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Reduction and Synthesis of Live and Bounded Free Choice Petri Nets*

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This paper provides reduction rules that make it possible to reduce all and only live and bounded Free Choice Petri nets to a circuit containing one place and one transition. The reduction algorithm is shown to require polynomial time in the size of the system. The reduction rules can be transformed into synthesis rules, which can be used for the stepwise construction of large systems. © 1994 Academic Press, Inc.

1. INTRODUCTION

Petri nets are one of the standard formal tools for the specification, analysis and synthesis of concurrent systems [18, 19]. In this paper we assume that the reader is familiar with a number of basic concepts of net theory. An annex contains a summary of the ones used in the text.

Reduction is one of the most interesting verification techniques for Petri nets. The verifier is given a kit of so called *reduction rules.* These rules transform a net system (a net with an initial marking) while preserving some properties of interest (i.e., the system obtained after the transformation has one of the properties if and only if the system before the transformation had it). Two properties which are very often considered are *boundedness* (absence of overflows in finite stores) and *liveness* (absence of partial or global deadlocks). The reason is that, in many cases, it is relatively easy to prove that the system is correct if it is live and bounded, while the most difficult part of the verification lies precisely in proving that these two properties hold.

The reduction rules are applied as long as possible. The properties are after that verified by means of other techniques (typically reachability analysis) on the reduced system, at a lower computational cost. Since the

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algorithms for the application of the rules are usually very efficient, the technique is very useful when the reduced system is much smaller than the original one.

Kits of rules which are known to reduce all and only the systems of a certain class to very simple systems—called in this paper *atomic systems*—are particularly interesting. The class is then said to be *completely reducible*. Membership in a completely reducible class can be decided by checking if the reduced system is atomic. In this paper, we give kits of rules that reduce all and only live and bounded Free Choice systems to atomic systems whose underlying net is isomorphic to

$$(\{s\}, \{t\}, \{(s, t), (t, s)\}).$$

It follows that liveness and boundedness of a Free Choice system can be decided by applying a reduction algorithm, which we show terminates in polynomial time in the size of the system.

Our result provides not only a verification but also a synthesis technique: we can "reverse" the reduction rules to obtain *synthesis rules*. Given a reduction rule that transforms a system Σ_1 into Σ_2, the corresponding synthesis rule transforms Σ_2 into Σ_1. The kits of reverse rules obtained make it possible to generate all and only live and bounded Free Choice systems starting from an atomic one by means of stepwise transformations. Two of our synthesis rules are typical refinements of places and transitions. The other two consist of the addition of certain new places and transitions, respectively.

Sources and Related Work. Free Choice systems were introduced in [13]. They make it possible to model both concurrency and nondeterminism, but constrain their interplay. They have been further studied in several papers (see [4] for a survey and, more recently, [3]).

Reduction techniques have been extensively studied by Berthelot [1, 2]. The rules described in his work make it possible to reduce to a system composed of just one transition two classes of net systems: the live and bounded *T*-systems (see the Annex for a definition) and a behavioural generalisation of them, namely the live, bounded, and persistent systems. Since *T*-systems are a subclass of Free Choice systems, our work generalises the first of these results.[1]

The paper by Genrich and Thiagarajan on Bipolar Schemes [12], as well as recent papers by Desel [6] and Kovalyov [16], extend Berthelot's results in different ways. Since these extensions are closely related to the results of this paper, we postpone a comparison to the conclusions.

[1] Not the second, because there exist persistent systems which are not Free Choice (and vice versa).

Other papers by Valette [23] and Suzuki and Murata [22] on reduction techniques do not provide results on complete reducibility.

The paper is organised as follows. Section 2 introduces the basic results on Free Choice systems used in the paper (more specific ones are introduced when needed), and Section 3 basic definitions on reductions. Section 4 describes the reduction rules, which are applied to an example in Section 5. In Section 6 the complete reducibility of the class of live and bounded Free Choice systems is proved. Section 7 shows that the reduction process terminates in polynomial time in the size of the system. Finally, Section 8 explains how to derive a synthesis procedure from the reduction rules. Some of the proofs are written in the proof style of W. H. J. Feijen.

