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An explicit formula for ndinv, a new statistic for two-shuffle parking functions

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ABSTRACT

In a recent paper, Duane, Garsia, and Zabrocki introduced a new statistic, "ndinv", on a family of parking functions. The definition was guided by their study of a recursion on $\langle \Delta_{h_m} C_{p_1} C_{p_2} \ldots C_{p_k} 1, e_n \rangle$ for Δ_{h_m} a Macdonald eigenoperator, C_{p_i} a modified Hall–Littlewood operator, and $(p_1, p_2, \ldots, p_k)$ a composition of n. Using their newly introduced statistic, one can give a new interpretation for $\langle \nabla e_n, h_j h_{n-j} \rangle$ as a sum of parking functions q, t counted by area and ndinv. This is a departure from the traditional sum, as stated by the shuffle conjecture, which q, t counts area and diagonal inversion number (dinv). Since their definition is necessarily recursive, they pose the problem of obtaining a non-recursive definition. In this paper, we solve this problem by giving an explicit formula for ndinv similar to the classical definition of dinv and prove it is equivalent to the ndinv of Duane, Garsia, and Zabrocki.

1. Introduction

We begin with a brief introduction to parking functions and some related algebraic expressions before giving our formula for ndinv.

1.1. Parking functions

Definition 1.1 *(Parking function).* A two-line array

$$\mathrm{PF} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{bmatrix}$$

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$$\mathrm{PF} = \begin{bmatrix} 2 & 3 & 1 & 5 & 4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

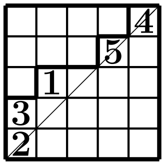


Fig. 1. PF as shown in the 5×5 lattice.

is a parking function exactly when

- The first line is a permutation of $\{1, 2, \ldots, n\}$.
- (Dyck path condition.) $d_1 = 0$ and when $i > 1$, $0 \leqslant d_i \leqslant d_{i-1} + 1$.
- (Increasing column condition.) If $d_i = d_{i-1} + 1$, then $c_i > c_{i-1}$.

The numbers in the first row of the array are referred to as cars, and car c_i is said to be in the d_ith diagonal, with the 0th diagonal being referred to as the main diagonal. We sometimes find it convenient to consider a parking function as a sequence of "dominoes" $\left[\begin{smallmatrix} c_i \\ d_i \end{smallmatrix}\right]$. Frequently, as first shown in [1], parking functions are also represented in an n by n lattice. The elements in the second row define a Dyck path in the n by n lattice, where d_i gives the number of full squares between the path and the main (southwest to northeast) diagonal. Car c_i is then placed directly to the right of the ith north step of the Dyck path. (See Fig. 1.) Note that, when $d_i = d_{i-1} + 1$, cars c_i and c_{i-1} are in the same column with c_i atop c_{i-1}.

Definition 1.2 *(Reading word).* The reading word of a parking function (word(PF)) is the permutation which records the cars by diagonals, reading cars in the highest diagonal first, then working downward. Cars within a diagonal are recorded from northeast to southwest.

Example 1.3. The reading word of the parking function in Fig. 1 is $(1, 3, 4, 5, 2)$.

For the purposes of this paper, we work with a subset of these parking functions. Recall that a permutation σ is a shuffle of $(1, \ldots, m)$ and $(m+1, \ldots, m+n)$ when if $i_1 < i_2 \leqslant m$ or $m < i_1 < i_2$, then i_1 occurs before i_2 in σ.

Definition 1.4 *(Two-shuffle parking functions).* A parking function PF is a two-shuffle parking function (as first defined in [2]) when for two integers m, n, we have:

1. word(PF) is a shuffle of $(1, \ldots, m)$ and $(m+1, \ldots, m+n)$;
2. $c_{n+m} > m$; and
3. $d_{n+m} = 0$.

Example 1.5. The parking function in Fig. 1 is a two-shuffle parking function for $m = 2$.

For the remainder of this paper, we call a car c a "big car" when $c > m$ and a "small car" when $c \leqslant m$. We may also use the symbols "c_s" or "c_b" to denote a small car or a big car (respectively).

Definition 1.6 *(Composition).* Let the set of f_i such that c_{f_i} is big and $d_{f_i} = 0$ be given in increasing order as $(f_1, f_2, \ldots, f_k)$. Then we say a car c_j is in the first part if $j \leqslant f_1$ and otherwise in the ith part if $f_{i-1} < j \leqslant f_i$. Then the "composition of PF" (comp(PF)) is the vector $(p_1, p_2, \ldots, p_k)$, where p_i gives the number of big cars in the ith part.

Example 1.7.

$$\mathrm{PF} = \begin{bmatrix} 3 & 4 & 1 & 6 & 2 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

then for $m = 3$ we have

$$\mathrm{comp}(\mathrm{PF}) = (2, 1).$$

There are two parking function statistics that have been extensively studied in the literature:

$$\mathrm{area}(\mathrm{PF}) = \sum_i d_i,$$

first described in [1] and

$$\mathrm{dinv}(\mathrm{PF}) = \sum_{i<j} \chi(d_i = d_j \text{ and } c_i < c_j) + \chi(d_i = d_j + 1 \text{ and } c_i > c_j),$$

first described in [3], where here, as elsewhere in this paper, χ gives the truth function. These two statistics are of interest because they play an essential role in a variety of results tying parking functions to the theory of Macdonald polynomials and the representation theory of the symmetric group; they are also core ingredients in the formulation of the "shuffle conjecture".

