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Higher-order generalized convexity and duality in nondifferentiable multiobjective mathematical programming ☆

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Abstract

In this paper, a class of generalized convexity is introduced and a unified higher-order dual model for nondifferentiable multiobjective programs is described, where every component of the objective function contains a term involving the support function of a compact convex set. Weak duality theorems are established under generalized convexity conditions. The well-known case of the support function in the form of square root of a positive semidefinite quadratic form and other special cases can be readily derived from our results.

1. Introduction

Consider the nonlinear programming problem

$$(\mathrm{P})\quad \text{Minimize } f(x) \quad \text{subject to} \quad g(x) \geqslant 0,$$

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where $f:\mathbb{R}^n \to \mathbb{R}$ and $g:\mathbb{R}^n \to \mathbb{R}^m$ are twice differentiable functions. The Mangasarian second-order dual [1] is

$$\text{(MD)} \quad \text{Maximize } f(u) - y^T g(u) - \frac{1}{2} p^T \nabla^2 \big[f(u) - y^T g(u) \big] p$$
$$\text{subject to} \quad \nabla \big[f(u) - y^T g(u) \big] + \nabla^2 \big[f(u) - y^T g(u) \big] p = 0,$$
$$y \geqslant 0.$$

By introducing two differentiable functions $h:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $k:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, Mangasarian [1] formulates the higher-order dual:

$$\text{(HD)} \quad \text{Maximize } f(u) - y^T g(u) + h(u,p) - y^T k(u,p)$$
$$\text{subject to} \quad \nabla_p h(u,p) = \nabla_p \big(y^T k(u,p) \big),$$
$$y \geqslant 0,$$

where $\nabla_p h(u,p)$ denotes the $n \times 1$ gradient of h with respect to p and $\nabla_p(y^T k(u,p))$ denotes the $n \times 1$ gradient of $y^T k$ with respect to p.

Recently, Mishra and Rueda [2] consider higher-order duality for the following nondifferentiable mathematical programming:

$$\text{(NP)} \quad \text{Minimize } f(x) + \big(x^T B x \big)^{1/2} \quad \text{subject to} \quad g(x) \geqslant 0,$$

where $f:\mathbb{R}^n \to \mathbb{R}$ and $g:\mathbb{R}^n \to \mathbb{R}^m$ are twice differentiable functions, and B is an $n \times n$ positive semidefinite (symmetric) matrix.

In [2], Mishra and Rueda generalize Zhang's Mangasarian type and Mond–Weir type higher-order duality [3] to higher-order type I functions. In this paper, we extend the results in [2] to a class of nondifferentiable multiobjective programming problems. A unified higher-order dual model for nondifferentiable multiobjective programs is presented, where every component of the objective function contains a term involving the support function of a compact convex set. Weak duality theorems are established under generalized convexity conditions. As a special case of these conditions appears repeatedly in the literature with the support function in the form of square root of a positive semidefinite quadratic form. Other special cases can be readily generated from our results.

Consider the nondifferentiable multiobjective programming problem:

$$\text{(NMP)} \quad \text{Minimize } \big(f_1(x) + s(x|C_1), f_2(x) + s(x|C_2), \ldots, f_p(x) + s(x|C_p) \big)$$
$$\text{subject to} \quad g(x) \geqslant 0, \quad x \in D, \tag{1}$$

where $f = [f_1, f_2, \ldots, f_p]^T$ and g are differentiable functions from $\mathbb{R}^n \to \mathbb{R}^p$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively, C_i is a compact convex set of $\mathbb{R}^n$ for each $i \in P = \{1, 2, \ldots, p\}$, and D is an open subset of $\mathbb{R}^n$.

