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Critical Hardy–Sobolev inequalities

S. Filippas [a,b,*], V. Maz'ya [c,d], A. Tertikas [e,b]

[a] Department of Applied Mathematics, University of Crete, 71409 Heraklion, Greece
[b] Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Greece
[c] Department of Mathematical Sciences, The University of Liverpool, M&O Building, Peach Street, Liverpool, L69 72L, UK
[d] Department of Mathematics, Ohio State University, Columbus, OH 43210, USA
[e] Department of Mathematics, University of Crete, 71409 Heraklion, Greece

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Abstract

We consider Hardy inequalities in $\mathbb{R}^n$, $n \geqslant 3$, with best constant that involve either distance to the boundary or distance to a surface of co-dimension $k < n$, and we show that they can still be improved by adding a multiple of a whole range of critical norms that at the extreme case become precisely the critical Sobolev norm.

Résumé

Nous étudions des inegalités de Hardy dans $\mathbb{R}^n$, $n \geqslant 3$, avec la meilleure constante, liée soit à la distance au bord, soit à la distance à une surface de codimension $k < n$. Nous obtenons des versions améliorées en ajoutant un certain nombre de normes critiques qui, dans le cas limite, sont précisement les normes critiques de Sobolev.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain and K be a compact, C^2 manifold without boundary embedded in $\mathbb{R}^n$, of co-dimension k, $1 \leqslant k < n$. When $k = 1$ we assume that $K = \partial\Omega$, whereas for $1 < k < n$ we assume that $K \cap \bar{\Omega} \neq \emptyset$. We set $d(x) = \mathrm{dist}(x, K)$.

We also recall for $1 < p$ and $p \neq k$ the following condition that was introduced in [4],

$$-\Delta_p d^{\frac{p-k}{p-1}} \geqslant 0 \quad \text{on } \Omega \setminus K, \tag{C}$$

where Δ_p is the p-Laplacian, that is $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. We note that for $k = 1$, condition (C) becomes $-\Delta d \geqslant 0$, which is equivalent to the convexity of the domain Ω for $n = 2$, but it is a much weaker condition than convexity of Ω for $n \geqslant 3$.

* Corresponding author.

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Under assumption (C) the following Hardy inequality holds true [4],

$$\int_\Omega |\nabla u|^p \, dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} \, dx \geqslant 0, \quad u \in C_0^\infty(\Omega \setminus K), \tag{1.1}$$

where $|\frac{p-k}{p}|^p$ is the best constant.

Here is our main result, which shows that inequality (1.1) can be improved by adding a multiple of a whole range of critical norms that at the extreme case become precisely the critical Sobolev norm.

Theorem 1.1. *Let $2 \leqslant p < n$, $p \neq k < n$ and $p < q \leqslant \frac{np}{n-p}$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and K is a compact, C^2 manifold without boundary embedded in $\mathbb{R}^n$, of co-dimension k, $1 \leqslant k < n$. When $k = 1$ we assume that $K = \partial\Omega$, whereas for $1 < k < n$ we assume that $K \cap \bar{\Omega} \neq \emptyset$.*

(i) *If in addition Ω and K satisfy condition* (C)*, then there exists a positive constant $c = c(\Omega, K)$ such that for all $u \in C_0^\infty(\Omega \setminus K)$, there holds*

$$\int_\Omega |\nabla u|^p \, dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} \, dx \geqslant c\left(\int_\Omega d^{-q+\frac{q-p}{p}n} |u|^q \, dx\right)^{\frac{p}{q}}. \tag{1.2}$$

(ii) *Without assuming condition* (C)*, there exist a positive constant $c = c(n, k, p, q)$ independent of Ω, K and a constant $M = M(\Omega, K)$, such that for all $u \in C_0^\infty(\Omega \setminus K)$, there holds*:

$$\int_\Omega |\nabla u|^p \, dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} \, dx + M\int_\Omega |u|^p \, dx \geqslant c\left(\int_\Omega d^{-q+\frac{q-p}{p}n} |u|^q \, dx\right)^{\frac{p}{q}}. \tag{1.3}$$

We note that the term in the right-hand side of (1.2) and (1.3) is optimal and in fact (1.2) is a scale invariant inequality. In the extreme case where $q = \frac{np}{n-p}$, the term in the right-hand side is precisely the critical Sobolev term.

The only result previously known, in the spirit of estimate (1.2), concerns the particular case where $\Omega = \mathbb{R}^n$, $p = 2$ and K is affine, that is, $K = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_k = 0\}$, $1 \leqslant k < n$, $k \neq 2$ and has been established in [18]. The case $p \neq 2$ was posed as an open question in [18].

On the other hand the nonnegativity of the left-hand side of (1.3) for $p = 2$ has been shown in [5] for $K = \partial\Omega$. Other improvements of the plain Hardy inequality involving any arbitrary subcritical L^q term are presented in [11] for the case where Ω is a convex domain and $K = \partial\Omega$. For earlier results involving improvements with some subcritical L^q terms see [8].

We emphasize that in our theorem the case $k = n$, which corresponds to taking distance from an interior point, is excluded. As a matter of fact estimate (1.2) fails in this case. Indeed in this case, the optimal improvement of the plain Hardy inequality involves the critical Sobolev exponent, but contrary to (1.2) it also has a logarithmic correction [12].

To establish Theorem 1.1 a crucial step is to obtain local estimates in a neighborhood of K, see Theorem 5.1.

For other directions in improving Hardy inequalities we refer to [1,4–7,9,13,16–18,20–22] and references therein.

The paper is organized as follows. In Section 2 we establish auxiliary weighted Sobolev type inequalities, in the special case where distance is taken from the boundary. We then use these inequalities in Section 3 to derive Hardy–Sobolev inequalities when distance is taken from the boundary. In Sections 4 and 5 we consider more general distance functions, where distance is taken from a surface K of co-dimension k, as well as other critical norms via interpolation.

Some preliminary results have been announced in [10].

2. Weighted inequalities involving the distance function

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary and $d(x) = \text{dist}(x, \partial\Omega)$. We denote by $\Omega_\delta := \{x \in \Omega:\ \text{dist}(x, \partial\Omega) \leqslant \delta\}$ a tubular neighborhood of $\partial\Omega$, for δ small. Then, for δ small we have that $d(x) \in C^2(\Omega_\delta)$. Also, if $x \in \Omega_\delta$ approaches $x_0 \in \partial\Omega \in C^2$ then clearly $d(x) \to 0$, and also

$$\Delta d(x) = (N-1)H(x_0) + \mathrm{O}\big(d(x)\big),$$

where $H(x_0)$ is the mean curvature of $\partial\Omega$ at x_0; see e.g., [14, Section 14.6]. As a consequence of this we have that there exists a δ^* sufficiently small and a positive constant c_0 such that

$$|d\Delta d| \leqslant c_0 d, \quad \text{in } \Omega_\delta, \text{ for } 0 < \delta \leqslant \delta^*. \tag{R}$$

We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies condition (R) if there exists a c_0 and a δ^* such that (R) holds. In case $d(x)$ is not a C^2 function we interpret the inequality in (R) in the weak sense, that is

$$\left|\int_{\Omega_\delta} d\Delta d\phi \, \mathrm{d}x\right| \leqslant c_0 \int_{\Omega_\delta} d\phi \, \mathrm{d}x, \quad \forall \phi \in C_0^\infty(\Omega), \phi \geqslant 0.$$

In our proofs, instead of assuming that Ω is a bounded domain of class C^2 we will sometimes assume that Ω satisfies condition (R). Thus, some of our results hold true for a larger class of domains. For instance, if Ω is a strip or an infinite cylinder, condition (R) is easily seen to be satisfied even though Ω is not bounded.

We first prove an L^1 estimate.

Lemma 2.1. *Let Ω be a bounded domain which satisfies condition* (R). *For any $S \in (0, \frac{1}{2} n\pi^{\frac{1}{2}}[\Gamma(1+n/2)]^{-\frac{1}{n}})$ and any $a > 0$, there exists $\delta_0 = \delta_0(a/c_0)$ such that for all $\delta \in (0, \delta_0]$ there holds*:

$$\int_{\Omega_\delta} d^a |\nabla v| \, \mathrm{d}x + \int_{\partial\Omega_\delta^c} d^a |v| \, \mathrm{d}S_x \geqslant S \left\| d^a v \right\|_{L^{\frac{N}{N-1}}(\Omega_\delta)}, \quad \forall v \in C^\infty(\Omega). \tag{2.1}$$

Proof. We will use the following inequality: If $V \subset \mathbb{R}^n$ is any bounded domain and $u \in C^\infty(V)$, then

$$S_n \|u\|_{L^{\frac{n}{n-1}}(V)} \leqslant \|\nabla u\|_{L^1(V)} + \|u\|_{L^1(\partial V)}, \tag{2.2}$$

where $S_n = n\pi^{\frac{1}{2}}[\Gamma(1+n/2)]^{-\frac{1}{n}}$; see [18, p. 189].