1.2. Some algebraic theorems and conjectures

The shuffle conjecture as given in [4] states in particular that

$$\nabla e_n = \Delta_{e_n} e_n = \sum_{\mathrm{PF}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} Q_{\mathrm{ides}(\mathrm{PF})},$$

where Q is the Gessel quasi-symmetric function, $\mathrm{ides}(\mathrm{PF}) = \mathrm{des}((\mathrm{word}(\mathrm{PF}))^{-1})$, and Δ_f is the linear operator defined by the following with $\tilde{H}_\mu[X; q, t]$ the modified Macdonald basis in [5]:

$$\Delta_f \tilde{H}_\mu[X; q, t] = f\left[\sum_{(i,j)\in\mu} t^{i-1} q^{j-1}\right] \tilde{H}_\mu[X; q, t].$$

A number of authors have given related expressions in terms of the area and dinv of particular families of parking functions. In [2], Haglund proved the identity

$$\langle \Delta_{h_m} E_{n,k}, e_n \rangle = \sum_{F(n,k,m)} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})}, \tag{1.1}$$

where $F(n, k, m)$ denotes the family of parking functions that start with a big car, have m small cars and n big cars, k of which are on the main diagonal and whose word is a shuffle of $1, 2, \ldots, m$ with $m + 1, m + 2, \ldots, m + n$. Note that here $E_{n,k}$ are the symmetric functions introduced by Garsia and Haglund in [6] with the property that

$$E_{n,1} + E_{n,2} + \cdots + E_{n,n} = e_n.$$

Recent work in [7] used modified Hall–Littlewood operators (represented here as C_a) to give a refinement of the shuffle conjecture. The identity

$$E_{n,k} = \sum_{(p_1, p_2, \ldots, p_k) \models n} C_{p_1} C_{p_2} \ldots C_{p_k} 1$$

suggested to Duane, Garsia, and Zabrocki (in [8]), that the polynomials

$$\langle \Delta_{h_m} C_{p_1} C_{p_2} \ldots C_{p_k} 1, e_n \rangle,$$

might yield a refinement of (1.1). In particular they found that

$$\langle \Delta_{h_m} C_{p_1} \dots C_{p_k} 1, e_n \rangle|_{q=1} = \sum_{\substack{\mathrm{PF} \in F(n,k,m) \\ \mathrm{comp(PF)}=(p_1,p_2,\dots,p_k)}} t^{\mathrm{area(PF)}}.$$

In an effort to obtain a combinatorial interpretation of the left hand side without the restriction "$q=1$", they were led to introduce a new statistic, which they called "ndinv". In [8] they only obtain an algorithmic construction of ndinv based on a recursion satisfied by the polynomial $\langle \Delta_{h_m} C_{p_1} \dots C_{p_k} 1, e_n \rangle$, and pose the problem of finding a non-recursive definition. In this paper, we solve this problem by giving an explicit formula for ndinv which has some analogy with the definition of the classical "dinv".

Example 1.8. We repeat a small example from [8] that illustrates the need for a new statistic. By explicit calculation, we know

$$\langle \Delta_{h_2} C_3 C_2 1, e_5 \rangle = t^3 q^4.$$

There is a single such two-shuffle parking function with comp(PF) $= (3,2)$, in particular

$$\mathrm{PF} = \begin{bmatrix} 7 & 2 & 5 & 4 & 6 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Since both the area and the dinv of PF is three (and in particular dinv(PF) $\neq 4$), the 'q' statistic must not be weighted by the dinv.

2. An explicit formula for ndinv

To compute the ndinv of a parking function PF explicitly, it is expedient to work with a slight modification of PF. (This modification was first explored in [8] in what the author refereed to as "Stage 1".) This modification is constructed as follows:

Procedure 2.1. Beginning with a parking function PF:

1. Working from left to right, if c_s is small, then shift $\begin{bmatrix} c_s \\ d_s \end{bmatrix}$ to the left past d_s big cars.
2. For every big car c_b, count the number of small cars which shifted past it in the previous step. Increase d_b by this number.

Use this modified parking function to define the first two lines of the following three-line array.

$$\Psi(\mathrm{PF}) = \begin{bmatrix} c_1^{\Psi} & c_2^{\Psi} & \dots & c_{n+m}^{\Psi} \\ d_1^{\Psi} & d_2^{\Psi} & \dots & d_{n+m}^{\Psi} \\ r_1^{\Psi} & r_2^{\Psi} & \dots & r_{n+m}^{\Psi} \end{bmatrix}.$$

Next, in a departure from Duane, Garsia, and Zabrocki's work we assign to each car c_i^{Ψ} an explicit statistic r_i^{Ψ} by setting:

$$r_i^{\Psi} = \begin{cases} 1, & i = 1, \\ r_{i-1}^{\Psi} + 1, & c_{i-1}^{\Psi} \leqslant m \text{ and } i > 1, \\ d_{i-1}^{\Psi} + 1, & c_{i-1}^{\Psi} > m \text{ and } i > 1. \end{cases} \tag{2.1}$$

Example 2.2. Again let $m = 3$ and

$$\mathrm{PF} = \begin{bmatrix} 3 & 4 & 1 & 6 & 2 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\Psi(\mathrm{PF}) = \begin{bmatrix} 3 & 1 & 4 & 6 & 2 & 5 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 1 & 2 \end{bmatrix}.$$

Now we define ndinv(PF) explicitly.

Definition 2.3 *(*ndinv*).* If m gives the number of small cars,

$$\mathrm{ndinv}(\mathrm{PF}) := \sum_{\substack{c_b^\Psi > m \\ c_s^\Psi \leqslant m}} \left(\chi(b<s)\chi\left(d_b^\Psi \leqslant r_s^\Psi < r_b^\Psi\right) + \chi(b>s)\chi\left(d_b^\Psi < r_s^\Psi \leqslant r_b^\Psi\right)\right) - m. \tag{2.2}$$

Mirroring previous conventions for dinv, we will say that:

Definition 2.4 *(Diagonal inversion).* A big car c_b and a small car c_s form a diagonal inversion in $\Psi(\mathrm{PF})$ exactly when they contribute to the sum in the above definition, that is if either $b<s$ and $d_b^\Psi \leqslant r_s^\Psi < r_b^\Psi$ or $b>s$ and $d_b^\Psi < r_s^\Psi \leqslant r_b^\Psi$.

Example 2.5. As in Example 1.7, $m=3$ and

$$\mathrm{PF} = \begin{bmatrix} 3 & 4 & 1 & 6 & 2 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Psi(\mathrm{PF}) = \begin{bmatrix} 3 & 1 & 4 & 6 & 2 & 5 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 1 & 2 \end{bmatrix}.$$

Then there are six pairs of cars which form diagonal inversions, namely (c_1^Ψ, c_4^Ψ), (c_1^Ψ, c_6^Ψ), (c_2^Ψ, c_4^Ψ), (c_2^Ψ, c_6^Ψ), (c_5^Ψ, c_4^Ψ), and (c_5^Ψ, c_6^Ψ).