Definition 1.1. A functional $F : D \times D \times \mathbb{R}^n \to \mathbb{R}$ is sublinear if, for any $x, u \in D$,

$$F(x,u;a_1 + a_2) \leqslant F(x,u;a_1) + F(x,u;a_2), \quad \forall a_1, a_2 \in \mathbb{R}^n \quad \text{and}$$
$$F(x,u;\alpha a) = \alpha F(x,u;a), \quad \forall \alpha \in \mathbb{R},\ \alpha \geqslant 0,\ \text{and } a \in \mathbb{R}^n.$$

2. Duality

We propose the following general dual (NMD) to (NMP):

$$\text{(NMD)}\quad \text{Maximize}\ \Big(f_1(u)+h_1(u,p)-p^T\nabla_p h_1(u,p)+u^T w_1 - \sum_{i\in I_0}\big[y_i g_i(u)+y_i k_i(u,p)-p^T\nabla_p y_i k_i(u,p)\big],$$
$$f_2(u)+h_2(u,p)-p^T\nabla_p h_2(u,p)+u^T w_2 - \sum_{i\in I_0}\big[y_i g_i(u)+y_i k_i(u,p)-p^T\nabla_p y_i k_i(u,p)\big],\ldots,$$
$$f_p(u)+h_p(u,p)-p^T\nabla_p h_p(u,p)+u^T w_p - \sum_{i\in I_0}\big[y_i g_i(u)+y_i k_i(u,p)-p^T\nabla_p y_i k_i(u,p)\big]\Big)$$

subject to

$$\lambda^T\nabla_p h(u,p)+\sum_{i=1}^{p}\lambda_i w_i=\nabla_p\big(y^T k(u,p)\big), \tag{2}$$

$$\sum_{i\in I_\alpha}\big[y_i g_i(u)+y_i k_i(u,p)-p^T\nabla_p\big(y_i k_i(u,p)\big)\big]\leqslant 0,\quad \alpha=1,2,\ldots,r, \tag{3}$$

$$y\geqslant 0, \tag{4}$$

$$w_i\in C_i,\quad i=1,2,\ldots,p,\quad \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_p)\in\Lambda^+, \tag{5}$$

where $\Lambda^+=\{\lambda\in\mathbb{R}^p\colon\ \lambda>0,\ \lambda^T e=1,\ e=\{1,1,\ldots,1\}\in\mathbb{R}^p\}$, $w=(w_1,w_2,\ldots,w_p)$, $I_\alpha\subset M=\{1,2,\ldots,m\}$, $\alpha=0,1,2,\ldots,\gamma$ with $\bigcup_{\alpha=0}^{\gamma} I_\alpha=M$ and $I_\alpha\cap I_\beta=\emptyset$ if $\alpha\neq\beta$.

Theorem 2.1 (Weak duality). *Let x be feasible for* (NMP) *and let (u,λ,w,y,p) be feasible for* (NMD). *Supposed that for all feasible (x,u,y,w,p), there exist a sublinear functional $F:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ such that*

$$\sum_{i\in I_\alpha}\big[y_i g_i(u)+y_i k_i(u,p)-p^T\nabla_p\big(y_i k_i(u,p)\big)\big]\leqslant 0$$
$$\Longrightarrow\quad F\Big(x,u,-\nabla_p\Big[\sum_{i\in I_\alpha}y_i k_i(u,p)\Big]\Big)\leqslant-\beta_\alpha d^2(x,u),\quad \alpha=1,2,\ldots,\gamma. \tag{6}$$

Furthermore, it is assumed that one of the following three conditions holds:

(a) *For $i\in P$,*

$$F\Big(x,u,\nabla_p h_i(u,p)+w_i-\nabla_p\Big(\sum_{i\in I_0}y_i k_i(u,p)\Big)\Big)\geqslant-\rho_i d^2(x,u)$$

$$\begin{aligned}
\Longrightarrow\quad & f_i(x)+x^T w_i-\Bigl(f_i(u)+u^T w_i-\sum_{i\in I_0} y_i g_i(u)\Bigr)\\
&-\Bigl(h_i(u,p)-\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\\
&+p^T\Bigl[\nabla_p h_i(u,p)-\nabla_p\Bigl(\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\Bigr]\geqslant 0;\\
& f_i(x)+x^T w_i-\Bigl(f_i(u)+u^T w_i-\sum_{i\in I_0} y_i g_i(u)\Bigr)\\
&-\Bigl(h_i(u,p)-\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\\
&+p^T\Bigl[\nabla_p h_i(u,p)-\nabla_p\Bigl(\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\Bigr]\leqslant 0\\
\Longrightarrow\quad & F\Bigl(x,u,\nabla_p h_i(u,p)+w_i-\nabla_p\Bigl(\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\Bigr)\leqslant -\rho_i d^2(x,u),
\end{aligned}$$

and

$$\sum_{\alpha=1}^{\gamma}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i\geqslant 0;$$