For $V = \Omega_\delta$ we apply (2.2) to $u = d^a v$, $v \in C^\infty(\Omega)$ to get:

$$S_n \left\| d^a v \right\|_{L^{\frac{N}{N-1}}(\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^a |\nabla v| \, \mathrm{d}x + a \int_{\Omega_\delta} d^{a-1} |v| \, \mathrm{d}x + \int_{\partial\Omega_\delta^c} d^a |v| \, \mathrm{d}S_x. \tag{2.3}$$

To estimate the middle term of the right-hand side, noting that $\nabla d \cdot \nabla d = 1$ a.e. and integrating by parts we have:

$$a \int_{\Omega_\delta} d^{a-1} |v| \, \mathrm{d}x = \int_{\Omega_\delta} \nabla d^a \cdot \nabla d |v| \, \mathrm{d}x = -\int_{\Omega_\delta} d^a \Delta d |v| \, \mathrm{d}x - \int_{\Omega_\delta} d^a \nabla d \cdot \nabla |v| \, \mathrm{d}x + \int_{\partial\Omega_\delta^c} d^a |v| \, \mathrm{d}S_x.$$

Under our condition (R) for δ small we have $|d\Delta d| < c_0 d$ in Ω_δ. It follows that

$$(a - c_0\delta) \int_{\Omega_\delta} d^{a-1} |v| \, \mathrm{d}x \leqslant \int_{\Omega_\delta} d^a |\nabla v| \, \mathrm{d}x + \int_{\partial\Omega_\delta^c} d^a |v| \, \mathrm{d}S_x. \tag{2.4}$$

From (2.3) and (2.4) we get:

$$\frac{a - c_0\delta}{2a - c_0\delta} S_n \|d^a v\|_{L^{\frac{n}{n-1}}(\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^a |\nabla v| \, \mathrm{d}x + \int_{\partial\Omega_\delta^c} d^a |v| \, \mathrm{d}S_x.$$

The result then follows by taking

$$\delta_0 = \frac{a(S_n - 2S)}{c_0(S_n - S)}. \qquad \square \tag{2.5}$$

We similarly have

Lemma 2.2. *Let Ω be a domain which satisfies condition* (R). *For any $S \in (0, \frac{1}{2} n v_n^{\frac{1}{n}})$ and $a > 0$ there exists $\delta_0 = \delta_0(a/c_0)$ such that for all $\delta \in (0, \delta_0]$ there holds*:

$$\int_{\Omega_\delta} d^a |\nabla v| \, dx \geqslant S \| d^a v \|_{L^{\frac{n}{n-1}}(\Omega_\delta)}, \quad \forall v \in C_0^\infty(\Omega_\delta). \tag{2.6}$$

The proof is quite similar to that of the previous lemma. Instead of (2.2) one uses the $(p = 1)$–Gagliardo–Nirenberg inequality valid for any $V \subset \mathbb{R}^n$, and any $u \in C_0^\infty(V)$,

$$\tilde{S}_n \|u\|_{L^{\frac{n}{n-1}}(V)} \leqslant \|\nabla u\|_{L^1(V)}, \tag{2.7}$$

where $\tilde{S}_n = n v_n^{\frac{1}{n}}$, and v_n denotes the volume of the unit ball in $\mathbb{R}^n$.

We next prove

Theorem 2.3. *Let Ω be a bounded domain of class C^2 and $1 < p < n$. Then there exists a $\delta_0 = \delta_0(\Omega, p, n)$ such that for all $\delta \in (0, \delta_0]$ there holds*:

$$\int_{\Omega_\delta} d^{p-1} |\nabla v|^p \, dx + \int_{\partial\Omega_\delta^c} |v|^p \, dS_x \geqslant C(n,p) \| d^{\frac{p-1}{p}} v \|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)}, \quad \forall v \in C^\infty(\Omega), \tag{2.8}$$

with a constant $C(n, p)$ depending only on n and p.

Proof. We will denote by $C(p)$, $C(n, p)$, etc. positive constants, not necessarily the same in each occurrence, which depend *only* on their arguments. As a first step we will prove the following estimate:

$$C(n,p) \| d^{\frac{p-1}{p}} v \|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^{p-1} |\nabla v|^p \, dx + \| d^{\frac{p-1}{p}} v \|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta^c)}. \tag{2.9}$$

To this end we apply estimate (2.1) to $w = |v|^s$, $s = \frac{(n-1)p}{n-p}$ with $a = \frac{(n-1)(p-1)}{n-p} > 0$. Then,

$$S(n,p) \left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}} |v|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-1}{n}} \leqslant s \int_{\Omega_\delta} d^{\frac{(n-1)(p-1)}{n-p}} |v|^{\frac{n(p-1)}{n-p}} |\nabla v| \, dx + \int_{\partial\Omega_\delta^c} d^{\frac{(n-1)(p-1)}{n-p}} |v|^{\frac{(n-1)p}{n-p}} \, dS_x.$$

We next estimate the middle term,

$$\begin{aligned} \int_{\Omega_\delta} d^{\frac{(n-1)(p-1)}{n-p}} |v|^{\frac{n(p-1)}{n-p}} |\nabla v| \, dx &\leqslant \left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}} |v|^{\frac{np}{n-p}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega_\delta} d^{p-1} |\nabla v|^p \, dx \right)^{\frac{1}{p}} \\ &\leqslant \varepsilon \left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}} |v|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-1}{n}} + c_\varepsilon \left(\int_{\Omega_\delta} d^{p-1} |\nabla v|^p \, dx \right)^{\frac{n-1}{n-p}}, \end{aligned}$$

whence,

$$\big(S(n,p) - \varepsilon s\big) \left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}} |v|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-1}{n}} \leqslant s c_\varepsilon \left(\int_{\Omega_\delta} d^{p-1} |\nabla v|^p \, dx \right)^{\frac{n-1}{n-p}} + \int_{\partial\Omega_\delta^c} d^{\frac{(n-1)(p-1)}{n-p}} |v|^{\frac{(n-1)p}{n-p}} \, dS_x.$$

Raising the above estimate to the power $\frac{n-p}{n-1}$ we easily obtain (2.9).

To prove (2.8) we need to combine (2.9) with the following estimate:

$$C(n,p) \| d^{\frac{p-1}{p}} v \|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^{p-1} |\nabla v|^p \, dx + \int_{\partial\Omega_\delta^c} |v|^p \, dS_x. \tag{2.10}$$

In the rest of the proof we will show (2.10) We note that the norm in the left-hand side is the critical trace norm of the function $d^{\frac{p-1}{p}}v$. To estimate it we will use the critical trace inequality [3, Proposition 1],

$$\|u\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \leqslant C(n,p)\|\nabla u\|^p_{L^p(\Omega_\delta)} + M\|u\|^p_{L^p(\Omega_\delta)}, \tag{2.11}$$

where $M = M(n,p,\Omega)$ in general depends on the domain Ω as well. For reasons that we will explain later we will apply this estimate not directly to $d^{\frac{p-1}{p}}v$ but to the function $u = d^{\frac{p-1}{p}+\theta}v$ with $\theta > 0$ instead. More specifically we have:

$$\begin{aligned}\big\|d^{\frac{p-1}{p}}v\big\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} &= \delta^{-\theta p}\big\|d^{\frac{p-1}{p}+\theta}v\big\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \\ &\leqslant \delta^{-\theta p}\big(C(n,p)\big\|\nabla(d^{\frac{p-1}{p}+\theta}v)\big\|^p_{L^p(\Omega_\delta)} + M\big\|d^{\frac{p-1}{p}+\theta}v\big\|^p_{L^p(\Omega_\delta)}\big).\end{aligned}$$

Now,

$$\big\|\nabla\big(d^{\frac{p-1}{p}+\theta}v\big)\big\|_{L^p(\Omega_\delta)} \leqslant \left(\frac{p-1}{p}+\theta\right)\big\|d^{-\frac{1}{p}+\theta}v\big\|_{L^p(\Omega_\delta)} + \big\|d^{\frac{p-1}{p}+\theta}\nabla v\big\|_{L^p(\Omega_\delta)},$$

and

$$\big\|d^{\frac{p-1}{p}+\theta}v\big\|_{L^p(\Omega_\delta)} \leqslant \delta\big\|d^{-\frac{1}{p}+\theta}v\big\|_{L^p(\Omega_\delta)}.$$

From the above three estimates we conclude that

$$\big\|d^{\frac{p-1}{p}}v\big\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \leqslant C(p)\delta^{-\theta p}\int_{\Omega_\delta} d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x + \big[C(n,p,\theta)+M\delta^p\big]\delta^{-\theta p}\int_{\Omega_\delta} d^{-1+p\theta}|v|^p\,\mathrm{d}x,$$

whence, by choosing δ sufficiently small,

$$\big\|d^{\frac{p-1}{p}}v\big\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \leqslant C(p)\delta^{-\theta p}\int_{\Omega_\delta} d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x + C(n,p,\theta)\delta^{-\theta p}\int_{\Omega_\delta} d^{-1+p\theta}|v|^p\,\mathrm{d}x. \tag{2.12}$$

To continue we will estimate the last term of the right-hand side of (2.12). Consider the identity:

$$\theta p d^{-1+\theta p} = -d^{\theta p}\Delta d + \operatorname{div}\big(d^{\theta p}\nabla d\big). \tag{2.13}$$

We multiply it by $|v|^p$ and integrate by parts over Ω_δ to get:

$$\theta p\int_{\Omega_\delta} d^{-1+\theta p}|v|^p\,\mathrm{d}x = -\int_{\Omega_\delta} d^{\theta p}\Delta d|v|^p\,\mathrm{d}x - p\int_{\Omega_\delta} d^{\theta p}|v|^{p-1}\nabla d\cdot\nabla|v|\,\mathrm{d}x + \int_{\partial\Omega_\delta^c} d^{\theta p}|v|^p\,\mathrm{d}S_x.$$

By our assumption (R) we have that $|d^{\theta p}\Delta d| \leqslant c_0\delta d^{-1+\theta p}$. On the other hand,

$$\begin{aligned}\left|p\int_{\Omega_\delta} d^{\theta p}|v|^{p-1}\nabla d\cdot\nabla|v|\,\mathrm{d}x\right| &\leqslant p\int_{\Omega_\delta} d^{\theta p}|v|^{p-1}|\nabla v|\,\mathrm{d}x \\ &\leqslant p\varepsilon\int_{\Omega_\delta} d^{-1+\theta p}|v|^p\,\mathrm{d}x + pc_\varepsilon\int_{\Omega_\delta} d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x.\end{aligned}$$

Putting together the last estimates we get:

$$(\theta p - c_0\delta - p\varepsilon)\int_{\Omega_\delta} d^{-1+\theta p}|v|^p\,\mathrm{d}x \leqslant pc_\varepsilon\int_{\Omega_\delta} d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x + \int_{\partial\Omega_\delta^c} d^{\theta p}|v|^p\,\mathrm{d}S_x, \tag{2.14}$$

whence, choosing δ, ε sufficiently small,

$$C(p,\theta)\int_{\Omega_\delta} d^{-1+p\theta}|v|^p\,\mathrm{d}x \leqslant C(p)\int_{\Omega_\delta} d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x + \int_{\partial\Omega_\delta^c} d^{p\theta}|v|^p\,\mathrm{d}S_x. \tag{2.15}$$

Combining (2.12) and (2.15) we obtain:

$$C(n,p,\theta)\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \leqslant \delta^{-\theta p}\int_{\Omega_\delta} d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x + \delta^{-\theta p}\int_{\partial\Omega_\delta^c} d^{p\theta}|v|^p\,\mathrm{d}S_x$$
$$\leqslant \int_{\Omega_\delta} d^{p-1}|\nabla v|^p\,\mathrm{d}x + \int_{\partial\Omega_\delta^c}|v|^p\,\mathrm{d}S_x. \tag{2.16}$$

By choosing a specific value of θ, e.g., $\theta = 1$, we get (2.10). We note that estimate (2.15) fails if $\theta = 0$, and this is the reason for introducing this artificial parameter. □

We next have:

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying* (R) *and $1 < p < n$. Then there exists a $\delta_0 = \delta_0(c_0, p, n)$ such that for all $\delta \in (0, \delta_0]$ there holds*

$$\int_{\Omega_\delta} d^{p-1}|\nabla v|^p\,\mathrm{d}x \geqslant C(n,p)\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)}, \quad \forall v \in C_0^\infty(\Omega_\delta), \tag{2.17}$$

with a constant $C(n,p)$ depending only on n and p.