Since there are 3 small cars, $\mathrm{ndinv}(\mathrm{PF}) = 6-3 = 3$.

The main results of this paper is that

Theorem 2.6. *For any* $(p_1, p_2, \ldots, p_k) \models n$ *and the preceding definition of ndinv we have*

$$\langle \Delta_{h_m} C_{p_1} \ldots C_{p_k} 1, e_n \rangle = \sum_{\substack{\mathrm{PF}\ \text{an } m,n \text{ two-shuffle parking function} \\ \mathrm{comp}(\mathrm{PF}) = (p_1, \ldots, p_k)}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{ndinv}(\mathrm{PF})}. \tag{2.3}$$

3. A recursion satisfied by ndinv

Since $\langle \Delta_{h_0} C_1 1, e_1 \rangle = 1$, to be consistent we must set

$$\mathrm{ndinv}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 0.$$

In [8], Duane, Garsia, and Zabrocki prove the following recursion:

$$\langle \Delta_{h_m} C_{p_1} \ldots C_{p_k} 1, e_n \rangle = \sum_{p' \models p_1} t^{p_1 - 1} q^{k-1} \langle \Delta_{h_{m-1}} C_{p_2} \ldots C_{p_k} C_{p'} 1, e_n \rangle + \chi(p_1 = 1) \langle \Delta_{h_m} C_{p_2} \ldots C_{p_k} 1, e_{n-1} \rangle$$

where $p' \models p_1$ denotes that $p' = (p'_1, \ldots, p'_{l(p')})$ is a composition of p_1 and we use $C_{p'}$ for $C_{p'_1} \ldots C_{p'_{l(p')}}$. Guided by this symmetric function recursion, Duane, Garsia, and Zabrocki give a recursive map on two-shuffle parking functions. We give a slightly modified version of their map below that we will use to show that, with ndinv as defined above, the right hand side of (2.3) satisfies the same recursion as the left hand side.

Procedure 3.1. We begin by modifying the first part:

$$\mathrm{PF} = \begin{bmatrix} c_1 & c_2 & \ldots & c_{f_1} & \ldots \\ d_1 & d_2 & \ldots & d_{f_1} & \ldots \end{bmatrix}.$$

1. Remove its first domino $\begin{bmatrix} c_1 \\ d_1 \end{bmatrix}$.
2. For each $1 < b < f_1$ such that $c_b > m$, replace $\begin{bmatrix} c_b \\ d_b \end{bmatrix}$ by $\begin{bmatrix} c_b \\ d_b - 1 \end{bmatrix}$.
3. If adjacent dominoes in the result are of the form $\begin{bmatrix} \ldots & c_b & c_s & \ldots \\ \ldots & d-1 & d & \ldots \end{bmatrix}$, with $c_b > m$ and $c_s \leqslant m$, then replace them by $\begin{bmatrix} \ldots & c_s & c_b & \ldots \\ \ldots & d-1 & d & \ldots \end{bmatrix}$.
4. Move the modified first part (all $f_1 - 1$ dominoes) to the end of the sequence.

We will call the resulting two-line array

$$\Phi(\mathrm{PF}) = \begin{bmatrix} \bar{c}_1 & \bar{c}_2 & \ldots & \bar{c}_{n+m-1} \\ \bar{d}_1 & \bar{d}_2 & \ldots & \bar{d}_{n+m-1} \end{bmatrix}.$$

Example 3.2. As in Example 1.7, let $m = 3$ and

$$\mathrm{PF} = \begin{bmatrix} 3 & 4 & 1 & 6 & 2 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\Phi(\mathrm{PF}) = \begin{bmatrix} 2 & 5 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Remark 3.3. Notice that the resulting parking function may no longer be a proper shuffle as written. While it is convenient to keep track of the original numbers of the cars in future proofs, it is easy to slightly modify our result to again get a two-shuffle parking function. Thus the two-line array $\Phi(\mathrm{PF})$ should represent the two-shuffle parking function $\overline{\mathrm{PF}}$ obtained by the following steps:

- Let $m' = m - 1$ if the removed car is small and $m' = m$ if the removed car is big.
- Replace in $\Phi(\mathrm{PF})$ all $\bar{c}_i \leqslant m$ by a "1" and all $\bar{c}_i > m$ by a "2".
- Next, from the highest to the lowest $\bar{d}_i$ values and from right to left replace all the "1′s" by $1, 2, \ldots, m'$ and all the "2′s" by $m' + 1, m' + 2, \ldots, n + m - 1$.

For a proof that $\overline{\mathrm{PF}}$ is always a two-shuffle parking function, we refer the reader to [8]. Notice next that by calling a car $\leqslant m$ "small" and a car $> m$ "big", we can freely apply the operation $\mathrm{PF} \to \Psi(\mathrm{PF})$ to the two-line array $\Phi(\mathrm{PF})$ and denote the result $\Psi(\Phi(\mathrm{PF}))$. Since the action of the map Ψ on a car domino depends only on whether the car is big or small, it follows that the second and third rows of $\Psi(\Phi(\mathrm{PF}))$ will be identical to those we would obtain by constructing $\Psi(\overline{\mathrm{PF}})$. Since the contents of these two rows together with the relative size of the corresponding cars (i.e. whether they are big or small) is the only information that will be used in the following, we will use $\Psi(\Phi(\mathrm{PF}))$ rather than $\Psi(\overline{\mathrm{PF}})$. This will substantially simplify the notational conventions we must adopt to carry out our arguments. Using these notational conventions, the recursive step used by Duane, Garsia, and Zabrocki in the algorithm giving their ndinv can be simply written in the following form:

Recursion 3.4. For a parking function PF with k parts,

$$\operatorname{ndinv}(\mathrm{PF}) = \begin{cases} 0 & \text{if } c_1 \text{ is big, } n = 1, \\ \operatorname{ndinv}(\overline{\mathrm{PF}}) + (k-1)\chi(c_1 \leqslant m) & \text{otherwise.} \end{cases} \tag{3.1}$$