(b) *There exists $k\in P$ such that*

$$\begin{aligned}
& F\Bigl(x,u,\nabla_p h_k(u,p)+w_k-\nabla_p\Bigl[\sum_{i\in I_0} y_i k_i(u,p)\Bigr]\Bigr)\geqslant -\rho_k d^2(x,u)\\
&\quad\Longrightarrow\quad f_k(x)+x^T w_k-\Bigl(f_k(u)+u^T w_k-\sum_{i\in I_0} y_i g_i(u)\Bigr)\\
&\qquad-\Bigl(h_k(u,p)-\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\\
&\qquad+p^T\Bigl[\nabla_p h_k(u,p)-\nabla_p\Bigl(\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\Bigr]\geqslant 0;
\end{aligned}$$

while

$$\begin{aligned}
& f_i(x)+x^T w_i-\Bigl(f_i(u)+u^T w_i-\sum_{i\in I_0} y_i g_i(u)\Bigr)-\Bigl(h_i(u,p)-\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\\
&\quad+p^T\Bigl[\nabla_p h_i(u,p)-\nabla_p\Bigl(\sum_{i\in I_0} y_i k_i(u,p)\Bigr)\Bigr]\leqslant 0
\end{aligned}$$

$$\Longrightarrow \quad F\left(x, u, \nabla_p h_i(u,p) + w_i - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u,p)\right)\right) \leqslant -\rho_i d^2(x,u),$$

for all $i \in P$

and

$$\sum_{\alpha=1}^{\gamma} \beta_\alpha + \sum_{i=1}^{p} \lambda_i \rho_i \geqslant 0;$$

(c) $$F\left(x, u, \lambda^T \nabla_p h(u,p) + \sum_{i=1}^{p} \lambda_i w_i - \nabla_p y^T k(u,p)\right) \geqslant -\rho d^2(x,u)$$

$$\Longrightarrow \quad \lambda^T f(x) + x^T \sum_{i=1}^{p} \lambda_i w_i - \left(\lambda^T f(u) + u^T \sum_{i=1}^{p} \lambda_i w_i - y^T g(u)\right)$$
$$- \left(\lambda^T h(u,p) - y^T k(u,p)\right) + p^T\left[\lambda^T \nabla_p h(u,p) - \nabla_p y^T k(u,p)\right]$$
$$\geqslant 0;$$

and

$$\sum_{\alpha=1}^{\gamma} \beta_\alpha + \rho \geqslant 0.$$

Then, the following relationships do not hold:

$$f_i(x) + s(x|C_i)$$
$$\leqslant f_i(u) + h_i(u,p) - p^T \nabla_p h_i(u,p) + u^T w_i$$
$$- \sum_{i\in I_0}\left[y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p y_i k_i(u,p)\right], \quad \textit{for all } i \in P, \tag{7}$$

and

$$f_k(x) + s(x|C_k)$$
$$< f_k(u) + h_k(u,p) - p^T \nabla_p h_k(u,p) + u^T w_k$$
$$- \sum_{i\in I_0}\left[y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p y_i k_i(u,p)\right], \quad \textit{for some } k \in P. \tag{8}$$

Proof. Since x is feasible for (NMP) and (u, λ, w, y, p) is feasible for (NMD), it follows that

$$\sum_{i\in I_\alpha}\left[y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p\left(y_i k_i(u,p)\right)\right] \leqslant 0$$
$$\Longrightarrow \quad F\left(x, u, -\nabla_p\left[\sum_{i\in I_\alpha} y_i k_i(u,p)\right]\right) \leqslant -\beta_\alpha d^2(x,u),$$
$$\alpha = 1, 2, \ldots, \gamma, \quad \text{by (6).} \tag{9}$$