Proof. One works as in the derivation of (2.9), using however (2.6) in the place of (2.1). We omit the details. □

We finally establish the following:

Theorem 2.5. *Let $1 < p < n$ and $D = \sup_{x\in\Omega} d(x) < \infty$. We assume that Ω is a domain satisfying both conditions* (C) *and* (R). *Then there exists a positive constant $C = C(n, p, c_0 D)$ such that for any $v \in C_0^\infty(\Omega)$,*

$$\int_\Omega d^{p-1}|\nabla v|^p\,\mathrm{d}x + \int_\Omega(-\Delta d)|v|^p\,\mathrm{d}x \geqslant C\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{np}{n-p}}(\Omega)}. \tag{2.18}$$

Proof. We first define suitable cutoff functions supported near the boundary. Let $\alpha(t) \in C^\infty([0,\infty))$ be a nondecreasing function such that $\alpha(t) = 1$ for $t \in [0, 1/2)$, $\alpha(t) = 0$ for $t \geqslant 1$ and $|\alpha'(t)| \leqslant C_0$. For δ small we define $\phi_\delta(x) := \alpha(\frac{d(x)}{\delta}) \in C_0^2(\Omega)$. Note that $\phi_\delta = 1$ on $\Omega_{\delta/2}$, $\phi_\delta = 0$ on Ω_δ^c and $|\nabla\phi_\delta| = |\alpha'(\frac{d(x)}{\delta})|\frac{|\nabla d(x)|}{\delta} \leqslant \frac{C_0}{\delta}$ with C_0 a universal constant.

For $v \in C_0^\infty(\Omega)$ we write $v = \phi_\delta v + (1-\phi_\delta)v$. The function $\phi_\delta v$ is compactly supported in Ω_δ, and by Lemma 2.2, we have:

$$S\left\|d^a\phi_\delta v\right\|_{L^{\frac{n}{n-1}}(\Omega_\delta)} \leqslant \int_\Omega d^a\left|\nabla(\phi_\delta v)\right|\mathrm{d}x. \tag{2.19}$$

On the other hand $(1-\phi_\delta)v$ is compactly supported in $\Omega^c_{\delta/2}$ and using (2.7), we have:

$$C(n)\left\|d^a(1-\phi_\delta)v\right\|_{L^{\frac{n}{n-1}}(\Omega)} \leqslant \left(\frac{2D}{\delta}\right)^a\int_\Omega d^a\left|\nabla\big((1-\phi_\delta)v\big)\right|\mathrm{d}x. \tag{2.20}$$

Combining (2.19) and (2.20) and using elementary estimates, we obtain the following L^1 estimate:

$$C\left(a,n,\frac{\delta}{D}\right)\left\|d^a v\right\|_{L^{\frac{n}{n-1}}(\Omega)} \leqslant \int_\Omega |d^a\nabla v|\,\mathrm{d}x + \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{a-1}|v|\,\mathrm{d}x. \tag{2.21}$$

We next derive the corresponding L^p, $p>1$ estimate. To this end we replace v by $|v|^s$ with $s=\frac{p(n-1)}{n-p}$ in (2.21) to obtain:

$$C\left(a,n,p,\frac{\delta}{D}\right)\left(\int_\Omega d^{\frac{an}{n-1}}|v|^{\frac{np}{n-p}}\,\mathrm{d}x\right)^{\frac{n-1}{n}} \leqslant s\int_\Omega d^a|v|^{\frac{n(p-1)}{n-p}}|\nabla v|\,\mathrm{d}x + \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{a-1}|v|^{1+\frac{n(p-1)}{n-p}}\,\mathrm{d}x.$$

Using Holders inequality in both terms of the right-hand side of this we get after simplifying,

$$C\left(a,n,p,\frac{\delta}{D}\right)\left(\int_\Omega d^{\frac{an}{n-1}}|v|^{\frac{np}{n-p}}\,\mathrm{d}x\right)^{\frac{n-p}{np}} \leqslant s\left(\int_\Omega d^{\frac{a(n-p)}{n-1}}|\nabla v|^p\right)^{1/p} + \left(\int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{\frac{a(n-p)}{n-1}-p}|v|^p\right)^{1/p}. \tag{2.22}$$

For $a=\frac{(n-1)(p-1)}{n-p}>0$, this yields:

$$C\left(n,p,\frac{\delta}{D}\right)\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{np}{n-p}}(\Omega)} \leqslant \int_\Omega d^{p-1}|\nabla v|^p\,\mathrm{d}x + \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{-1}|v|^p\,\mathrm{d}x. \tag{2.23}$$

We note that condition (C) has not been used so far and therefore all previous estimates are valid even for general domains.

To complete the proof we will estimate the last term in (2.23). For $\theta>0$, we clearly have:

$$\left(\frac{\delta}{2}\right)^{p\theta}\int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{-1}|v|^p\,\mathrm{d}x \leqslant \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{-1+p\theta}|v|^p\,\mathrm{d}x \leqslant \int_\Omega d^{-1+p\theta}|v|^p\,\mathrm{d}x. \tag{2.24}$$

To estimate the last term we work as in (2.13)–(2.15). Thus, we start from the identity (2.13), multiply by $|v|^p$ and integrate by parts in Ω. Now there are no boundary terms and also the term containing Δd is not a lower order term anymore and has to be kept. Notice however that because of condition (C) we have that $-\Delta d\geqslant 0$ in the distributional sense. Without reproducing the details we write the analogue of (2.15) which is:

$$C(p,\theta)\int_\Omega d^{-1+p\theta}|v|^p\,\mathrm{d}x \leqslant C(p)\int_\Omega d^{p-1+p\theta}|\nabla v|^p\,\mathrm{d}x + \int_\Omega d^{p\theta}(-\Delta d)|v|^p\,\mathrm{d}x. \tag{2.25}$$

Combining (2.24) and (2.25) and recalling that $d\leqslant D$, we get:

$$C(p,\theta)\left(\frac{\delta}{D}\right)^{p\theta}\int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{-1}|v|^p\,\mathrm{d}x \leqslant \int_\Omega d^{p-1}|\nabla v|^p\,\mathrm{d}x + \int_\Omega(-\Delta d)|v|^p\,\mathrm{d}x. \tag{2.26}$$

Choosing e.g., $\theta=1$ and combining (2.26) and (2.23) the result follows. The dependence of the constant C in (2.18) on the domain Ω enters through the ratio δ/D. By Lemma 2.2 (cf. (2.5)) we obtain that the dependence of C on Ω enters through c_0D. We also note that $C(n,p,\infty)=0$. □

3. Hardy–Sobolev inequalities

Here we will prove various Hardy Sobolev inequalities. Let $d(x)=\operatorname{dist}(x,\partial\Omega)$ and $V\subset\Omega$. For $p>1$, and $u\in C_0^\infty(\Omega)$ we set:

$$I_p[u](V):=\int_V|\nabla u|^p\,\mathrm{d}x-\left(\frac{p-1}{p}\right)^p\int_V\frac{|u|^p}{d^p}\,\mathrm{d}x. \tag{3.1}$$

For simplicity we also write $I_p[u]$ instead of $I_p[u](\Omega)$. We next put:

$$u(x)=d^{\frac{p-1}{p}}(x)v(x). \tag{3.2}$$

We first prove an auxiliary inequality:

Lemma 3.1. *For $p \geqslant 2$, there exists positive constant $c = c(p)$ such that*

$$I_p[u](V) \geqslant c(p) \int_V d^{p-1} |\nabla v|^p \,\mathrm{d}x + \left(\frac{p-1}{p}\right)^{p-1} \int_V \nabla d \cdot \nabla |v|^p \,\mathrm{d}x. \tag{3.3}$$

Proof. We have that

$$\nabla u = \frac{p-1}{p} d^{\frac{p-1}{p}-1} v \nabla d + d^{\frac{p-1}{p}} \nabla v =: a + b.$$

For $p \geqslant 2$ we have that for $a, b \in \mathbb{R}^n$,

$$|a+b|^p - |a|^p \geqslant c(p)|b|^p + p|a|^{p-2} a \cdot b.$$

Using this we obtain:

$$I_p[u](V) \geqslant c(p) \int_V d^{p-1} |\nabla v|^p \,\mathrm{d}x + (\frac{p-1}{p})^{p-1} \int_V \nabla d \cdot \nabla |v|^p \,\mathrm{d}x \tag{3.4}$$

which is the sought for estimate. □

We first establish estimates in Ω_δ.

Theorem 3.2. *Let $2 \leqslant p < n$. We assume that Ω is a bounded domain of class C^2. Then, there exists a $\delta_0 = \delta_0(p, n, \Omega)$ such that for $0 < \delta \leqslant \delta_0$ and all $u \in C_0^\infty(\Omega)$,*

$$\int_{\Omega_\delta} |\nabla u|^p \,\mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_{\Omega_\delta} \frac{|u|^p}{d^p} \,\mathrm{d}x \geqslant C\left(\int_{\Omega_\delta} |u|^{\frac{np}{n-p}} \,\mathrm{d}x\right)^{\frac{n-p}{n}}, \tag{3.5}$$

where $C = C(n, p) > 0$ depends only on n and p.