Thus, if Ψ is applied to the two-line array $\Phi(\mathrm{PF})$, the result is the three-line array

$$\Psi\big(\Phi(\mathrm{PF})\big) = \begin{bmatrix} \bar{c}_1^{\Psi} & \bar{c}_2^{\Psi} & \dots & \bar{c}_{n+m-1}^{\Psi} \\ \bar{d}_1^{\Psi} & \bar{d}_2^{\Psi} & \dots & \bar{d}_{n+m-1}^{\Psi} \\ \bar{r}_1^{\Psi} & \bar{r}_2^{\Psi} & \dots & \bar{r}_{n+m-1}^{\Psi} \end{bmatrix}.$$

Then to show that their ndinv and our explicit formulation are identical we need only prove that (3.1) holds true with ndinv(PF) replaced by (2.2) and $\operatorname{ndinv}(\overline{\mathrm{PF}})$ replaced by

$$\sum_{\substack{\bar{c}_b^{\Psi} > m \\ \bar{c}_s^{\Psi} \leqslant m}} \big(\chi(b<s)\chi\big(\bar{d}_b^{\Psi} \leqslant \bar{r}_s^{\Psi} < \bar{r}_b^{\Psi}\big) + \chi(b>s)\chi\big(\bar{d}_b^{\Psi} < \bar{r}_s^{\Psi} \leqslant \bar{r}_b^{\Psi}\big)\big) - m' \tag{3.2}$$

with m' as defined in Remark 3.3. Notice that to calculate ndinv using the recursion in (3.1), as Duane, Garsia and Zabrocki do in their paper, we need to apply Procedure 3.1 repeatedly. Every time we apply the procedure once, we will remove the first domino and move the resulting first part to the end. Suppose there are k parts in PF. If we apply Procedure 3.1 k times, the first car of each part will be removed. We call this the first round. Let k_1 be the number of parts after the first round. Again, applying Procedure 3.1 k_1 times removes the first car of each of these k_1 parts. We call this the second round.

Definition 3.5 *(Round).* We then define the ith round as applying Procedure 3.1 an additional k_{i-1} times, where k_{i-1} is the number of parts after the $(i-1)$st round.

This notion of "round" beautifully enlightens the relation between our definition of ndinv with the definition of Duane, Garsia, and Zabrocki. In fact, it will follow from our proofs that the d_i^{Ψ} gives the round at which a big car c_i^{Ψ} first appears in the main diagonal and $\bar{r}_i^{\Psi}$ gives the round at which car c_i^{Ψ} is removed. Using this it is not difficult to derive that for any given small car c_s^{Ψ} the expression

$$-1 + \sum_{\bar{c}_b^{\Psi} > m} \big(\chi(b<s)\chi\big(\bar{d}_b^{\Psi} \leqslant \bar{r}_s^{\Psi} < \bar{r}_b^{\Psi}\big) + \chi(b>s)\chi\big(\bar{d}_b^{\Psi} < \bar{r}_s^{\Psi} \leqslant \bar{r}_b^{\Psi}\big)\big)$$

gives precisely the number of big cars that are to the right of c_s^{Ψ} at the round of its removal in the recursive algorithm of Duane, Garsia and Zabrocki.

4. Our ndinv and Recursion 3.4

To show that our ndinv satisfies Recursion 3.4 we need to further examine the combination of Φ and Ψ, as it occurs in the following diagram:

$$\begin{array}{ccc} \mathrm{PF} & \xrightarrow{\Psi} & \Psi(\mathrm{PF}) \\ \big\downarrow{\scriptstyle\Phi} & & \\ \Phi(\mathrm{PF}) & \xrightarrow{\Psi} & \Psi(\Phi(\mathrm{PF})). \end{array}$$

Let us say that a certain car c_b is in position i in $\Psi(\mathrm{PF})$ and position j in $\Psi(\Phi(\mathrm{PF}))$. More precisely suppose that $c_i^{\Psi} = c_b$ and $\bar{c}_j^{\Psi} = c_b$ then, using the symbol "ind" to denote an index, we will, simply write "$\operatorname{ind}^{\Psi}(c_b) = i$" and "$\operatorname{ind}^{\Psi\Phi}(c_b) = j$". It will also be convenient to have an alternate notation for the $d, \bar{d}$, and $r, \bar{r}$ values. For instance, if we have $i = \mathrm{ind}^{\Psi}(c_b)$, $d_i^{\Psi} = 3$, and $r_i^{\Psi} = 5$, we will simply express this by writing $d^{\Psi}(c_b) = 3$ and $r^{\Psi}(c_b) = 5$. On the other hand if $j = \mathrm{ind}^{\Psi\Phi}(c_b)$, $\bar{d}_j^{\Psi} = 3$, and $\bar{r}_j^{\Psi} = 5$, we will write $\bar{d}^{\Psi}(c_b) = 3$ and $\bar{r}^{\Psi}(c_b) = 5$. The entries of PF and Φ(PF) will be handled in an analogous manner. Thus if $d_i = 4$, we may also write $\mathrm{ind}(c_i) = i$ and $d(c_i) = 4$. Similarly if $\bar{c}_i = c_b$ and $\bar{d}_i = 4$, we may write $\mathrm{ind}^{\Phi}(c_b) = i$ and $\bar{d}(c_b) = 4$ etc. Finally, it will also be convenient to write $c_1 \to c_2$ to state that car c_1 is to the left of car c_2 in a given expression. Using this notation, we can give an overview of the path we will follow to establish that our ndinv and the ndinv of Duane, Garsia and Zabrocki satisfy the same recursion. To be precise, we plan to establish the following facts:

Theorem 4.1. *With Ψ(PF) and $\Psi(\Phi(\mathrm{PF}))$ as defined above, we have for any cars c, c_1, c_2:*

- *Fact (1). If $f_1 < \mathrm{ind}^{\Psi}(c_1), \mathrm{ind}^{\Psi}(c_2)$, then $c_1 \to c_2$ in Ψ(PF) if and only if $c_1 \to c_2$ in $\Psi(\Phi(\mathrm{PF}))$.*
- *Fact (2). If $1 < \mathrm{ind}^{\Psi}(c_1), \mathrm{ind}^{\Psi}(c_2) \leqslant f_1$, then $c_1 \to c_2$ in Ψ(PF) if and only if $c_1 \to c_2$ in $\Psi(\Phi(\mathrm{PF}))$.*
- *Fact (3). If $1 < \mathrm{ind}^{\Psi}(c_1) \leqslant f_1 < \mathrm{ind}^{\Psi}(c_2)$, then $\mathrm{ind}^{\Psi\Phi}(c_1) > \mathrm{ind}^{\Psi\Phi}(c_2)$.*
- *Fact (4). For $\mathrm{ind}^{\Psi}(c) > f_1$ we have $\bar{d}^{\Psi}(c) = d^{\Psi}(c)$ and $\bar{r}^{\Psi}(c) = r^{\Psi}(c)$.*
- *Fact (5). For $1 < \mathrm{ind}^{\Psi}(c) < f_1$ we have $\bar{d}^{\Psi}(c) = d^{\Psi}(c) - 1$ if c is a big car and $\bar{r}^{\Psi}(c) = r^{\Psi}(c) - 1$, whether c is big or small.*