2. Some Results on Free Choice Systems

Basic definitions on Petri nets used in this and the following sections are contained in the Annex. For the reader familiar with Petri nets, the only points worth mentioning here are that, for technical reasons:

- nets are assumed to be connected, and
- in a net system (N, M_0) with $N=(S, T, F)$, S and T are assumed to be nonempty.

Definition 2.1. A net $N=(S, T, F)$ is *Free Choice* iff $\forall s \in S$, $\forall t \in s^\bullet$: $s^\bullet = \{t\} \vee {}^\bullet t = \{s\}$.

We denote the class of all live and bounded Free Choice systems by LBFC. WFFC (Well Formed Free Choice) denotes the class of nets underlying LBFC systems. More formally: $N \in$ WFFC if and only if there exists a marking M_0 such that $(N, M_0) \in$ LBFC.

This paper makes extensive use of known results about Free Choice systems. Those used throughout the whole paper are contained in this section. We start however with a result that also holds for non-Free Choice nets:

Theorem 2.2 [17]. *Let N be a structurally live and structurally bounded net. Then N is conservative and consistent.*

A net is said to be *covered by S-components* iff every node of it belongs to some S-component. For WFFC nets we have the following result:

Theorem 2.3 [13, 3]. *Decomposition theorems.*
Let $N \in$ WFFC. *Then*:

(a) *N is covered by S-components*

(b) *N is covered by T-components.*

Remark 2.4. Since S-components and T-components are strongly connected nets, and all nets are assumed to be connected, Theorem 2.3 implies that the nets in WFFC are strongly connected. This result can be extended to the underlying nets of arbitrary live and bounded systems (see, for instance, [3]).

Using Theorem 2.3, the following characterisation of the class WFFC is easy to derive.

THEOREM 2.5. [10]. *Characterisation of the class* WFFC.
$N \in$ WFFC *iff* N *is Free Choice, structurally live, and structurally bounded.*

Proof. ($\Leftarrow$) Follows from the definitions.

($\Rightarrow$) N is Free Choice and structurally live by definition. We show that N is structurally bounded. Let M_0 be an arbitrary marking and s and arbitrary place of N. By Theorem 2.3, there exists an S-component $N_1 = (S_1, T_1, F_1)$ of N such that $s \in S_1$. Since N_1 is an S-graph, we have

$$\forall M \in [M_0\rangle: M(s) \leqslant \sum_{s_1 \in S_1} M_0(s_1).$$

Since this holds for an arbitrary place s, (N, M_0) is bounded. Since this holds for an arbitrary marking M_0, N is structurally bounded. ∎

The net of Fig. 1 is in WFFC (the marking shown in the figure makes the net live and bounded). This net is covered by the two S-components shown in Fig. 2, and by the two T-components shown in Fig. 3.

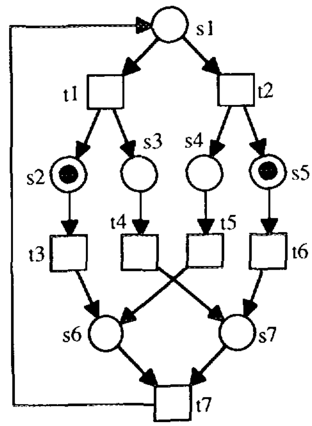

FIG. 1. A live and bounded Free Choice system.

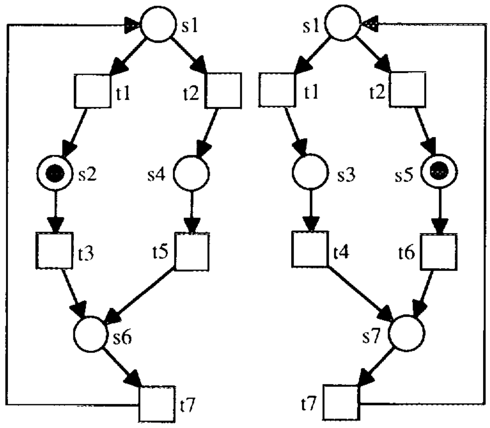

FIG. 2. The net of Fig. 1 is covered by S-components.