Proof. Using Lemma 3.1 we have that

$$C(p) I_p[u](\Omega_\delta) \geqslant \int_{\Omega_\delta} d^{p-1} |\nabla v|^p \,\mathrm{d}x + \int_{\Omega_\delta} \nabla d \cdot \nabla |v|^p \,\mathrm{d}x.$$

Integrating by parts the last term, we get:

$$C(p) I_p[u](\Omega_\delta) \geqslant \int_{\Omega_\delta} d^{p-1} |\nabla v|^p \,\mathrm{d}x + \int_{\Omega_\delta} (-\Delta d) |v|^p \,\mathrm{d}x + \int_{\partial \Omega_\delta^c} |v|^p \,\mathrm{d}S_x. \tag{3.6}$$

We next estimate the middle term of the right-hand side. By condition (R), we have:

$$\left| \int_{\Omega_\delta} (-\Delta d) |v|^p \,\mathrm{d}x \right| \leqslant c_0 \int_{\Omega_\delta} |v|^p \,\mathrm{d}x. \tag{3.7}$$

Starting from the identity $1 + d\Delta d = \operatorname{div}(d \nabla d)$, we multiply it by $|v|^p$ and integrate by parts over Ω_δ to get:

$$\int_{\Omega_\delta} |v|^p \,\mathrm{d}x + \int_{\Omega_\delta} d \Delta d |v|^p \,\mathrm{d}x = -p \int_{\Omega_\delta} d |v|^{p-1} \nabla d \cdot \nabla |v| \,\mathrm{d}x + \delta \int_{\partial \Omega_\delta^c} |u|^p \,\mathrm{d}S.$$

Using once more (R) and standard inequalities we get:

$$(1 - \delta c_0 - \varepsilon p) \int_{\Omega_\delta} |v|^p \,\mathrm{d}x \leqslant \delta p C_\varepsilon \int_{\Omega_\delta} d^{p-1} |\nabla v|^p \,\mathrm{d}x + \delta \int_{\partial \Omega_\delta^c} |u|^p \,\mathrm{d}S,$$

whence for ε, δ sufficiently small,

$$\int_{\Omega_\delta} |v|^p \,\mathrm{d}x \leqslant C(p)\delta \int_{\Omega_\delta} d^{p-1}|\nabla v|^p \,\mathrm{d}x + C(p)\delta \int_{\partial\Omega_\delta^c} |u|^p \,\mathrm{d}S. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8) we obtain:

$$C(p)I_p[u](\Omega_\delta) \geqslant \int_{\Omega_\delta} d^{p-1}|\nabla v|^p \,\mathrm{d}x + \int_{\partial\Omega_\delta^c} |v|^p \,\mathrm{d}S_x. \tag{3.9}$$

To complete the proof we now use Theorem 2.3, that is,

$$\int_{\Omega_\delta} d^{p-1}|\nabla v|^p \,\mathrm{d}x + \int_{\partial\Omega_\delta^c} |v|^p \,\mathrm{d}S_x \geqslant C(n,p)\big\|d^{\frac{p-1}{p}} v\big\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)} = C(n,p)\|u\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)}. \tag{3.10}$$

The result then follows from (3.9) and (3.10). □

Next we prove:

Theorem 3.3. *Let $2 \leqslant p < n$. We assume that Ω is a bounded domain of class C^2. Then there exist positive constants $M = M(n,p,\Omega)$ and $C = C(n,p)$ such that for all $u \in C_0^\infty(\Omega)$, there holds*:

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p}\,\mathrm{d}x + M\int_\Omega |u|^p \,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^{\frac{np}{n-p}}\,\mathrm{d}x\right)^{\frac{n-p}{n}}. \tag{3.11}$$

We emphasize that $C(n,p)$ is independent of Ω.

Proof. Clearly we have:

$$I_p[u](\Omega) = I_p[u](\Omega_\delta) + I_p[u]\big(\Omega_\delta^c\big). \tag{3.12}$$

By Theorem 3.2, for δ small, we have:

$$I_p[u](\Omega_\delta) \geqslant C(n,p)\|u\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)}. \tag{3.13}$$

Since $d(x) \geqslant \delta$ in Ω_δ^c,

$$I_p[u](\Omega_\delta^c) \geqslant \int_{\Omega_\delta^c} |\nabla u|^p \,\mathrm{d}x - \left(\frac{p-1}{p\delta}\right)^p \int_{\Omega_\delta^c} |u|^p \,\mathrm{d}x. \tag{3.14}$$

Using the Sobolev embedding of $L^{\frac{np}{n-p}}(\Omega_\delta^c)$ into $W^{1,p}(\Omega_\delta^c)$, see [15, Theorem 4.1], we get:

$$\|u\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta^c)} \leqslant C(n,p)\int_{\Omega_\delta^c} |\nabla u|^p \,\mathrm{d}x + C(n,p,\Omega)\int_{\Omega_\delta^c} |u|^p \,\mathrm{d}x.$$

From this and (3.14), we get:

$$I_p[u](\Omega_\delta^c) \geqslant C(n,p)\|u\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta^c)} - C(n,p,\Omega)\int_\Omega |u|^p \,\mathrm{d}x. \tag{3.15}$$

The result follows from (3.12), (3.14) and (3.15). □

We finally show:

Theorem 3.4. *Let $2 \leqslant p < n$ and $D = \sup_{x\in\Omega} d(x) < \infty$. We assume that Ω is a domain satisfying both conditions* (C) *and* (R). *Then there exists a positive constant $C = C(n, p, c_0 D)$ such that for any $u \in C_0^\infty(\Omega)$ there holds*:

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p}\,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^{\frac{np}{n-p}}\,\mathrm{d}x\right)^{\frac{n-p}{n}}. \tag{3.16}$$

Proof. Working as in the derivation of (3.6), we get:

$$C(p) I_p[u](\Omega) \geqslant \int_\Omega d^{p-1}|\nabla v|^p \,\mathrm{d}x + \int_\Omega (-\Delta d)|v|^p \,\mathrm{d}x.$$

The result then follows from Theorem 2.5. □

4. Extensions

Here we will extend the previous inequalities in two directions. First by considering different distant functions and secondly by interpolating between the Sobolev $L^{\frac{pn}{n-p}}$ norm and the L^p norm. This way we will obtain new scale invariant inequalities.

We denote by K a surface embedded in $\mathbb{R}^n$, of codimension k, $1 < k < n$. We also allow for the extreme cases $k = n$ or 1, with the following convention. In case $k = n$, K is identified with the origin, that is $K = \{0\}$, assumed to be in the interior of Ω. In case $k = 1$, K is identified with $\partial\Omega$.

From now on distance is taken from K, that is, $d(x) = \operatorname{dist}(x, K)$. We also set $K_\delta := \{x \in \Omega:\ \operatorname{dist}(x, K) \leqslant \delta\}$ is a tubular neighborhood of K, for δ small, and $K_\delta^c := \Omega \setminus K_\delta$.

We say that K satisfies condition (R) whenever there exists a δ^* sufficiently small and a positive constant c_0 such that

$$|d\Delta d + 1 - k| \leqslant c_0 d, \quad \text{in } K_\delta,\ \text{for } 0 < \delta \leqslant \delta^*. \tag{R}$$

For $k = 1$ this coincides with condition (R) of Section 2. For $k > 1$, if K is a compact, C^2 surface without boundary, then condition (R) is satisfied; see, e.g., [2, Theorem 3.2] or [19, Section 3].

We next present an interpolation lemma:

Lemma 4.1. *Let a, b, p and q be such that*

$$1 \leqslant p < n, \quad p < q \leqslant \frac{pn}{n-p}, \quad \textit{and} \quad b = a - 1 + \frac{q-p}{qp} n. \tag{4.1}$$

Then for any $\eta > 0$, there holds:

$$\left\|d^b v\right\|_{L^q(\Omega)} \leqslant \lambda \eta^{-\frac{1-\lambda}{\lambda}} \left\|d^a v\right\|_{L^{\frac{pn}{n-p}}(\Omega)} + (1-\lambda)\eta \left\|d^{a-1} v\right\|_{L^p(\Omega)}, \quad \forall v \in C^\infty(\Omega), \tag{4.2}$$

where

$$0 < \lambda := \frac{n(q-p)}{qp} \leqslant 1. \tag{4.3}$$

Proof. For $p_s := \frac{pn}{n-p}$ and λ as in (4.3) we use Holder's inequality to obtain:

$$\int_\Omega d^{qb}|v|^q \,\mathrm{d}x = \int_\Omega \left(d^{a\lambda q}|v|^{\lambda q}\right)\left(d^{q(b-a\lambda)}|v|^{q(1-\lambda)}\right)\mathrm{d}x$$
$$\leqslant \left(\int_\Omega d^{ap_s}|v|^{p_s}\,\mathrm{d}x\right)^{\frac{\lambda q}{p_s}} \left(\int_\Omega d^{p(a-1)}|v|^p\,\mathrm{d}x\right)^{\frac{(1-\lambda)q}{p}},$$

that is,

$$\left\|d^b v\right\|_{L^q(\Omega)} \leqslant \left\|d^a v\right\|^{\lambda}_{L^{\frac{pn}{n-p}}(\Omega)} \left\|d^{a-1} v\right\|^{1-\lambda}_{L^p(\Omega)}.$$

Combining this with Young's inequality,

$$X^{\lambda}Y^{1-\lambda} \leqslant \lambda\eta^{-\frac{1-\lambda}{\lambda}}X + (1-\lambda)\eta Y, \quad \eta > 0, \tag{4.4}$$

the result follows. □

We first prove inequalities in K_δ.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and K a C^2 surface of codimension k, satisfying condition* (R). *We also assume that*

$$p = 1 < q \leqslant \frac{n}{n-1}, \quad b = a - 1 + \frac{q-1}{q}n, \quad \text{and} \quad a \neq 1-k. \tag{4.5}$$

Then there exists a $\delta_0 = \delta_0(\frac{|a+k-1|}{c_0})$ and $C = C(a,q,n,k) > 0$ such that for all $\delta \in (0, \delta_0]$, there holds:

$$\int_{K_\delta} d^a|\nabla v|\,\mathrm{d}x + \int_{\partial K_\delta} d^a|v|\,\mathrm{d}S_x \geqslant C\|d^b v\|_{L^q(K_\delta)}, \quad \forall v \in C_0^\infty(\Omega \setminus K). \tag{4.6}$$

Proof. Using the interpolation inequality (4.2) in K_δ with $\eta = 1$, we get:

$$\begin{aligned}\|d^b v\|_{L^q(K_\delta)} &\leqslant \frac{n(q-1)}{q}\|d^a v\|_{L^{\frac{N}{N-1}}(K_\delta)} + \frac{q-n(q-1)}{q}\|d^{a-1}v\|_{L^1(K_\delta)} \\ &\leqslant C(n,q)\left(\|d^a v\|_{L^{\frac{N}{N-1}}(K_\delta)} + \int_{K_\delta} d^{a-1}|v|\,\mathrm{d}x\right).\end{aligned} \tag{4.7}$$