By the sublinearity of F, we have

$$F\left(x, u, -\nabla_p\left[\sum_{i\in M\setminus I_0} y_i k_i(u, p)\right]\right) \leqslant -\sum_{\alpha}^{\gamma} \beta_\alpha d^2(x, u). \tag{10}$$

From (2), (10) and the sublinearity of F, we obtain

$$F\left(x, u; \nabla_p \lambda^T h(u, p) + \sum_{i=1}^{p} \lambda_i w_i - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u, p)\right)\right) \geqslant \left(\sum_{\alpha=1}^{\gamma} \beta_\alpha\right) d^2(x, u). \tag{11}$$

Now on the contrary, we suppose that (7) and (8) hold. Since $x^T w_i \leqslant s(x|C_i)$, $i = 1, 2, \ldots, p$, we have

$$\begin{aligned} f_i(x) + x^T w_i &\leqslant f_i(x) + s(x|C_i) \\ &\leqslant f_i(u) + h_i(u, p) - p^T \nabla_p h_i(u, p) + u^T w_i \\ &\quad - \sum_{i\in I_0}\left[y_i g_i(u) + y_i k_i(u, p) - p^T \nabla_p y_i k_i(u, p)\right], \quad \forall i \in P, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &f_k(x) + x^T w_k \\ &\quad\leqslant f_k(x) + s(x|C_k) < f_k(u) + h_k(u, p) - p^T \nabla_p h_k(u, p) + u^T w_k \\ &\quad - \sum_{i\in I_0}\left[y_i g_i(u) + y_i k_i(u, p) - p^T \nabla_p y_i k_i(u, p)\right], \quad \text{for some } k \in P. \end{aligned} \tag{13}$$

If case (a) is satisfied, then we obtain

$$F\left(x, u; \nabla_p h_i(u, p) + w_i - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u, p)\right)\right) \leqslant -\rho_i d^2(x, u), \quad \forall i \in P, \tag{14}$$

and

$$F\left(x, u; \nabla_p h_k(u, p) + w_k - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u, p)\right)\right) < -\rho_k d^2(x, u), \quad \text{for some } k \in P. \tag{15}$$

Since $\lambda \in \Lambda^+$, it follows from (14), (15) and the sublinearity of F that

$$F\left(x, u; \sum_{i=1}^{p} \lambda_i\left(\nabla_p h_i(u, p) + w_i - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u, p)\right)\right)\right) < \left(-\sum_{i=1}^{p} \lambda_i \rho_i\right) d^2(x, u). \tag{16}$$

Since $\sum_{\alpha=1}^{\gamma}\beta_\alpha + \sum_{i=1}^{p}\lambda_i\rho_i \geqslant 0,$ it is clear from (16) that

$$F\left(x, u; \sum_{i=1}^{p}\lambda_i\left[\nabla_p h_i(u, p) + w_i - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u, p)\right)\right]\right) < \left(\sum_{\alpha=1}^{\gamma}\beta_\alpha\right)d^2(x, u), \tag{17}$$

which contradicts (11). Hence, (7) and (8) cannot hold.

If case (b) is satisfied, then we note that (14) holds and that (13) implies

$$F\left(x, u; \nabla_p h_k(u, p) + w_k - \nabla_p\left(\sum_{i\in I_0} y_i g_i(u)\right)\right) < -\rho_k d^2(x, u), \quad \text{for some } k \in P. \tag{18}$$

Since (18) and (14) imply (17), it is clear that (7) and (8) cannot hold.

