matroid. From now on, we write $\chi(M, x) = \sum_{k=0}^{r} (-1)^{r-k} b_k x^k$. Note that if M is a graphic matroid $M(G)$ for a connected graph G, then $b_k = a_{k+1}$ $(0 \leq k \leq r)$ as $P_G(x) = x\chi(M(G), x)$. The following results give bounds for the value b_k. The first is an upper bound from Rota [6], and the second is a lower bound due to Björner [1].

Theorem 1.3 (*[6]*). *Let M be a rank-r matroid with n elements. Then*

$$b_k \leq \binom{n}{r-k}.$$

Theorem 1.4 (*[1]*). *Suppose that M is a simple matroid on n elements with girth g. Then for $0 \leq k \leq r$,*

$$b_k \geq \sum_{i=0}^{g-1} \binom{n-r+i-1}{i} \binom{r-i}{k} - c_g \binom{r-g+1}{k}.$$

In this paper, using the inductive techniques of [3] for graphs, we bound the absolute value of the coefficients of characteristic polynomials of simple binary matroids. The following are our main results; the proofs will be given in Section 2. Our proofs are similar to those in [3]; the generalization from graphs to binary matroids does not affect the inductive arguments we use to prove our bounds.

Theorem 1.5. *Let M be a simple binary matroid on n elements with girth $g \geq 3$. Let c_g denote the number of circuits of size g. Then for $0 \leq k \leq r$,*

$$b_k \leq \binom{n}{r-k} - \binom{n-g+2}{r-k-g+2} + \binom{n-c_g-g+2}{r-k-g+2}. \quad (1)$$

Theorem 1.1 is an immediate consequence of the above result. This result also extends Theorem 1.3 for simple binary matroids. It is easily checked that for $n \geq 3$, $\chi_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$, where C_n is a cycle of size n. Since $\chi_{C_n}(x) = x\chi(M(C_n), x)$, it is straightforward to verify that each coefficient of $\chi(M(C_n), x)$ attains the bound in Theorem 1.5. Thus, our upper bound is best possible for the class of simple binary matroids. Furthermore, our upper bound does not hold in general for nonbinary matroids. Indeed, $\chi(U_{2,n}, x) = x^2 - nx + (n-1)$; therefore, the bound fails for $U_{2,n}$ when $n \geq 5$.

Theorem 1.6. *Let M be a simple binary matroid with $|E(M)| = n$ and $r(M) = r$. Let c_i be the number of i-element circuits of M for $3 \leq i \leq l$, where l is an integer such that $3 \leq l \leq r+1$. Then for $0 \leq k \leq r$,*

$$b_k \geq \sum_{i=0}^{l-1} \binom{n-r-1+i}{i} \binom{r-i}{k} - \sum_{j=3}^{l} c_j \sum_{i=0}^{l-j} \binom{n-r-1+i}{i} \binom{r+1-j-i}{k}. \quad (2)$$

This result generalizes Theorem 1.2 from graphs to binary matroids, and it extends Theorem 1.4 (let $l = g$) for simple binary matroids. We do not know if this bound holds for non-binary matroids. As the flow polynomial of a graph G is the characteristic polynomial of the dual matroid $M^*(G)$, new bounds for the coefficients of the flow polynomial of a graph G are obtained as a direct consequence of the last two theorems in Section 3.

2. Proofs of the main results

The following two elementary lemmas are used in [3] and will also be used in our proof.

Lemma 2.1. *For non-negative integers m_1, m_2, x, t $(m_1 \geq m_2)$ such that $x \leq m_1 - m_2$,*

$$\binom{m_1-x}{t} + \binom{m_2+x}{t} \leq \binom{m_1}{t} + \binom{m_2}{t}.$$

Lemma 2.2. *For non-negative integers a, b and c where $a \geq b$,*

$$\sum_{i=0}^{a-b} \binom{c+i}{i} \binom{a-i}{b} = \binom{a+c+1}{a-b}.$$

The following deletion-contraction formula for the characteristic polynomial can be found in [7,8].