For $V = K_\delta$ we apply (2.2) to $u = d^a v$, $v \in C^\infty(\Omega)$ to get,

$$S_n\|d^a v\|_{L^{\frac{n}{n-1}}(K_\delta)} \leqslant \int_{K_\delta} d^a|\nabla v|\,\mathrm{d}x + |a|\int_{K_\delta} d^{a-1}|v|\,\mathrm{d}x + \int_{\partial K_\delta} d^a|v|\,\mathrm{d}S_x. \tag{4.8}$$

Combining (4.7) and (4.8) we get the analogue of (2.3) which is,

$$C(a,q,n)\|d^b v\|_{L^q(K_\delta)} \leqslant \int_{K_\delta} d^a|\nabla v|\,\mathrm{d}x + \int_{K_\delta} d^{a-1}|v|\,\mathrm{d}x + \int_{\partial K_\delta} d^a|v|\,\mathrm{d}S_x. \tag{4.9}$$

It remains to estimate the middle term of the right-hand side. Noting that $\nabla d \cdot \nabla d = 1$ a.e. and integrating by parts in K_δ, we have:

$$a\int_{K_\delta} d^{a-1}|v|\,\mathrm{d}x = \int_{K_\delta} \nabla d^a \cdot \nabla d|v|\,\mathrm{d}x = -\int_{K_\delta} d^a \Delta d|v|\,\mathrm{d}x - \int_{K_\delta} d^a \nabla d \cdot \nabla|v|\,\mathrm{d}x + \int_{\partial K_\delta} d^a|v|\,\mathrm{d}S_x,$$

whence,

$$(a+k-1)\int_{K_\delta} d^{a-1}|v|\,\mathrm{d}x - \int_{K_\delta} d^{a-1}(d\Delta d + 1 - k)|v|\,\mathrm{d}x - \int_{K_\delta} d^a\nabla d \cdot \nabla|v|\,\mathrm{d}x + \int_{\partial K_\delta} d^a|v|\,\mathrm{d}S_x.$$

Using (R) we easily arrive at the analogue of (2.4), that is,

$$\big(|a+k-1| - c_0\delta\big)\int_{K_\delta} d^{a-1}|v|\,\mathrm{d}x \leqslant \int_{K_\delta} d^a|\nabla v|\,\mathrm{d}x + \int_{\partial K_\delta} d^a|v|\,\mathrm{d}S_x. \tag{4.10}$$

For estimate (4.10) to be useful we need $|a+k-1| > 0$, whence the restriction $a \neq 1-k$. The result then follows from (4.9) and (4.10), taking e.g., $\delta_0 = \frac{|a+k-1|}{2c_0}$. □

We next present the analogue of Lemma 2.2:

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and K a surface of co-dimension k, satisfying condition* (R)*. We also assume*

$$p = 1 < q \leqslant \frac{n}{n-1}, \quad b = a - 1 + \frac{q-1}{q} n, \quad \text{and} \quad a \neq 1 - k.$$

Then, there exists a $\delta_0 = \delta_0(\frac{|a+k-1|}{c_0})$ and a $C = C(a, q, n, k) > 0$, such that for all $\delta \in (0, \delta_0]$ there holds:

$$\int_{K_\delta} d^a |\nabla v| \, \mathrm{d}x \geqslant C \| d^b v \|_{L^q(K_\delta)}, \quad \forall v \in C_0^\infty(K_\delta). \tag{4.11}$$

The proof is quite similar to that of the previous lemma. The only difference is that instead of (2.2) one uses (2.7). We omit the details.

We next have:

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and K a C^2 surface of co-dimension k, with $1 \leqslant k < n$, satisfying condition* (R)*. We also assume*:

$$1 \leqslant p < n, \quad p < q \leqslant \frac{pn}{n-p}, \quad \text{and} \quad b = a - 1 + \frac{q-p}{qp} n, \tag{4.12}$$

and set $a = \frac{p-k}{p}$. Then there exists a $\delta_0 = \delta_0(p, q, \Omega, K)$ and $C = C(p, q, n, k) > 0$ such that for all $\delta \in (0, \delta_0]$ and all $v \in C_0^\infty(\Omega \setminus K)$, there holds:

$$\int_{K_\delta} d^{p-k} |\nabla v|^p \, \mathrm{d}x + \int_{\partial K_\delta} d^{1-k} |v|^p \, \mathrm{d}S_x \geqslant C \| d^b v \|^p_{L^q(K_\delta)}; \tag{4.13}$$

in particular the constant C is independent of Ω, K.

Proof. We will use Lemma 4.2. Since in this lemma the parameters a, b, p, q have a different meaning, to avoid confusion, we will use capital letters for the parameters a, b, p, q appearing in the statement of the present theorem. That is, we suppose that

$$1 \leqslant P < n, \quad P < Q \leqslant \frac{Pn}{n-P}, \quad \text{and} \quad B = A - 1 + \frac{Q-P}{QP} n, \tag{4.14}$$

and for $A = \frac{P-k}{P}$, we will prove that the following estimate holds true,

$$\int_{K_\delta} d^{P-k} |\nabla v|^P \, \mathrm{d}x + \int_{\partial K_\delta} d^{1-k} |v|^P \, \mathrm{d}S_x \geqslant C \| d^B v \|^P_{L^Q(K_\delta)}. \tag{4.15}$$

We will argue in a similar way, as in the proof of Theorem 2.3. We first prove the following $L^Q - L^P$ estimate:

$$C(P, Q, n, k) \| d^B v \|^P_{L^Q(K_\delta)} \leqslant \int_{K_\delta} d^{P-k} |\nabla v|^P \, \mathrm{d}x + \int_{\partial K_\delta} d^{1-k} |v|^P \, \mathrm{d}S_x + \| d^{\frac{P-k}{P}} v \|^P_{L^{\frac{(n-1)P}{n-P}}(\partial K_\delta)}. \tag{4.16}$$

To this end we replace in (4.6) v by $|v|^s$ with

$$s = Q \frac{P-1}{P} + 1. \tag{4.17}$$

Also, for A, B, P and Q as in (4.14), we set:

$$q = Q s^{-1}, \quad b = Bs, \quad a = b + 1 - \frac{q-1}{q} N = BQ \frac{P-1}{P} + A. \tag{4.18}$$

It is easy to check that a, b, q thus defined satisfy (4.5). Then, from (4.6), we have:

$$\| d^B v \|^{1 + \frac{P-1}{P} Q}_{L^Q(K_\delta)} = \| d^b |v|^s \|_{L^q(K_\delta)} \leqslant Cs \int_{K_\delta} d^a |v|^{s-1} |\nabla v| \, \mathrm{d}x + C \int_{\partial K_\delta} d^a |v|^s \, \mathrm{d}x, \tag{4.19}$$

with $C = C(a, q, n, k) = C(P, Q, A, n, k)$. Using Holder's inequality in the middle term of the right-hand side, we get:

$$\int_{K_\delta} d^a |v|^{s-1} |\nabla v| \,\mathrm{d}x = \int_{K_\delta} d^A |\nabla v| d^{BQ\frac{P-1}{P}} |v|^{Q\frac{P-1}{P}} \,\mathrm{d}x$$
$$\leqslant \big\| d^A |\nabla v| \big\|_{L^P(K_\delta)} \big\| d^B v \big\|_{L^Q(K_\delta)}^{\frac{P-1}{P}Q}$$
$$\leqslant c_\varepsilon \big\| d^A |\nabla v| \big\|_{L^P(K_\delta)}^{1+\frac{P-1}{P}Q} + \varepsilon \big\| d^B v \big\|_{L^Q(K_\delta)}^{1+\frac{P-1}{P}Q}. \tag{4.20}$$

From now on we use the specific value of $A = \frac{P-k}{P}$. For this choice of A a straightforward calculation shows that

$$a - 1 + k = \frac{P-1}{P} \frac{Q-P}{P} (n-k) \neq 0, \tag{4.21}$$

and therefore it corresponds to an acceptable value of a, see (4.5). Because of (4.21) the case $k = n$ is excluded.

We next estimate the last term of (4.19). Using Holder's inequality (similarly as in Lemma 4.1), we get:

$$\int_{\partial K_\delta} d^a |v|^s \,\mathrm{d}x = \int_{\partial K_\delta} d^\mu |v|^{\lambda(Q\frac{P-1}{P}+1)} d^{BQ\frac{P-1}{P}+A-\mu} |v|^{(1-\lambda)(Q\frac{P-1}{P}+1)} \,\mathrm{d}x$$
$$\leqslant \bigg(\int_{\partial K_\delta} d^{\frac{(P-k)(n-1)}{n-P}} |v|^{\frac{P(n-1)}{n-P}} \,\mathrm{d}x \bigg)^{\frac{\lambda(n-P)}{(n-1)P}(Q\frac{P-1}{P}+1)} \bigg(\int_{\partial K_\delta} d^{1-k} |v|^P \,\mathrm{d}x \bigg)^{\frac{1-\lambda}{P}(Q\frac{P-1}{P}+1)},$$

where,

$$\lambda = \frac{(n-1)(Q-P)}{Q(P-1)+P}, \quad \text{and} \quad \mu = \frac{(n-1)(Q-P)(P-k)}{P^2}.$$

Using then Young's inequality (cf. (4.4)) we obtain for a positive constant $C = C(P, Q, n)$,

$$C \int_{\partial K_\delta} d^a |v|^s \,\mathrm{d}x \leqslant \big(\big\| d^{\frac{P-k}{P}} v \big\|_{L^{\frac{P(n-1)}{n-P}}(\partial K_\delta)} + \big\| d^{\frac{1-k}{P}} v \big\|_{L^P(\partial K_\delta)} \big)^{Q\frac{P-1}{P}+1}. \tag{4.22}$$

From (4.19), (4.20) and (4.22) we easily obtain (4.16).

To complete the proof of the theorem we will show that

$$C \big\| d^{\frac{P-k}{P}} v \big\|_{L^{\frac{P(n-1)}{n-P}}(\partial K_\delta)}^P \leqslant \int_{K_\delta} d^{P-k} |\nabla v|^P \,\mathrm{d}x + \int_{\partial K_\delta} d^{1-k} |v|^P \,\mathrm{d}S_x, \tag{4.23}$$

for a positive constant $C = C(P, Q, n, k)$. The proof of (4.23) parallels that of (2.10). In particular, for $k = 1$ this is precisely estimate (2.10). In the sequel we will sketch the proof of (4.23).