Before we immerse ourselves in the technicalities required for a proof of all these facts, it will be good to see how they give all that is needed to establish our desired goal. Recall that, in the present notation, by definition, a big car c_b and a small car c_s form a diagonal inversion in Ψ(PF) if either $c_b \to c_s$ and $d^{\Psi}(c_b) \leqslant r^{\Psi}(c_s) < r^{\Psi}(c_b)$ or $c_s \to c_b$ and $d^{\Psi}(c_b) < r^{\Psi}(c_s) \leqslant r^{\Psi}(c_b)$.

Theorem 4.2. *In Ψ(PF), if the first car is small, then it forms a diagonal inversion with a big car c_b only when the big car is on the main diagonal ($d^{\Psi}(c_b) = 0$). If the first car is big, it forms no diagonal inversions.*

Proof. By definition $r_1^{\Psi} = 1$. We look at the two cases separately.

- ($c_1^{\Psi} = c_s \leqslant m$.) Then we want all big cars (c_b) such that $d^{\Psi}(c_b) < 1 \leqslant r^{\Psi}(c_b)$. Those are exactly the big cars on the main diagonal.
- ($c_1^{\Psi} = c_b > m$.) Then we want all small cars (c_s) such that $d^{\Psi}(c_b) \leqslant r^{\Psi}(c_s) < 1$. Since $r^{\Psi}(c_s) \geqslant 1$, there are no such cars. □

Keeping this in mind let us see how these diagonal inversions change after we apply Φ.

Theorem 4.3. *For $\mathrm{ind}^{\Psi}(c_s) > 1$ and $\mathrm{ind}^{\Psi}(c_b) > 1$, a small car c_s and a big car c_b form a diagonal inversion in Ψ(PF) if and only if they form a diagonal inversion in $\Psi(\Phi(\mathrm{PF}))$.*

Proof. We split the argument into cases:

- ($\mathrm{ind}^{\Psi}(c_s), \mathrm{ind}^{\Psi}(c_b) > f_1$.) Fact (1) and Fact (4) make this case trivial.
- ($1 < \mathrm{ind}^{\Psi}(c_s), \mathrm{ind}^{\Psi}(c_b) \leqslant f_1$.) Fact (2) gives $c_b \to c_s$ or $c_s \to c_b$ in both Ψ(PF) and $\Psi(\Phi(\mathrm{PF}))$ and Fact (5) gives that $\bar{d}^{\Psi}(c_b) \leqslant \bar{r}^{\Psi}(c_s) < \bar{r}^{\Psi}(c_b)$ is $d^{\Psi}(c_b) - 1 \leqslant r^{\Psi}(c_s) - 1 < r^{\Psi}(c_b) - 1$ in the first case and $\bar{d}^{\Psi}(c_b) < \bar{r}^{\Psi}(c_s) \leqslant \bar{r}^{\Psi}(c_b)$ is $d^{\Psi}(c_b) - 1 < r^{\Psi}(c_s) - 1 \leqslant r^{\Psi}(c_b) - 1$ in the second case.
- ($1 < \mathrm{ind}^{\Psi}(c_s) \leqslant f_1 < \mathrm{ind}^{\Psi}(c_b)$.) Then $c_s \to c_b$ in Ψ(PF) but Fact (3) gives $c_b \to c_s$ in $\Psi(\Phi(\mathrm{PF}))$. Nevertheless, Facts (4) and (5) give $d^{\Psi}(c_b) < r^{\Psi}(c_s) \leqslant r^{\Psi}(c_b)$, or better $d^{\Psi}(c_b) \leqslant r^{\Psi}(c_s) - 1 < r^{\Psi}(c_b)$ and thus $\bar{d}^{\Psi}(c_b) \leqslant \bar{r}^{\Psi}(c_s) < \bar{r}^{\Psi}(c_b)$ as desired.
- ($1 < \mathrm{ind}^{\Psi}(c_b) \leqslant f_1 < \mathrm{ind}^{\Psi}(c_s)$.) Then $c_b \to c_s$ in Ψ(PF) but Fact (3) gives $c_s \to c_b$ in $\Psi(\Phi(\mathrm{PF}))$. Nevertheless, again Facts (4) and (5) give $d^{\Psi}(c_b) \leqslant r^{\Psi}(c_s) < r^{\Psi}(c_b)$, or better $d^{\Psi}(c_b) - 1 < r^{\Psi}(c_s) \leqslant r^{\Psi}(c_b) - 1$ and thus $\bar{d}^{\Psi}(c_b) < \bar{r}^{\Psi}(c_s) \leqslant \bar{r}^{\Psi}(c_b)$ as desired. □

As we can clearly see, Theorem 4.2 accounts for the second term in the second case of (3.1) and Theorem 4.3 accounts for the first term in the second case of (3.1), when we replace ndinv($\overline{\mathrm{PF}}$) by the expression in (3.2). Since our ndinv and the ndinv of Duane, Garsia, and Zabrocki are equal in the base case, Theorem 4.1 is all that is needed to show that these two ndinvs satisfy the same recursion and that, consequently, they must be identical.