Lemma 2.3. *Let e be an element of a matroid M such that e is neither a loop nor a coloop. Then*

$$\chi(M, x) = \chi(M\backslash e, x) - \chi(M/e, x).$$

Lemma 2.4. *Let e be an element of a matroid M such that e is neither a loop nor a coloop and $\chi(M, x) = \sum_{k=0}^{r}(-1)^{r-k} b_k x^k$. Then*

(i) $\chi(M, x) = \chi(M\backslash e, x) - \chi(si(M/e), x)$, *and*
(ii) *Let b'_k be the absolute value of the coefficient of x^k in $\chi(M\backslash e, x)$ and b''_k be the absolute value of the coefficient of x^k in $\chi(si(M/e), x)$. Then $b_k = b'_k + b''_k$.*

Proof. It is easily verified by the definition of the characteristic polynomial that $\chi(M/e, x) = \chi(si(M/e), x)$. So by Lemma 2.3, (i) holds. (ii) is an immediate consequence of (i). □

Let e be an element of a matroid M. Recall that we use $d_i(e)$ to denote the number of i-element circuits of M containing e. Let c'_i denote the number of i-element circuits of $M\backslash e$ and c''_i denote the number of i-element circuits of $si(M/e)$.

Lemma 2.5. *Let M be a simple matroid. Then*

(i) $c'_i = c_i - d_i(e)$,
(ii) $c''_i \leq c_i - d_i(e) + d_{i+1}(e)$, *and*
(iii) $|E(si(M/e))| = n - d_3(e) - 1$ *if M is binary.*

Proof. Note that (i) and (ii) are easily verified. Suppose that M is a simple binary matroid and $e \in E(M)$. Since M is binary, each line containing e has at most three elements. Hence $|E(si(M/e))| = n - d_3(e) - 1$. □

Proof of Theorem 1.5. We use induction on the girth g of M. Let c_i be the number of i-circuits. Suppose $g = 3$. Let e be an element of a three-element circuit of M. Then $r(M\backslash e) = r(M)$ and $r(si(M/e)) = r(M) - 1$. We now use induction on c_3. First assume that $c_3 = 1$. Note that $\chi(M, x) = \chi(M\backslash e, x) - \chi(si(M/e), x)$. By Theorem 1.3, $b'_k \leq \binom{n-1}{r-k}$ and $b''_k \leq \binom{n-2}{r-1-k}$. Thus,

$$\begin{aligned} b_k &= b'_k + b''_k \leq \binom{n-1}{r-k} + \binom{n-2}{r-k-1} \\ &= \binom{n}{r-k} - \binom{n-1}{r-k-1} + \binom{n-2}{r-k-1}. \end{aligned}$$

So the inequality (1) is satisfied when $g = 3$ and $c_3 = 1$.

Now assume the result holds for all simple binary matroids of girth $g = 3$ containing fewer than c_3 three-element circuits. Again we let e be an element of a three-element circuit of M. Note that $M\backslash e$ contains $c_3 - d_3(e)$ triangles. Now by inductive hypothesis,

$$b'_k \leq \binom{n-1}{r-k} - \binom{(n-1)-1}{r-k-1} + \binom{n-c_g+d_3(e)-2}{r-k-1}.$$

As M is binary, $|E(si(M/e))| = n - d_3(e) - 1$. By Theorem 1.3, $b''_k \leq \binom{n-d_3(e)-1}{r-1-k}$. Therefore,

$$\begin{aligned} b_k &= b'_k + b''_k \\ &\leq \binom{n-1}{r-k} - \binom{n-2}{r-k-1} + \binom{(n-c_g-1)+(d_3(e)-1)}{r-k-1} + \binom{n-d_3(e)-1}{r-k-1} \\ &\leq \binom{n-1}{r-k} - \binom{n-2}{r-k-1} + \binom{n-c_g-1}{r-k-1} + \binom{n-2}{r-k-1} \quad \text{by Lemma 2.1} \\ &= \binom{n}{r-k} - \binom{n-1}{r-k-1} + \binom{n-c_g-1}{r-k-1}. \end{aligned}$$

Hence (1) holds for all simple binary matroids of girth 3.

Next suppose that $g(M) = g > 3$ and the result holds for all simple binary matroids with girth less than g. First we examine the case where $c_g = 1$. Since $g(M/e) = g - 1$, $c_{g(M/e)} = 1$. By Theorem 1.3, $b'_k \leq \binom{n-1}{r-k}$ and by the inductive hypothesis,

$$b''_k \leq \binom{n-1}{r-k-1} - \binom{n-g+2}{r-k-g+2} + \binom{n-g+1}{r-k-g+2}.$$

Hence, $b_k = b'_k + b''_k \leq \binom{n}{r-k} - \binom{n-g+2}{r-k-g+2} + \binom{n-g+1}{r-k-g+2}$.