Applying the critical trace inequality (2.11) to $d^{\frac{P-k}{P}+\theta} v$, $\theta > 0$, in the domain K_δ we obtain for δ sufficiently small the analogue of (2.12), that is

$$\big\| d^{\frac{P-k}{P}} v \big\|_{L^{\frac{P(n-1)}{n-P}}(\partial K_\delta)}^P \leqslant C(P,k)\delta^{-\theta P} \int_{K_\delta} d^{P-k+P\theta} |\nabla v|^P \,\mathrm{d}x + C(n,P,k,\theta)\delta^{-\theta P} \int_{K_\delta} d^{-k+P\theta} |v|^P \,\mathrm{d}x. \tag{4.24}$$

We next estimate the last term of (4.24). Starting from the identity,

$$(1 - k + \theta P) d^{-k+\theta P} = -d^{1-k+\theta P} \Delta d + \operatorname{div}\big(d^{1-k+\theta P} \nabla d\big), \tag{4.25}$$

we multiply it by $|v|^P$ and integrate by parts over K_δ to get:

$$(1 - k + \theta P) \int_{K_\delta} d^{-k+\theta P} |v|^P \,\mathrm{d}x$$
$$= -\int_{K_\delta} d^{1-k+\theta P} \Delta d |v|^P \,\mathrm{d}x - P \int_{K_\delta} d^{1-k+\theta P} |v|^{P-1} \nabla d \cdot \nabla |v| \,\mathrm{d}x + \int_{\partial K_\delta} d^{1-k+\theta P} |v|^P \,\mathrm{d}S_x,$$

or, equivalently,

$$\theta P\int_{K_\delta} d^{-k+\theta P}|v|^P\,\mathrm{d}x = -\int_{K_\delta} d^{k+\theta P}(d\Delta d+1-k)|v|^P\,\mathrm{d}x - P\int_{K_\delta} d^{1-k+\theta P}|v|^{P-1}\nabla d\cdot\nabla|v|\,\mathrm{d}x + \int_{\partial K_\delta} d^{1-k+\theta P}|v|^P\,\mathrm{d}S_x.$$

By our condition (R) we have that $|d\Delta d+1-k|\leqslant c_0 d$. On the other hand,

$$\left|P\int_{K_\delta} d^{1-k+\theta P}|v|^{P-1}\nabla d\cdot\nabla|v|\,\mathrm{d}x\right| \leqslant P\int_{K_\delta} d^{1-k+\theta P}|v|^{P-1}|\nabla v|\,\mathrm{d}x \leqslant P\varepsilon\int_{K_\delta} d^{-k+\theta P}|v|^P\,\mathrm{d}x + Pc_\varepsilon\int_{K_\delta} d^{P-k+\theta P}|\nabla v|^P\,\mathrm{d}x.$$

Putting together the last estimates we obtain, for ε, δ small the analogue of (2.15) that is

$$C(P,\theta)\int_{K_\delta} d^{-k+P\theta}|v|^P\,\mathrm{d}x \leqslant C(P)\int_{K_\delta} d^{P-k+P\theta}|\nabla v|^P\,\mathrm{d}x + \int_{\partial K_\delta} d^{1-k+P\theta}|v|^P\,\mathrm{d}S_x. \tag{4.26}$$

Combining (4.24), (4.26) and using the fact that $d(x)\leqslant\delta$ when $x\in K_\delta$, we complete the proof of (4.23) as well as of the theorem. □

Remark 1. We note that estimate (4.13) fails when $k=n$ (see (4.21)). This is not accidental as we shall see in the next section.

Remark 2. The choice $a=\frac{p-k}{p}$ corresponds to the Hardy–Sobolev inequality as it will become clear in the next section. We note that the corresponding estimate for $a\in\mathbb{R}$ and b, p, q as in (4.12) remains true. Thus, there exists a positive constant $C=C(a,n,p,q,k)$ such that for all $v\in C_0^\infty(\Omega\setminus K)$ there holds:

$$\int_{K_\delta} d^{ap}|\nabla v|^p\,\mathrm{d}x + \int_{\partial K_\delta} d^{(a-1)p+1}|v|^p\,\mathrm{d}S_x \geqslant C\|d^b v\|_{L^q(\Omega)}. \tag{4.27}$$

The proof of (4.27) in case $a\neq\frac{p-k}{p}$ is much simpler than in the case $a=\frac{p-k}{p}$. We also note that if $a\neq\frac{p-k}{p}$ then (4.27) is true even if $k=n$.

We will finally prove the analogue of Theorem 2.5.

Theorem 4.5. *Let $\Omega\subset\mathbb{R}^n$ be a domain and K a surface of codimension k, $1\leqslant k<n$, satisfying both conditions* (R)*. In addition we assume that $D=\sup_{x\in\Omega} d(x)<\infty$, condition* (C) *is satisfied, and*

$$1\leqslant p<n,\quad p<q\leqslant\frac{pn}{n-p},\quad \textit{and}\quad b=a-1+\frac{q-p}{qp}n. \tag{4.28}$$

We set $a=\frac{p-k}{p}$. Then there exists a positive constant $C=C(p,n,\Omega,K)$ such that for all $v\in C_0^\infty(\Omega\setminus K)$ there holds:

$$\int_\Omega d^{p-k}|\nabla v|^p\,\mathrm{d}x + \left|\int_\Omega d^{-k}(-d\Delta d-1+k)|v|^p\,\mathrm{d}x\right| \geqslant C\|d^b v\|^p_{L^q(\Omega)}. \tag{4.29}$$

Proof. As before, to avoid confusion in the proof, we will use capital letters for the parameters a, b, p, q appearing in the statement of the Theorem. That is, we suppose that

$$1\leqslant P<n,\quad P<Q\leqslant\frac{Pn}{n-P},\quad \text{and}\quad B=A-1+\frac{Q-P}{QP}n,$$

and for $A = \frac{P-k}{P}$, we will prove that

$$\int_\Omega d^{P-k}|\nabla v|^P\,\mathrm{d}x + \left|\int_\Omega d^{-k}(-d\Delta d - 1 + k)|v|^P\,\mathrm{d}x\right| \geqslant C\left\|d^B v\right\|^P_{L^Q(\Omega)}. \tag{4.30}$$

Let $\alpha(t) \in C^\infty([0,\infty))$ be the nondecreasing function defined at the beginning of the proof of Theorem 2.4 and $\phi_\delta(x) := \alpha(\frac{d(x)}{\delta}) \in C_0^2(\Omega)$, so that $\phi_\delta = 1$ on $K_{\delta/2}$, $\phi_\delta = 0$ on K_δ^c and $|\nabla\phi_\delta| \leqslant \frac{C_0}{\delta}$ with C_0 a universal constant.

For $v \in C_0^\infty(\Omega)$ we write $v = \phi_\delta v + (1-\phi_\delta)v$. The function $\phi_\delta v$ is compactly supported in K_δ, and by Lemma 4.3, we have:

$$C(a,n,q)\left\|d^b v\right\|_{L^q(K_\delta)} \leqslant \int_{K_\delta} d^a|\nabla v|\,\mathrm{d}x. \tag{4.31}$$

On the other hand $(1-\phi_\delta)v$ is compactly supported in $K^c_{\delta/2}$ and using (2.7) we easily get

$$\left\|d^b(1-\phi_\delta)v\right\|_{L^q(K^c_{\delta/2})} \leqslant C(\Omega)\frac{D^{|b|}}{\delta^{|a|}}\left\|d^a\left|\nabla\big((1-\phi_\delta)v\big)\right|\right\|_{L^1(K^c_{\delta/2})}. \tag{4.32}$$

Combining (4.31) and (4.32) we obtain the analogue of (2.21) which is

$$C\left\|d^a v\right\|_{L^{\frac{n}{n-1}}(\Omega)} \leqslant \int_\Omega \left|d^a\nabla v\right|\mathrm{d}x + \int_{K_\delta\setminus K_{\delta/2}} d^{a-1}|v|\,\mathrm{d}x. \tag{4.33}$$

We next pass to L^Q–L^P estimates. We replace in (4.33) v by $|v|^s$ with s as in (4.17). Also, for $A = \frac{P-k}{P}$ and B, P, Q as in (4.18), we get (cf. (4.19)):

$$C\left\|d^B v\right\|^{1+\frac{P-1}{P}Q}_{L^Q(K_\delta)} \leqslant s\int_{K_\delta} d^a|v|^{s-1}|\nabla v|\,\mathrm{d}x + \int_{K_\delta\setminus K_{\delta/2}} d^{a-1}|v|^s\,\mathrm{d}x. \tag{4.34}$$

Using Holder's inequality in both terms of the right-hand side we get

$$\begin{aligned}\int_\Omega d^a|v|^{s-1}|\nabla v|\,\mathrm{d}x &= \int_\Omega d^A|\nabla v| d^{BQ\frac{P-1}{P}}|v|^{Q\frac{P-1}{P}}\,\mathrm{d}x\\ &\leqslant \left\|d^A|\nabla v|\right\|_{L^P(\Omega)}\left\|d^B v\right\|^{\frac{P-1}{P}Q}_{L^Q(\Omega)},\end{aligned}$$

and

$$\begin{aligned}\int_{K_\delta\setminus K_{\delta/2}} d^{a-1}|v|^s\,\mathrm{d}x &= \int_{K_\delta\setminus K_{\delta/2}} d^{A-1}|v| d^{BQ\frac{P-1}{P}}|v|^{Q\frac{P-1}{P}}\,\mathrm{d}x\\ &\leqslant \left\|d^{A-1}|v|\right\|_{L^P(K_\delta\setminus K_{\delta/2})}\left\|d^B v\right\|^{\frac{P-1}{P}Q}_{L^Q(\Omega)}.\end{aligned}$$

Substituting into (4.34) we get after simplifying,

$$C\left\|d^B v\right\|^P_{L^Q(\Omega)} \leqslant \int_\Omega d^{P-k}|\nabla v|^P\,\mathrm{d}x + \int_{K_\delta\setminus K_{\delta/2}} d^{-k}|v|^P\,\mathrm{d}x. \tag{4.35}$$