Now suppose that case (c) is satisfied. Since $\lambda \in \Lambda^+$, it follows from (12) and (13) that

$$\sum_{i=1}^{p}\lambda_i\left(f_i(x) + x^T w_i\right) < \sum_{i=1}^{p}\lambda_i\Big\{f_i(u) + h_i(u, p) - p^T\nabla_p h_i(u, p) + u^T w_i - \sum_{i\in I_0}\left[y_i g_i(u) + y_i k_i(u, p) - p^T\nabla_p y_i k_i(u, p)\right]\Big\}.$$

Thus, by (c),

$$F\left(x, u; \sum_{i=1}^{p}\lambda_i\left[\nabla_p h_i(u, p) + w_i - \nabla_p\left(\sum_{i\in I_0} y_i g_i(u)\right)\right]\right) < -\rho d^2(x, u). \tag{19}$$

Since $\sum_{\alpha=1}^{\gamma}\beta_\alpha + \rho \geqslant 0$, it follows from (19) that

$$F\left(x, u; \sum_{i=1}^{p}\lambda_i\left[\nabla_p h_i(u, p) + w_i - \nabla_p\left(\sum_{i\in I_0} y_i g_i(u)\right)\right]\right) < \left(\sum_{\alpha=1}^{\gamma}\beta_\alpha\right)d^2(x, u),$$

which contradicts (11). Hence, (7) and (8) cannot hold. □

3. Special cases

Let us consider the case, where $C_i = \{B_i w\colon\ w^T B_i w \leqslant 1\}$. It is easily shown that $(x^T B_i x)^{1/2} = s(x|C_i)$ and that the sets C_i, $i = 1, 2, \ldots, p$, are compact and convex. Then, the primal problem (NMP) and the dual problem (NMD) become, respectively,

$$(\text{NMP})_1 \quad \text{Minimize } \left(f_1(x) + \left(x^T B_1 x\right)^{1/2}, \ldots, f_p(x) + \left(x^T B_p x\right)^{1/2}\right)$$
$$\text{subject to} \quad g(x) \geqslant 0, \quad x \in D,$$

and

$$\begin{aligned}
(\text{NMD})_1 \quad \text{Maximize } \Big(& f_1(u) + h_1(u,p) - p^T \nabla_p h_1(u,p) + u^T B_1 w \\
& - \sum_{i \in I_0} \big[y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p y_i k_i(u,p) \big], \\
& \ldots, f_p(u) + h_p(u,p) - p^T \nabla_p h_p(u,p) + u^T B_p w \\
& - \sum_{i \in I_0} \big[y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p y_i k_i(u,p) \big] \Big)
\end{aligned}$$

subject to

$$\lambda^T \nabla_p h(u,p) + \sum_{i=1}^{p} \lambda_i B_i w = \nabla_p \big(y^T k(u,p) \big),$$

$$\sum_{i \in I_\alpha} \big[y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p \big(y_i^T k_i(u,p) \big) \big] \leqslant 0,$$

$$\alpha = 1, 2, \ldots, r, \quad y \geqslant 0,$$

$$w^T B_i w \leqslant 1, \quad i = 1, 2, \ldots, p, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Lambda^+.$$

Remark 3.1.

(i) If $p = 1$, then $(\text{NMP})_1$ and $(\text{NMD})_1$ reduce, respectively, to (NDP) and (NDHGD) considered in [2].
(ii) If $p = 1$, $I_0 = M$, $I_\alpha = \emptyset$ $(1 \leqslant \alpha \leqslant \gamma)$, then $(\text{NMP})_1$ and $(\text{NMD})_1$ reduce, respectively, to (NDP) and (NDHMD) considered in [2].
(iii) If $p = 1$, $I_0 = \emptyset$, $I_1 = M$, $I_\alpha = \emptyset$ $(2 \leqslant \alpha \leqslant \gamma)$, then $(\text{NMP})_1$ and $(\text{NMD})_1$ reduce, respectively, to (NDP) and (NDHD) considered in [2].
(iv) If $h(u,p) = p^T \nabla f(u)$ and $k_i(u,p) = p^T \nabla g_i(u)$, $i = 1, 2, \ldots, m$, then $(\text{NMD})_1$ reduce to (VD) considered in [4].

We note that if $F(x,u;\nabla\phi(u)) = \nabla\phi(u)^T \eta(x,u)$, where $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, then the conditions of Theorem 2.1 reduce to higher-order generalized invexity considered in [2]. Thus, our results in Theorem 2.1 improve, extend and unify the results obtained in [2,4].

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