5. Proof of Theorem 4.1

In this section, Facts (1)–(5) will be progressively established by a combination of claims and auxiliary lemmas. To begin to understand the relation between the arrays $\Psi(\mathrm{PF})$ and $\Psi(\Phi(\mathrm{PF}))$, we will start by showing that unlike when we apply Φ and reorder the parts, applying Ψ does not change the part containing any particular car. Before we can proceed, we need an observation. Suppose that for two diagonal numbers of a Dyck path we have $d_{i_1} < d_{i_2}$ for some $i_1 < i_2$ and notice that the “slow growth” condition $d_i \leqslant d_{i-1} + 1$ assures that for any $d_{i_1} < d \leqslant d_{i_2}$ there must be an index $i_1 < i \leqslant i_2$ such that $d_i = d$ and $d_{i-1} = d - 1$. This simple fact immediately implies that, in any one of our two-shuffle parking functions, between any two cars c_1 and c_2 with $d(c_1) < d(c_2)$ there must be at least $d(c_2) - d(c_1)$ big cars. Let us keep this in mind.

Claim 5.1. *If a car c is in the jth part of* PF, *then c is in the jth part of $\Psi(\mathrm{PF})$.*

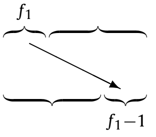
Proof. Since only small cars are shifted by Ψ, we only need to show that if a small car c_s is to the right of big car c_b on the main diagonal, then c_s does not move past c_b. Since $d(c_b) = 0$ our above observation shows that between c_b and c_s there are at least $d(c_s)$ big cars; this insures that c_s will remain to the right of c_b after applying Ψ. □

Next we show that if two cars are adjacent in the first part of $\Psi(\mathrm{PF})$, they are also adjacent in $\Psi(\Phi(\mathrm{PF}))$. We begin with a useful claim.

Claim 5.2. *If c_{s_1} and c_{s_2} are both small cars in* PF *and* $\mathrm{ind}(c_{s_1}) < \mathrm{ind}(c_{s_2})$, *then c_{s_2} does not move past c_{s_1} when we form $\Psi(\mathrm{PF})$.*

Proof. The proof is similar to that of Claim 5.1. Assume $d(c_{s_1}) < d(c_{s_2})$ or else we are done. Then our prior observation shows that there are at least $d(c_{s_2}) - d(c_{s_1})$ big cars between c_{s_1} and c_{s_2}. Thus c_{s_2} moves past at least $d(c_{s_2}) - d(c_{s_1})$ big cars before reaching the original position of car c_{s_1}. Thus c_{s_2} moves past at most $d(c_{s_1})$ big cars that were originally to the left of c_{s_1} in PF and must remain to the right of c_{s_1} in $\Psi(\mathrm{PF})$. □

To better understand the next lemma, notice that when we apply Φ to our parking function PF, elements in the first part of PF are moved to the end, as below:

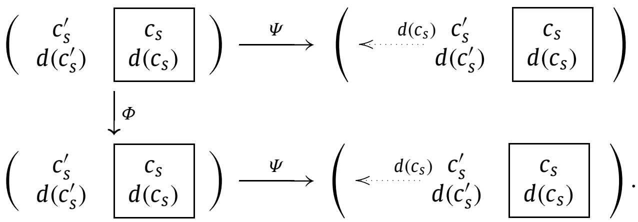


Lemma 5.3. *If two cars are adjacent in the first part of $\Psi(\mathrm{PF})$, then they are adjacent also in $\Psi(\Phi(\mathrm{PF}))$. Thus the relative order of cars in the first part of $\Psi(\mathrm{PF})$ (excluding the first car) is the same as the relative order of the last $f_1 - 1$ cars in $\Psi(\Phi(\mathrm{PF}))$ or stated differently, if $1 < \mathrm{ind}^{\Psi}(c) = j \leqslant f_1$ then*

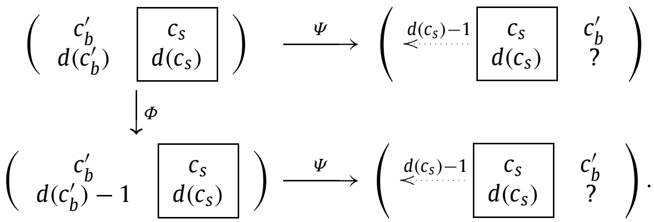
$$\mathrm{ind}^{\Psi\Phi}(c) = j - f_1 + n + m - 1.$$

Proof. Again observe that a car c in PF with $1 < \mathrm{ind}(c) \leqslant f_1$ will be shifted to the end by Φ. Thus the position J it occupies in $\Phi(\mathrm{PF})$ must satisfy $n+m-f_1 < J \leqslant n+m-1$. We would like to show that $J = j+n+m-f_1-1$. By Claim 5.2, no small car moves past another small car under Ψ. Since no two small cars change their relative order in Φ, we have that the relative order of small cars within the first part of PF is the same as the relative order of small cars in the last f_1-1 cars in $\Psi(\Phi(\mathrm{PF}))$. Clearly the relative order of any two big cars in the first part of PF is the same as their relative order in the last f_1-1 cars in $\Psi(\Phi(\mathrm{PF}))$. Thus we need to show that if a small car c_s in PF moves past t big cars when we apply Ψ, then it moves past a total of t big cars when we apply Φ and then Ψ. We split the remaining proof into three cases, for convenience let c'_s or c'_b denote the car immediately preceding c_s (or c_b respectively) in PF:

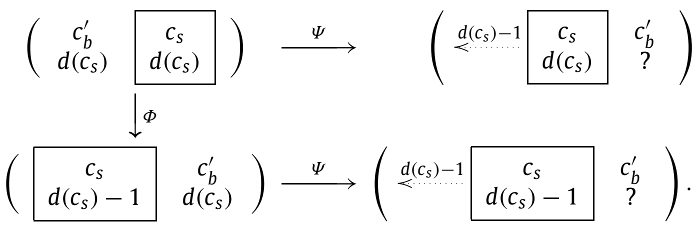
- $c'_s \leqslant m$. Then since c_s has a small car to its left, it will not switch places with any car in the construction of $\Phi(\mathrm{PF})$. Thus c_s occupies position $\mathrm{ind}(c_s)+n+m-f_1-1$ in $\Phi(\mathrm{PF})$ and its diagonal number remains $d(c_s)$. Thus c_s moves past $d(c_s)$ big cars when we form $\Psi(\mathrm{PF})$ and $\Psi(\Phi(\mathrm{PF}))$ respectively, as below[2]:



- $c'_b > m$ and $d(c_s) \neq d(c'_b)$. Since $d(c_s) \neq d(c'_b)$, after we replace $d(c'_b)$ with $d(c'_b)-1$ in step (2) of Procedure 3.1, we do not need to interchange car c_s and c'_b in step (3) of Procedure 3.1, as below:



- $c'_b > m$ and $d(c_s) = d(c'_b)$. Since $d(c_s) = d(c'_b)$, after we replace $d(c'_b)$ with $d(c'_b)-1$ in step (2) of Procedure 3.1, we need to interchange cars c_s and c'_b in step (3) of Procedure 3.1. These then will occupy positions $\mathrm{ind}(c_s)+n+m-f_1-2$ and $\mathrm{ind}(c'_b)+n+m-f_1$ in $\Phi(\mathrm{PF})$. When we form $\Psi(\mathrm{PF})$, we move c_s past $d(c_s)$ cars, including c'_b. (We may assume $d(c_s) = d(c'_b) \neq 0$ or else c_s and c'_b are in two different parts and we are done by Claim 5.1.) When we move c_s in Procedure 2.1, we move it past $d(c'_b)-1$ big cars. In addition, as we previously discussed, we moved car c_s past car c'_b, meaning that in total we shifted the car $d(c'_b)-1+1 = d(c_s)$ times, as required. Thus we have:



□

[2] Here, as in the following diagrams, the variable above the left pointing arrow gives the number of big cars not shown in the diagram that the boxed car must pass to form $\Psi(\mathrm{PF})$ or $\Psi(\Phi(\mathrm{PF}))$. Note that we give the relative order of the two cars in the right hand diagram, but there may be additional cars between them.

Theorem 5.4. *For all cars c we have*

$$\mathrm{ind}^{\Psi\Phi}(c) = \mathrm{ind}^{\Psi}(c) - f_1 + \chi(i \leqslant f_1)(n+m-1).$$

Proof. Lemma 5.3 deals with the case that c is in the first part of our parking function. It remains to consider cars that are not in the first part. The only part whose interior is altered by Φ is the first part so the only thing that the map Φ does to the last $k-1$ parts is to rigidly shift them to the beginning of the parking function (i.e. for $i > f_1$, the ith car of PF becomes $(i - f_1)$th car in $\Phi(\mathrm{PF})$). That means for $i > f_1$, $c_i = \bar{c}_{i-f_1}$ and $d_i = \bar{d}_{i-f_1}$. By Claim 5.1, the map Ψ moves small cars within each part, so when we consider the effects of Ψ on all but the first part of PF or on the first $k-1$ parts of $\Phi(\mathrm{PF})$, we can ignore its effect on the remaining parts. We are applying Ψ to pairs of objects which are locally identical, and thus we are done. □

Notice that at this point we have completed the proof of Facts (1) and (2). Since Fact (3) is an immediate consequence of the definition of the map Φ, it remains to prove Facts (4) and (5) which together express the relationship between $d^{\Psi}(c)$ and $\bar{d}^{\Psi}(c)$ for a big car c and between $r^{\Psi}(c)$ and $\bar{r}^{\Psi}(c)$ for any car c. We begin with the easiest case:

Lemma 5.5. *If $\mathrm{ind}(c) > f_1$, then $d^{\Psi}(c) = \bar{d}^{\Psi}(c)$ and $r^{\Psi}(c) = \bar{r}^{\Psi}(c)$.*

Proof. As we observed in the proof of Theorem 5.4, the map Φ alters only the interior of the first part; thus it is clear that $d^{\Psi}(c) = \bar{d}^{\Psi}(c)$. Thus we only need to prove that $r^{\Psi}_{f_1+1} = 1$ since $\bar{r}^{\Psi}_1$ should be 1 by definition of Ψ. The f_1th car in PF is the last car of the first part in PF, so $c_{f_1} > m$ and $d_{f_1} = 0$. Thus by definition, $r^{\Psi}_{f_1+1} = 1$. □

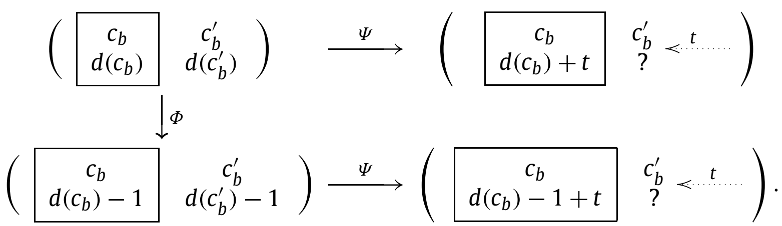
Next, we again consider elements from the first part of PF.

Lemma 5.6. *If $1 < \mathrm{ind}^{\Psi}(c_b) \leqslant f_1$ and $c_b > m$, then*

$$\bar{d}^{\Psi}(c_b) = d^{\Psi}(c_b) - 1.$$

Proof. Recall that if t small cars move past a big car c_b under Ψ, then t cars move past c_b when we apply the combination of Φ and Ψ. As in Lemma 5.3, we split the argument into cases but now we use c'_s and c'_b to denote the car that immediately follows c_s and c_b (respectively) in PF.

- $c'_b > m$. Then $d(c_b)$ is replaced by $d(c_b) - 1$ in step (2) of Procedure 3.1. Since $c'_b > m$, c_b and c'_b do not switch places in the following step. Thus if t cars move past the car c_b when we apply the combination of Φ and Ψ, t cars move past c_b when we apply Ψ to $\Phi(\mathrm{PF})$. Thus $d^{\Psi}(c_b) = d(c_b) + t$ and $\bar{d}^{\Psi}(c_b) = (d(c_b) - 1) + t$. Thus we have[3]:



[3] In the following diagrams, the variable above the left pointing arrow gives the number of small cars moved past the boxed car to form $\Psi(\mathrm{PF})$ or $\Psi(\Phi(\mathrm{PF}))$. As before, we give the relative order of the two cars in the right hand diagram, but there may be additional cars between them.