Let $N=(S, T, F)$ be a net. The net $N^{\text{rd}}=(T, P, F^{-1})$ is the *reverse dual* of N. Loosely speaking, N^{rd} is obtained by interchanging places and transitions and reversing the arcs in N.

THEOREM 2.6 [13, 3]. *Duality theorem.*
$N \in$ WFFC *iff* $N^{\text{rd}} \in$ WFFC.

The reverse dual of the net of Fig. 1 is in WFFC (in fact, the two nets happen to be isomorphic).

THEOREM 2.7 [10]. *Characterisation of* LBFC *in terms of* WFFC.
$(N, M_0) \in$ LBFC *iff the following two conditions hold*:

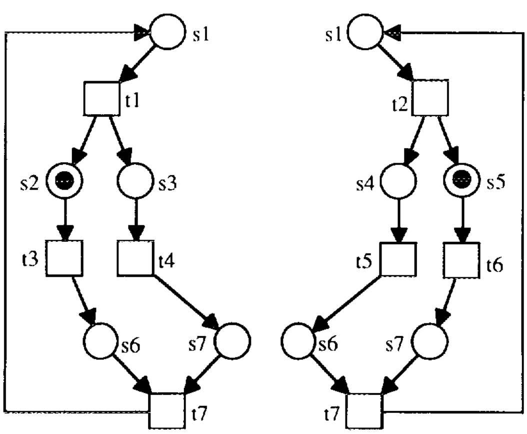

FIG. 3. The net of Fig. 1 is covered by T-components.

(i) $N \in \text{WFFC}$

(ii) *Every nonempty siphon of N is marked at M_0 (i.e., at least one of its places contains a token at M_0).*

The reader can check that all nonempty siphons of the net system shown in Fig. 1 are marked.

THEOREM 2.8 [14]. *Connection between liveness and deadlock-freeness. Let (N, M_0) be a bounded and strongly connected Free Choice system. Then $(N, M_0) \in \text{LBFC}$ iff (N, M_0) is deadlock free.*

3. REDUCTION RULES: BASIC DEFINITIONS

A *transformation rule* T is a binary relation on the class of all net systems. Given $(\Sigma, \tilde{\Sigma}) \in T$, Σ is called the *source system* and $\tilde{\Sigma}$ the *target system*. $(\Sigma, \tilde{\Sigma}) \in T$ is read: the rule T can transform Σ into $\tilde{\Sigma}$. The transformation rule T is *applicable* to Σ iff there exists a system $\tilde{\Sigma}$ such that $(\Sigma, \tilde{\Sigma}) \in T$. A finite set $\{T_1, ..., T_a\}$ of transformation rules is called a *kit*. A system Σ can be transformed into Σ' by a kit $\{T_1, ..., T_a\}$ of transformation rules iff $(\Sigma, \Sigma') \in (\bigcup_{i=1}^{a} T_i)^*$.

A *reduction rule* transforms a source system into a simpler target system, according to some criterion (in our case, the target system will have fewer nodes). Let $\mathscr{A}$ be a class of particularly simple systems, called *atomic systems*. We say that a system Σ can be *completely reduced*, or just reduced by a kit $\{R_1, ..., R_a\}$ iff there exists $\Sigma' \in \mathscr{A}$ such that Σ is reduced to Σ' by the kit. The class of systems reduced by the kit is denoted by $\mathscr{R}(R_1, ..., R_a)$.

Our goal is to give reduction rules that preserve certain properties. We identify a property of systems (e.g. liveness) with the class of systems that satisfy it (e.g., the class of all live systems). We can now formalise the idea that a rule preserves a property:

Let $\mathscr{C}$ be a class of net systems. A reduction rule R is *sound with respect to* $\mathscr{C}$ iff

$$((\Sigma, \tilde{\Sigma}) \in R \wedge \Sigma \in \mathscr{C}) \Rightarrow \tilde{\Sigma} \in \mathscr{C}.$$