Now suppose that result holds for binary matroids with girth g having fewer than c_g circuits of size g. Let C be a circuit of M with $|C| = g$ and choose an element e from C. Then by the inductive hypothesis, and by Lemma 2.1 using $x = d_g(e) - 1$,

$$\begin{aligned}
b_k &= b'_k + b''_k \\
&\leq \binom{n-1}{r-k} - \binom{n-g+1}{r-k-g+2} + \binom{n-g-c_g+d_g(e)+1}{r-k-g+2} \\
&\quad + \binom{n-1}{r-k-1} - \binom{n-g+2}{r-k-g+2} + \binom{n-d_g(e)-g+2}{r-k-g+2} \\
&\leq \binom{n}{r-k} + \binom{n-c_g-g+2}{r-k-g+2} + \binom{n-g+1}{n-k-g+2} \\
&\quad - \binom{n-g+1}{n-k-g+2} - \binom{n-g+2}{r-k-g+2} \\
&= \binom{n}{r-k} - \binom{n-g+2}{r-k-g+2} + \binom{n-c_g-g+2}{r-k-g+2}.
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 1.6. We use induction on the integer $s = \sum_{i=3}^{l} ic_i$ where $3 \leq l \leq r + 1$. Suppose $s = 0$. Then any circuit of M must contain at least $l + 1$ elements. First consider the case where M has no circuits. Then $M \cong U_{n,n}$. By Zaslavsky [8, 7.2.2], $\chi(U_{n,n}, x) = (x-1)^n$. Hence $b_k = \binom{n}{r-k}$. Observe that

$$\sum_{i=0}^{l-1} \binom{n-r-1+i}{i} \binom{r-i}{k} = \sum_{i=0}^{r-k} \binom{n-r-1+i}{i} \binom{r-i}{k} = \binom{n}{r-k}$$

by Lemma 2.2. So the result holds for this case.

Now suppose that $s = 0$, but M has at least one circuit. Note that $g(M) \geq l + 1 > 3$ as $c_i = 0$ for all $3 \leq i \leq l$. Hence $M/e = si(M/e)$. Evidently the result holds for $n = 4$ (in this case $M \cong U_{3,4}$, and the bound is easy to verify). Suppose that $n > 4$ and the result holds for all simple binary matroids on fewer than n elements. Take an element e from a circuit of M. Let b'_k and b''_k be the absolute value of the coefficient of x^k in $\chi(M\backslash e, x)$ and $\chi(si(M/e), x)$, respectively. Then by the inductive hypothesis, the theorem holds for b'_k and b''_k. Since $b_k = b'_k + b''_k$, we have

$$\begin{aligned}
b_k &\geq \sum_{i=1}^{l-1} \left[\binom{n-r-2+i}{i} \binom{r-i}{k} + \binom{n-r+i-2}{i-1} \binom{r-i}{k} \right] + \binom{r}{k} \\
&= \sum_{i=1}^{l-1} \binom{r-i}{k} \binom{n-r+i-1}{i} + \binom{r}{k} \\
&= \sum_{i=0}^{l-1} \binom{r-i}{k} \binom{n-r+i-1}{i}.
\end{aligned}$$

Thus, the result holds for $s = 0$.

Now suppose that $s = \sum_{i=3}^{l} ic_i > 0$ and the results holds for all simple binary matroids with $s < \sum_{i=3}^{l} ic_i$. Let e be an element of a t-circuit of M, where $3 \leq t \leq l$. Let c'_i and c''_i be the i-circuits of $M\backslash e$ and $si(M/e)$, respectively. By Lemma 2.5, $c'_i = c_i - d_i(e)$, and $c''_i \leq c_i - d_i(e) + d_{i+1}(e)$. We consider the value of s for $M\backslash e$ and $si(M/e)$.