- $c'_s \leqslant m$ and $d(c'_s) \neq d(c_b)$. Then $d(c_b)$ is replaced by $d(c_b) - 1$ in step (2) of Procedure 3.1, but in the following step c_b and c'_s do not switch places. Thus by the same argument as in the previous case, $d^{\Psi}(c_b) = d(c_b) + t$ and $\bar{d}^{\Psi}(c_b) = (d(c_b) - 1) + t$. Thus we have:

$$\begin{array}{ccc} \left(\boxed{\begin{matrix} c_b \\ d(c_b) \end{matrix}} \quad \begin{matrix} c'_s \\ d(c'_s) \end{matrix} \right) & \xrightarrow{\ \Psi\ } & \left(\boxed{\begin{matrix} c_b \\ d(c_b)+t \end{matrix}} \quad \begin{matrix} c'_s \\ d(c'_s) \end{matrix} \overset{t}{<\cdots} \right) \\ \big\downarrow \Phi & & \\ \left(\boxed{\begin{matrix} c_b \\ d(c_b)-1 \end{matrix}} \quad \begin{matrix} c'_s \\ d(c'_s) \end{matrix} \right) & \xrightarrow{\ \Psi\ } & \left(\boxed{\begin{matrix} c_b \\ d(c_b)-1+t \end{matrix}} \quad \begin{matrix} c'_s \\ d(c'_s) \end{matrix} \overset{t}{<\cdots} \right). \end{array}$$

- $c'_s \leqslant m$ and $d(c'_s) = d(c_b)$. Then $d(c_b)$ is replaced by $d(c_b) - 1$ in step (2) of Procedure 3.1, and in the following step c_b and c'_s switch places. Notice that if t cars move past car c_b when we apply the combination of Φ and Ψ, only $t-1$ cars move past c_b when we apply Ψ to $\Phi(\mathrm{PF})$. Thus $d^{\Psi}(c_b) = d(c_b) + t$ and $\bar{d}^{\Psi}(c_b) = d(c_b) + (t-1)$. Finally, we have:

$$\begin{array}{ccc} \left(\boxed{\begin{matrix} c_b \\ d(c_b) \end{matrix}} \quad \begin{matrix} c'_s \\ d(c_b) \end{matrix} \right) & \xrightarrow{\ \Psi\ } & \left(\begin{matrix} c'_s \\ d(c_b) \end{matrix} \quad \boxed{\begin{matrix} c_b \\ d(c_b)+t \end{matrix}} \overset{t}{<\cdots} \right) \\ \big\downarrow \Phi & & \\ \left(\begin{matrix} c'_s \\ d(c_b)-1 \end{matrix} \quad \boxed{\begin{matrix} c_b \\ d(c_b) \end{matrix}} \right) & \xrightarrow{\ \Psi\ } & \left(\begin{matrix} c'_s \\ d(c_b)-1 \end{matrix} \quad \boxed{\begin{matrix} c_b \\ d(c_b)+t-1 \end{matrix}} \overset{t-1}{<\cdots} \right). \end{array}$$ □

Lemma 5.7. *If* $1 < \mathrm{ind}^{\Psi}(c) \leqslant f_1$, *then*

$$\bar{r}^{\Psi}(c) = r^{\Psi}(c) - 1.$$

Proof. For $1 < \mathrm{ind}^{\Psi}(c) \leqslant f_1$, by Theorem 5.4, the relative order of the first $f_1 - 1$ cars in $\Psi(\mathrm{PF})$ is same as of the last $f_1 - 1$ cars in $\Psi(\Phi(\mathrm{PF}))$. We split the argument into cases and denote by c'_s or c'_b the car immediately preceding c in $\Psi(\mathrm{PF})$:

- ($c'_b > m$.) Then c'_b immediately precedes c in $\Psi(\Phi(\mathrm{PF}))$ and by the definition of Ψ, $r^{\Psi}(c) = d^{\Psi}(c'_b) + 1$ and $\bar{r}^{\Psi}(c) = \bar{d}^{\Psi}(c'_b) + 1$. By Theorem 5.6, however, $\bar{d}^{\Psi}(c'_b) = d^{\Psi}(c'_b) - 1$. Thus $\bar{r}^{\Psi}(c) = d^{\Psi}(c'_b) = r^{\Psi}(c) - 1$.
- ($c'_s \leqslant m$ and $\mathrm{ind}^{\Psi}(c) = 2$.) Then $\bar{r}^{\Psi}(c) = 1$ since it follows the last car in the previous part of $\Psi(\Phi(\mathrm{PF}))$, which by definition is a big car on the main diagonal. Clearly $r^{\Psi}(c) = 2$ as required.
- ($c'_s \leqslant m$ and $\mathrm{ind}^{\Psi}(c) > 2$.) Inductively, we may assume $\bar{r}^{\Psi}(c'_s) = r^{\Psi}(c'_s) - 1$. Then by definition $\bar{r}^{\Psi}(c) = \bar{r}^{\Psi}(c'_s) + 1 = r^{\Psi}(c'_s) = r^{\Psi}(c) - 1$. □

6. Conclusion

We have seen that Theorems 4.1, 4.2 and 4.3 prove that our ndinv satisfies Recursion 3.4. In particular, this combinatorial result combined with the symmetric function results of Duane, Garsia, and Zabrocki in [8] proves that

Theorem 6.1. *With the ndinv defined in* (3.2) *and* $(p_1, p_2, \ldots, p_k) \vdash n$ *for any integer* $m \geqslant 0$ *we have*

$$\langle \Delta_{h_m} C_{p_1} \ldots C_{p_k} 1, e_n \rangle = \sum_{\substack{\mathrm{PF}\ \text{an } m,n \text{ two-shuffle parking function} \\ \mathrm{comp}(\mathrm{PF}) = (p_1, \ldots, p_k)}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{ndinv}(\mathrm{PF})}.$$

It would be interesting to consider if the ndinv statistic could be extended to give a statistic on all parking functions. We end now with the proof of a prior statement about the sequence r_i^{Ψ}.

Theorem 6.2. *For each car c, $\bar{r}^{\Psi}(c)$ gives the number of the round in which car c is removed when we apply Procedure 3.1 repeatedly.*

Proof. Suppose $\bar{r}^{\Psi}(c) = 1$ for some car c. This happens if and only if for the car c'_b preceding c is a big car and $\bar{d}(c'_b) = 0$, in other words c'_b is a big car on the main diagonal of PF. This is true exactly when c is the first car in some part of PF and will be removed in the first round. Moreover, by Theorem 5.7, the r_j^{Ψ} value of any car will decreased by 1 in any round where it is not removed. This completes the proof by induction. □

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