R is *strongly sound with respect to* $\mathscr{C}$ iff

$$(\Sigma, \tilde{\Sigma}) \in R \Rightarrow (\Sigma \in \mathscr{C} \Leftrightarrow \tilde{\Sigma} \in \mathscr{C}).$$

PROPOSITION 3.1. *Let $\mathscr{C}$ be a class of systems such that $\mathscr{A} \subseteq \mathscr{C}$. If $\{R_1, ..., R_a\}$ is a kit of reduction rules strongly sound with respect to $\mathscr{C}$, then $\mathscr{R}(R_1, ..., R_a) \subseteq \mathscr{C}$.*

Proof. Let Σ be an arbitrary system of $\mathscr{R}(R_1, ..., R_a)$. There exists by definition a sequence $(\Sigma_0, \Sigma_1, ..., \Sigma_n)$ with $\Sigma = \Sigma_0$ and $\Sigma_n \in \mathscr{A}$ such that

$$\forall i, 0 \leqslant i \leqslant (n-1) : (\Sigma_i, \Sigma_{(i+1)}) \in \bigcup_{j=1}^{a} R_j$$

Since $\mathscr{A} \subseteq \mathscr{C}$, we have $\Sigma_n \in \mathscr{C}$. Since the rules $R_1, ..., R_a$ are strongly sound, if $\Sigma_{i+1} \in \mathscr{C}$ then $\Sigma_i \in \mathscr{C}$. Therefore, every element of the sequence is contained in $\mathscr{C}$, in particular Σ. ▮

The intended use of reduction rules is as verification tools: given a system, we reduce it to a simpler one which enjoys some properties if and only if the original system enjoyed them. Since we cannot assume that the original system satisfies the properties, a reduction rule should be strongly sound.

A kit $\{R_1, ..., R_a\}$ of reduction rules is *complete with respect to a class* $\mathscr{C}$ of systems iff $\mathscr{C} \subseteq \mathscr{R}(S_1, ..., S_a)$. By Proposition 3.1, if $\{R_1, ..., R_a\}$ is strongly sound and complete with respect to $\mathscr{C}$, then $\mathscr{C} = \mathscr{R}(S_1, ..., S_a)$.

The last concept we introduce is that of a structural rule. A *structural rule* is a binary relation on the class of all nets $\mathscr{N}$. Every rule T has an underlying structural rule ST, obtained projecting the binary relation T on the class $\mathscr{N}$. All definitions above can be easily extended to structural rules.

4. The Reduction Rules

Our reduction rules are introduced in this section. The format for their description is similar to that used in [12], and corresponds to our interpretation that $(\Sigma, \tilde{\Sigma}) \in R$ is read "R can transform Σ into $\tilde{\Sigma}$". First, we give the *conditions of application* of the rule, which specify to which systems Σ is R applicable, or, in other words, which systems have some image under the rule. Then, we specify the target systems $\tilde{\Sigma}$ corresponding to a source system.

Note that not every pair composed by a net and a marking is a net system; this is the case only if the net is connected and contains some place and some transition. Therefore, every rule has to be shown to be well defined; that is, we have to show that given a source system Σ, $\tilde{\Sigma}$ is a system as well.

4.1. *Abstraction Rules*

The two rules we introduce here merge two places (R_1), respectively two transitions (R_2). R_2 is a particular case of the post-fusion rule of [2]. R_1

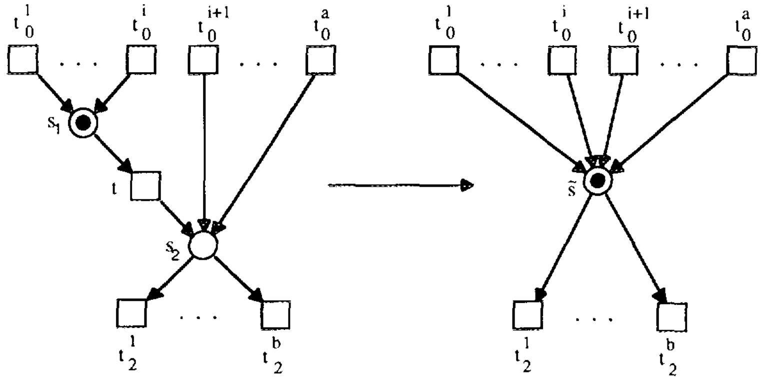

FIG. 4. The rule R_1.

is a similar pre-fusion rule for places. We can consider them as typical abstractions: in the case of R_2, two events are considered no longer distinguishable, and in the case of R_1 two local states are merged into one.