Observe that

$$\sum_{i=3}^{l} ic'_i = \sum_{i=3}^{l} ic_i - \sum_{i=3}^{l} id_i(e) < \sum_{i=3}^{l} ic_i$$

and

$$\sum_{i=3}^{l-1} ic_i'' \leq \sum_{i=3}^{l-1} ic_i - \sum_{i=3}^{l-1} id_i(e) + \sum_{i=3}^{l-1} [(i+1)d_{i+1}(e) - d_{i+1}(e)]$$
$$= \sum_{i=3}^{l} ic_i - 3d_3(e) - \sum_{i=4}^{l} d_i(e) < \sum_{i=3}^{l} ic_i.$$

The last inequality is strict since $d_t(e) > 0$ for some $3 \leq t \leq l$. So by the inductive hypothesis, the result holds for b_k' and b_k''. Note that $r(M\backslash e) = r(M)$, $|E(M\backslash e)| = n-1$; $r(si(M/e)) = r(M) - 1$, and $|E(si(M/e))| = n - d_3(e) - 1$ by Lemma 2.5. Hence, the theorem holds for b_k' and

$$\begin{aligned}
b_k'' \geq & \sum_{i=0}^{l-2} \binom{n - d_3(e) - 1 - r + 1 - 1 + i}{i} \binom{r-1-i}{k} \\
& - \sum_{j=3}^{l-1} c_j'' \sum_{i=0}^{l-j-1} \binom{n - d_3(e) - 1 - r + 1 - 1 + i}{i} \binom{r-j-i}{k} \\
\geq & \sum_{i=0}^{l-2} \binom{n - r - 1 - d_3(e) + i}{i} \binom{r-1-i}{k} \\
& - \sum_{j=3}^{l-1} (c_j - d_j(e) + d_{j+1}(e)) \sum_{i=0}^{l-j-1} \binom{n-r-1+i}{i} \binom{r-i-j}{k} \\
= & \binom{r-1}{k} + \sum_{i=0}^{l-3} \binom{n - r - d_3(e) + i}{i+1} \binom{r-2-i}{k} \\
& - \sum_{j=3}^{l-1} (c_j - d_j(e)) \sum_{i=1}^{l-j} \binom{n-r-2+i}{i-1} \binom{r+1-j-i}{k} \\
& - \sum_{j=4}^{l} d_j(e) \sum_{i=0}^{l-j} \binom{n-r-1+i}{i} \binom{r+1-j-i}{k}.
\end{aligned}$$

Therefore,
$$\begin{aligned}
b_k = b_k' + b_k'' \geq & \sum_{i=0}^{l-1} \binom{n-r-2+i}{i} \binom{r-i}{k} \\
& - \sum_{j=3}^{l} (c_j - d_j(e)) \sum_{i=0}^{l-j} \binom{n-r-2+i}{i} \binom{r+1-j-i}{k} \\
& + \binom{r-1}{k} + \sum_{i=0}^{l-3} \binom{n-r-d_3(e)+i}{i+1} \binom{r-2-i}{k} \\
& - \sum_{j=3}^{l-1} (c_j - d_j(e)) \sum_{i=1}^{l-j} \binom{n-r-2+i}{i-1} \binom{r+1-j-i}{k} \\
& - \sum_{j=4}^{l} d_j(e) \sum_{i=0}^{l-j} \binom{n-r-1+i}{i} \binom{r+1-j-i}{k}.
\end{aligned}$$

Regrouping and using the identity $\binom{x}{i} = \binom{x-1}{i} + \binom{x-1}{i-1}$, we obtain

$$\begin{aligned}
b_k \geq & \sum_{i=0}^{l-1} \binom{n-r-2+i}{i} \binom{r-i}{k} \\
& + \binom{r-1}{k} + \sum_{i=0}^{l-3} \binom{n-r-d_3(e)+i}{i+1} \binom{r-2-i}{k} \\
& - \sum_{j=3}^{l} c_j \sum_{i=0}^{l-j} \binom{n-r+i-1}{i} \binom{r-i-j+1}{k}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=3}^{l} d_j(e) \sum_{i=0}^{l-j} \binom{n-r+i-1}{i} \binom{r-i-j+1}{k} \\
&-\sum_{j=4}^{l} d_j(e) \sum_{i=0}^{l-j} \binom{n-r+i-1}{i} \binom{r-i-j+1}{k} \\
=\ & \binom{r-1}{k} - \sum_{j=3}^{l} c_j \sum_{i=0}^{l-j} \binom{n-r+i-1}{i} \binom{r-i-j+1}{k} \\
&+\sum_{i=0}^{l-1} \binom{n-r-2+i}{i} \binom{r-i}{k} + \sum_{i=0}^{l-3} d_3(e) \binom{n-r+i-1}{i} \binom{r-i-2}{k} \\
&+\sum_{i=0}^{l-3} \binom{n-d_3(e)-r+i}{i+1} \binom{r-i-2}{k}.
\end{aligned}
$$