Here we have also used the specific value of $A = \frac{P-k}{P}$. To conclude we need to estimate the last term in (4.35). For $\theta > 0$, we clearly have:

$$\left(\frac{\delta}{2}\right)^{p\theta}\int_{K_\delta\setminus K_{\delta/2}} d^{-k}|v|^P\,\mathrm{d}x \leqslant \int_{K_\delta\setminus K_{\delta/2}} d^{-k+P\theta}|v|^P\,\mathrm{d}x \leqslant \int_\Omega d^{-k+P\theta}|v|^P\,\mathrm{d}x. \tag{4.36}$$

To estimate the last term we work as in (2.24)–(2.25) (see also (4.25)–(4.26)) to finally get

$$\int_\Omega d^{-k+P\theta}|v|^P\,\mathrm{d}x \leqslant C(p)\int_\Omega d^{P-k+P\theta}|\nabla v|^P\,\mathrm{d}x + \left|\int_\Omega d^{-k+P\theta}(-d\Delta d + 1 - k)|v|^P\,\mathrm{d}x\right|. \tag{4.37}$$

We not that we also used the fact that

$$p \neq k, \quad \text{and} \quad (p-k)(d\Delta d + 1 - k) \leqslant 0, \quad \text{on } \Omega \setminus K, \tag{4.38}$$

which is a direct consequence of condition (C); see [4]. Combining (4.36) and (4.37) and recalling that $d \leqslant D$, we get:

$$C\left(P, \theta, \frac{\delta}{D}\right) \int_{K_\delta \setminus K_{\delta/2}} d^{-k}|v|^P \,\mathrm{d}x \leqslant \int_\Omega d^{P-k}|\nabla v|^P \,\mathrm{d}x + \left|\int_\Omega d^{-k}\big(-d\Delta d + 1 - k\big)|v|^P \,\mathrm{d}x\right|, \tag{4.39}$$

and the result follows easily. □

Remark 1. As in Theorem 4.4 the case $k = n$ is excluded.

Remark 2. In case $k = 1$ or in case $q = \frac{np}{n-p}$, the dependence of the constant C in (4.29) from Ω, K is the same as in Theorem 2.5, that is, $C = C(n, p, q, c_0 D)$.

Remark 3. In case $a \neq \frac{p-k}{p}$ the analogue of (4.29) remains true. That is, for b, p, q as in (4.28),

$$\int_\Omega d^{ap}|\nabla v|^p \,\mathrm{d}x + \left|\int_\Omega d^{(a-1)p}(-d\Delta d - 1 + k)|v|^p \,\mathrm{d}x\right| \geqslant C\left\|d^b v\right\|_{L^q(\Omega)}, \tag{4.40}$$

for a constant $C = C(p, q, n, k, a) > 0$. The case $k = n$ is not excluded.

5. Extended Hardy–Sobolev inequalities

In this section we will use the v-inequalities of the previous section to prove new Hardy–Sobolev inequalities. For $V \subset \mathbb{R}^n$ we set:

$$I_{p,k}[u](V)\colon \quad \int_V |\nabla u|^p \,\mathrm{d}x - \left|\frac{p-k}{p}\right|^p \int_V \frac{|u|^p}{d^p} \,\mathrm{d}x. \tag{5.1}$$

Then for $u(x) = d^H(x)v(x)$ with

$$H := \frac{p-k}{p},$$

we have for $p \geqslant 2$,

$$I_{p,k}[u](V) \geqslant c(p) \int_V d^{p-k}|\nabla v|^p \,\mathrm{d}x + H|H|^{p-2} \int_V d^{1-k}\nabla d \cdot \nabla |v|^p \,\mathrm{d}x. \tag{5.2}$$

The proof of (5.2) is quite similar to the proof of (3.3).

As in the previous section,

$$1 \leqslant p < n, \quad p < q \leqslant \frac{pn}{n-p}, \quad \text{and} \quad b = a - 1 + \frac{q-p}{qp} n. \tag{5.3}$$

We will be interested in the specific value $a = \frac{p-k}{p}$ which corresponds to the critical Hardy Sobolev inequalities.

We first present estimates in K_δ.

Theorem 5.1. *Let $2 \leqslant p < n$ and $p < q \leqslant \frac{np}{n-p}$. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and K a C^2 surface of co-dimension k, with $1 \leqslant k < n$, satisfying condition* (R). *Then, there exist positive constants $C = C(n, k, p, q)$ and $\delta_0 = \delta_0(p, n, \Omega, K)$ such that for $0 < \delta \leqslant \delta_0$ and $u \in C_0^\infty(\Omega \setminus K)$ we have:*

(a) *If $p > k$ then*

$$\int_{K_\delta} |\nabla u|^p \,\mathrm{d}x - |H|^p \int_{K_\delta} \frac{|u|^p}{d^p} \,\mathrm{d}x \geqslant C\left(\int_{K_\delta} d^{-q+\frac{q-p}{p}n}|u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}. \tag{5.4}$$

(b) *If $p < k$, the Hardy inequality,*

$$\int_{K_\delta} |\nabla u|^p \, dx - |H|^p \int_{K_\delta} \frac{|u|^p}{d^p} \, dx \geqslant 0, \tag{5.5}$$

in general fails. However, there exists a positive constant M such that

$$\int_{K_\delta} |\nabla u|^p \, dx - |H|^p \int_{K_\delta} \frac{|u|^p}{d^p} \, dx + M \int_{K_\delta} |u|^p \, dx \geqslant C \left(\int_{K_\delta} d^{-q+\frac{q-p}{p}n} |u|^q \, dx \right)^{\frac{p}{q}}. \tag{5.6}$$

We emphasize that $C = C(n, k, p, q) > 0$ is independent of Ω, K.

(c) *If in addition, u is supported in K_δ, that is $u \in C_0^\infty(K_\delta \setminus K)$ then,* (5.4) *holds true even for $p < k$.*

Proof. Using (5.1) and integrating by parts once we have that

$$I_{p,k}[u](K_\delta) \geqslant C(p) \int_{K_\delta} d^{p-k} |\nabla v|^p \, dx + H|H|^{p-2} \int_{K_\delta} d^{-k}(-d\Delta d + k - 1)|v|^p \, dx$$
$$+ H|H|^{p-2} \int_{\partial K_\delta} d^{1-k} |v|^p \, dS_x. \tag{5.7}$$

At first we estimate the middle term of the right-hand side. We have that

$$|d\Delta d + 1 - k| \leqslant c_0 d, \quad \text{for } x \in K_\delta, \tag{5.8}$$

and therefore

$$\left| \int_{K_\delta} d^{-k}(-d\Delta d + k - 1)|v|^p \, dx \right| \leqslant c_0 \int_{K_\delta} d^{1-k} |v|^p \, dx. \tag{5.9}$$

At this point we will derive some general estimates that we will use in the sequel. Our goal is to prove (5.11) and (5.12) below. For $a \in \mathbb{R}$ we consider the identity $(1 + a)d^a + d^{1+a}\Delta d = \operatorname{div}(d^{1+a}\nabla d)$. Multiply by $|v|^p$ and integrate by parts to get:

$$(a+1) \int_{K_\delta} d^a |v|^p \, dx + \int_{K_\delta} d^{a+1} \Delta d |v|^P \, dx = -p \int_{K_\delta} d^{a+1} \nabla d \cdot \nabla |v| |v|^{p-1} \, dx + \int_{\partial K_\delta} d^{a+1} |v|^p \, dS_x,$$

or, equivalently,

$$(a+k) \int_{K_\delta} d^a |v|^p \, dx + \int_{K_\delta} d^a (d\Delta d + 1 - k)|v|^p \, dx$$
$$= -p \int_{K_\delta} d^{a+1} \nabla d \cdot \nabla |v| |v|^{p-1} \, dx + \int_{\partial K_\delta} d^{a+1} |v|^p \, dS_x. \tag{5.10}$$

We next estimate the first term of the right-hand side of (5.10),

$$p \int_{K_\delta} d^{a+1} \nabla d \cdot \nabla |v| |v|^{p-1} \, dx \leqslant \left(\int_{K_\delta} d^a |v|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_{K_\delta} d^{a+p} |\nabla v|^p \, dx \right)^{\frac{1}{p}}$$
$$\leqslant \varepsilon(p-1) \int_{K_\delta} d^a |v|^p \, dx + \varepsilon^{-(p-1)} \int_{K_\delta} d^{a+p} |\nabla v|^p \, dx.$$

From this, (5.8) and (5.10) we easily obtain the following two estimates:

$$\big(|a+k| - c_0\delta - \varepsilon(p-1) \big) \int_{K_\delta} d^a |v|^p \, dx \leqslant \varepsilon^{-(p-1)} \int_{K_\delta} d^{a+p} |\nabla v|^p \, dx + \int_{\partial K_\delta} d^{a+1} |v|^p \, dS_x, \tag{5.11}$$

and

$$\int_{\partial K_\delta} d^{a+1}|v|^p \,\mathrm{d}S_x \leqslant \varepsilon^{-(p-1)} \int_{K_\delta} d^{a+p}|\nabla v|^p \,\mathrm{d}x + \big(|a+k| + c_0\delta + \varepsilon(p-1)\big) \int_{K_\delta} d^a |v|^p \,\mathrm{d}x. \tag{5.12}$$

From (5.11) taking $a = 1-k$ we get that

$$\int_{K_\delta} d^{1-k}|v|^p \,\mathrm{d}x \leqslant C(p)\delta \int_{K_\delta} d^{p-k}|\nabla v|^p \,\mathrm{d}x + C(p)\delta \int_{\partial K_\delta} d^{1-k}|u|^p \,\mathrm{d}S_x. \tag{5.13}$$

At this point we distinguish two cases according to whether $p > k$ or $p < k$. Assume first that $p > k$, or equivalently, $H > 0$. Then from (5.7) and (5.13) we get that

$$I_{p,k}[u](K_\delta) \geqslant C(p) \int_{K_\delta} d^{p-k}|\nabla v|^p \,\mathrm{d}x + C(p,k) \int_{\partial K_\delta} d^{1-k}|v|^p \,\mathrm{d}S_x. \tag{5.14}$$

Using Theorem 4.4 as well as the fact that

$$\|d^b v\|^p_{L^q(K_\delta)} = \bigg(\int_{K_\delta} d^{-q+\frac{q-p}{p}n} |u|^q \,\mathrm{d}x \bigg)^{\frac{p}{q}},$$

we easily obtain (5.4).