Now

$$
\begin{aligned}
d_3(e)\binom{n-r+i-1}{i} + \binom{n-d_3(e)-r+i}{i+1} &\geq \sum_{j=0}^{d_3(e)-1} \binom{n-r+i-1-j}{i} + \binom{n-d_3(e)-r+i}{i+1} \\
&= \binom{n-r+i}{i+1}.
\end{aligned}
$$

Using this in the last inequality for b_k, regrouping and simplifying, one can easily show that

$$
b_k \geq \sum_{i=0}^{l-1} \binom{n-r-1+i}{i} \binom{r-i}{k} - \sum_{j=3}^{l} c_j \sum_{i=0}^{l-j} \binom{n-r+i-1}{i} \binom{r-i-j+1}{k}.
$$

This completes the proof of the theorem. □

3. Consequences on the flow polynomials of graphs

Let G be a graph. It is well known that the flow polynomial of a graph G is the characteristic polynomial of the dual matroid $M^*(G)$: $F_G(x) = P_{M^*(G)}(x)$. For more information on the flow polynomials, see [2,5]. A *cocircuit* of a matroid M is a circuit in the dual matroid M^*. Note that a cocircuit in a graph is a non-empty minimal edge-cut. The cogirth of G is the size of a smallest cocircuit. A graph is cosimple if it does not have any edge-cut of size one or two. The following are immediate consequences of our main results.

Theorem 3.1. *Let G be a cosimple graph on p vertices and q edges with cogirth g^* and c_g^* cocircuits of size g^*. Suppose that the flow polynomial of G is $F_G(x) = \sum_{k=0}^{r}(-1)^{r-k} f_k x^k$, where $r = q - p + \omega(G)$.*

Then for $0 \leq k \leq r$,

$$
f_k \leq \binom{q}{r-k} - \binom{q-g^*+2}{r-k-g^*+2} + \binom{q-c_g^*-g^*+2}{r-k-g^*+2}. \tag{3}
$$

Theorem 3.2. *Let G be a cosimple graph on p vertices and q edges. Suppose that the flow polynomial of G is $F_G(x) = \sum_{k=0}^{r}(-1)^{r-k} f_k x^k$, where $r = q - p + \omega(G)$. Let c_i^* be the number of i-element cocircuits of G for $3 \leq i \leq l$, where l is an integer such that $3 \leq l \leq r+1$.*

Then for $0 \leq k \leq r$,

$$
f_k \geq \sum_{i=0}^{l-1} \binom{q-r-1+i}{i} \binom{r-i}{k} - \sum_{j=3}^{l} c_j^* \sum_{i=0}^{l-j} \binom{q-r-1+i}{i} \binom{r+1-j-i}{k}. \tag{4}
$$

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References

[1] A. Björner, Some matroid inequalities, Discrete Math. 31 (1980) 101–103.
[2] F.M. Dong, K.M. Koh, Bounds for the coefficients of flow polynomials, J. of Comb. Theory, Ser. B 3 (2007) 413–420.
[3] W.X. Li, F. Tian, Some problems concerning the chromatic polynomials of graphs, Acta Math. Sinica 21 (1978) 223–230 (in Chinese).
[4] J.G. Oxley, Matroid Theory, Second ed., Oxford University Press, New York, 2011.
[5] J.G. Oxley, T. Brylawski, The Tutte polynomial and its applications, in: N. White (Ed.), Matroid Applications, Cambridge University Press, Cambridge, 1992, pp. 123–225.
[6] G.-C. Rota, On the foundations of combinatorial theory (I), theory of Möbius functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 (1964) 340–368.
[7] D.J.A. Welsh, Matroid Theory, in: L. M. S. Monographs, vol. 8, Academic Press, London-New York, 1976.
[8] T. Zaslavsky, The Möbius function and the characteristic polynomial, in: N. White (Ed.), Combinatorial Geometries, Cambridge University Press, Cambridge, 1987, pp. 214–238.