If $u \in C_0^\infty(K_\delta \setminus K)$ then the boundary terms in (5.7) and (5.13) are absent and the same argument yields (5.4) even if $p < k$.

Suppose now that $p < k$, that is, $H < 0$. Using again (5.7) and (5.13) we get that

$$I_{p,k}[u](K_\delta) \geqslant C(p) \int_{K_\delta} d^{p-k}|\nabla v|^p \,\mathrm{d}x - C(p,k) \int_{\partial K_\delta} d^{1-k}|v|^p \,\mathrm{d}S_x. \tag{5.15}$$

To estimate the last term of this we will use (5.12) with $a = p-k$ in the following way,

$$\begin{aligned} \int_{\partial K_\delta} d^{1-k}|v|^p \,\mathrm{d}S_x &= \delta^{-p} \int_{\partial K_\delta} d^{1+p-k}|v|^p \,\mathrm{d}S_x \\ &\leqslant \varepsilon^{-(p-1)} \int_{K_\delta} d^{p-k}|\nabla v|^p \,\mathrm{d}x + C(\varepsilon,p)\delta^{-p} \int_{K_\delta} d^{p-k}|v|^p \,\mathrm{d}x. \end{aligned} \tag{5.16}$$

From (5.15) and (5.16) choosing ε big we get:

$$I_{p,k}[u](K_\delta) \geqslant C(p) \int_{K_\delta} d^{p-k}|\nabla v|^p \,\mathrm{d}x - M \int_{K_\delta} d^{p-k}|v|^p \,\mathrm{d}x. \tag{5.17}$$

On the other hand from (5.16) and Theorem 4.4 we get that

$$C(p,q,n,k)\|d^b v\|^p_{L^q(K_\delta)} \leqslant C(p) \int_{K_\delta} d^{p-k}|\nabla v|^p \,\mathrm{d}x + M \int_{K_\delta} d^{p-k}|v|^p \,\mathrm{d}x. \tag{5.18}$$

From (5.17) and (5.18) we easily conclude (5.6).

It remains to explain why when $p < k$ and $u \in C_0^\infty(\Omega \setminus K)$ the simple Hardy (5.5) in general fails. Let us consider the case where K and therefore K_δ are strictly contained in Ω. In this case the function $u_\varepsilon = d^{H+\varepsilon}$, for $\varepsilon > 0$ is in $W^{1,p}(K_\delta)$. On the other hand for $p < k$ a simple density argument shows that $W^{1,p}(K_\delta \setminus K) = W^{1,p}(K_\delta)$. An easy calculation shows that

$$\int_{K_\delta} |\nabla u_\varepsilon|^p \,\mathrm{d}x - |H|^p \int_{K_\delta} \frac{|u_\varepsilon|^p}{d^p} \,\mathrm{d}x = \big(|H+\varepsilon|^p - |H|^p\big) \int_{K_\delta} d^{-k+p\varepsilon} \,\mathrm{d}x < 0, \tag{5.19}$$

by taking $\varepsilon > 0$ small and noting that $H < 0$. □

Remark. The result is not true in case $k = n$, as discussed in the introduction.

We next prove estimates in Ω.

Theorem 5.2. *Let $2 \leqslant p < n$ and $p < q \leqslant \frac{np}{n-p}$. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and K a C^2 surface of co-dimension k, with $1 \leqslant k < n$, satisfying condition* (R). *Then, there exist positive constants $C = C(n,k,p,q)$ and M such that for all $u \in C_0^\infty(\Omega \setminus K)$, there holds*:

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} \,\mathrm{d}x + M \int_\Omega |u|^p \,\mathrm{d}x \geqslant C\left(\int_\Omega d^{-q+\frac{q-p}{p}n} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}. \tag{5.20}$$

We note that $C(n,k,p,q)$ is independent of Ω, K.

Proof. Clearly we have:

$$I_{p,k}[u](\Omega) = I_{p,k}[u](K_\delta) + I_{p,k}[u]\big(K_\delta^c\big). \tag{5.21}$$

By Theorem 5.1 for δ small, we have:

$$I_{p,k}[u](K_\delta) \geqslant C(n,k,p,q)\left(\int_{K_\delta} d^{-q+\frac{q-p}{p}n} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}} - M \int_{K_\delta} |u|^p \,\mathrm{d}x. \tag{5.22}$$

Since $d(x) \geqslant \delta$ in K_δ^c,

$$I_{p,k}[u](K_\delta^c) \geqslant \int_{K_\delta^c} |\nabla u|^p \,\mathrm{d}x - C(p,k,\delta) \int_{K_\delta^c} |u|^p \,\mathrm{d}x. \tag{5.23}$$

From the Sobolev embedding of $L^{\frac{np}{n-p}}(K_\delta^c)$ into $W^{1,p}(K_\delta^c)$ we get:

$$\|u\|^p_{L^{\frac{np}{n-p}}(K_\delta^c)} \leqslant C(p,n) \int_{K_\delta^c} |\nabla u|^p \,\mathrm{d}x + C(p,n,\Omega,K) \int_{K_\delta^c} |u|^p \,\mathrm{d}x.$$

Using the interpolation Lemma 4.1 (with $a = 0$) we have:

$$\begin{aligned} C(n,p,q)\left(\int_{K_\delta^c} d^{-q+\frac{q-p}{p}n} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}} &\leqslant \|u\|^p_{L^{\frac{pn}{n-p}}(K_\delta^c)} + \big\|d^{-1}u\big\|^p_{L^p(K_\delta^c)} \\ &\leqslant \|u\|^p_{L^{\frac{pn}{n-p}}(K_\delta^c)} + \delta^{-p} \|u\|^p_{L^p(K_\delta^c)}. \end{aligned} \tag{5.24}$$

From (5.23)–(5.24) we get for $M = M(n,p,q,\Omega,K)$,

$$I_{p,k}[u]\big(K_\delta^c\big) \geqslant C(n,p,q)\left(\int_{K_\delta^c} d^{-q+\frac{q-p}{p}n} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}} - M \int_{K_\delta^c} |u|^p \,\mathrm{d}x. \tag{5.25}$$

The result follows from (5.21), (5.22) and (5.25). □

Our final result reads:

Theorem 5.3. *Let $2 \leqslant p < n$ and $p < q \leqslant \frac{np}{n-p}$. We assume that $\Omega \subset \mathbb{R}^n$ is a domain and K a surface of co-dimension k, $1 \leqslant k < n$, satisfying condition* (R). *In addition we assume that $D = \sup_{x \in \Omega} d(x) < \infty$ and condition* (C) *is satisfied. Then for all $u \in C_0^\infty(\Omega)$ there holds*

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} \,\mathrm{d}x \geqslant C\left(\int_\Omega d^{-q+\frac{q-p}{p}n} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}, \tag{5.26}$$

for $C = C(n,P,Q,\Omega,K) > 0$.

Proof. Working as in the derivation of (5.7) we get:

$$C(p,k)I_{p,k}[u](\Omega) \geqslant \int_\Omega d^{p-k}|\nabla v|^p \,\mathrm{d}x + H\int_\Omega d^{-k}(-d\Delta d + 1 - k)|v|^p \,\mathrm{d}x. \tag{5.27}$$

Because of condition (C) we have that $H(-d\Delta d + 1 - k) \geqslant 0$, see (4.38). The result then follows from Theorem 4.5. □

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References

[1] Adimurthi, M.J. Esteban, An improved Hardy–Sobolev inequality in $W^{1,p}$ and its application to Schrödinger operators, NoDEA Nonlinear Differential Equations Appl. 12 (2) (2005) 243–263.
[2] L. Ambrosio, H.M. Soner, Level set approach to mean curvature flow in arbitrary codimension, J. Differential Geom. 43 (1996) 693–737.
[3] R.J. Biezuner, Best constants in Sobolev trace inequalities, Nonlinear Anal. 54 (3) (2003) 575–589.
[4] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, Trans. Amer. Math. Soc. 356 (6) (2004) 2169–2196.
[5] H. Brezis, M. Marcus, Hardy's inequalities revisited, Ann. Scuola Norm. Pisa 25 (1997) 217–237.
[6] H. Brezis, J.-L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Comp. Madrid 10 (1997) 443–469.
[7] X. Cabré, Y. Martel, Existence versus instantaneous blowup for linear heat equations with singular potentials, C. R. Acad. Sci. Paris Ser. I Math. 329 (1999) 973–978.
[8] J. Dávila, L. Dupaigne, Hardy-type inequalities, J. Eur. Math. Soc. 6 (3) (2004) 335–365.
[9] J. Dolbeault, M.J. Esteban, M. Loss, L. Vega, An analytical proof of Hardy-like inequalities related to the Dirac operator, J. Funct. Anal. 216 (1) (2004) 1–21.
[10] S. Filippas, V.G. Maz'ya, A. Tertikas, A sharp Hardy Sobolev inequality, C. R. Acad. Sci. Paris Ser. I Math. 339 (2004) 483–486.
[11] S. Filippas, V.G. Maz'ya, A. Tertikas, On a question of Brezis and Marcus, Calc. Var. Partial Differential Equations 25 (4) (2006) 491–501.
[12] S. Filippas, A. Tertikas, Optimizing improved Hardy inequalities, J. Funct. Anal. 192 (2002) 186–233.
[13] J.P. Garcia, I. Peral, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998) 441–476.
[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1983.
[15] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, vol. 5, Amer. Math. Soc., Providence, RI, 1999.
[16] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, A geometrical version of Hardy's inequality, J. Funct. Anal. 189 (2002) 539–548.
[17] M. Marcus, V.J. Mizel, Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^n$, Trans. Amer. Math. Soc. 350 (1998) 3237–3255.
[18] V.G. Maz'ya, Sobolev Spaces, Springer-Verlag, Berlin/New York, 1985.
[19] D. Slepcev, On level set approach to motion of manifolds of arbitrary codimension, Interfaces and Free Boundaries 5 (2003) 417–458.
[20] J. Tidblom, A geometrical version of Hardy's inequality for $W_0^{1,p}(\Omega)$, Proc. Amer. Math. Soc. 132 (8) (2004) 2265–2271.
[21] J. Tidblom, A Hardy inequality in the half-space, J. Funct. Anal. 221 (2) (2005) 482–495.
[22] J.L. Vásquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal. 173 (2000) 103–153.