R_1 is informally described in Fig. 4. For its textual description, we need a preliminary definition. If F is the flow relation of a net N, then $F(x \leftarrow y)$ denotes the relation obtained by replacing all appearances of the node x in the pairs of F by the node y. $F \mid x$ denotes the relation obtained by removing from F all the pairs containing the node x (this notation is extended to a set of nodes X in the obvious way).

Rule 1. Let $\Sigma = (N, M_0)$ be a system. $(\Sigma, \tilde{\Sigma}) \in R_1$, where $\tilde{\Sigma} = (\tilde{N}, \tilde{M}_0)$, iff:

Conditions on Σ. There exists $t \in T$ such that:

1. $|{}^\bullet t| = |t^\bullet| = 1, {}^\bullet t \neq t^\bullet$
2. ${}^\bullet({}^\bullet t) \neq \varnothing$
3. $({}^\bullet t)^\bullet = \{t\}$.

Changes in Σ to produce $\tilde{\Sigma}$. Let $\{s_1\} = {}^\bullet t$ and $\{s_2\} = t^\bullet$.

1. $\tilde{S} = (S \setminus \{s_1, s_2\}) \cup \{\tilde{s}\}$ (where $\tilde{s} \notin S$)
2. $\tilde{T} = T \setminus \{t\}$
3. $\tilde{F} = F(s_1 \leftarrow \tilde{s}, s_2 \leftarrow \tilde{s}) \mid t$
4. $\forall s \in \tilde{S}\colon \tilde{M}(s) = \begin{cases} M(s) & \text{if } s \neq \tilde{s} \\ M(s_1) + M(s_2) & \text{if } s = \tilde{s}. \end{cases}$

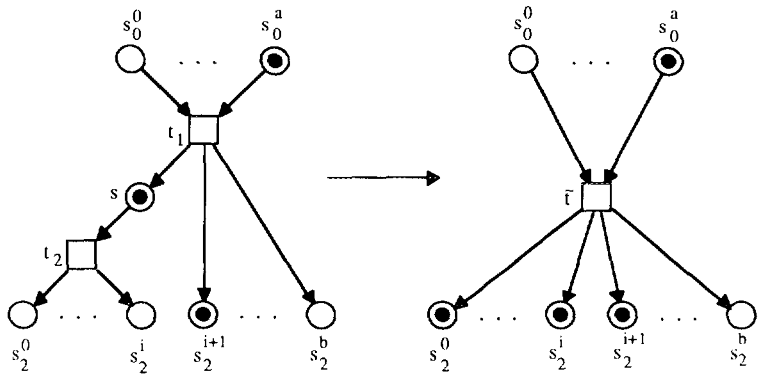

FIG. 5. The rule R_2.

The rule is well defined. First, N contains at least two places and two transitions by conditions 1 and 2, and therefore $\tilde{N}$ has at least one place and one transition. Second, $\tilde{N}$ is connected because N is connected.

The second reduction rule is graphically described in Fig. 5. Its textual description is as follows.

Rule 2. Let $\Sigma = (N, M_0)$ be a system. $(\Sigma, \tilde{\Sigma}) \in R_2$, where $\tilde{\Sigma} = (\tilde{N}, \tilde{M}_0)$, iff:

Conditions on Σ. There exists $s \in S$ such that:

1. $|{}^\bullet s| = |s^\bullet| = 1, {}^\bullet s \neq s^\bullet$
2. $(s^\bullet)^\bullet \neq \emptyset$
3. ${}^\bullet(s^\bullet) = \{s\}$.

Changes in Σ to produce $\tilde{\Sigma}$. Let $\{t_1\} = {}^\bullet s$ and $\{t_2\} = s^\bullet$.

1. $\tilde{S} = S \backslash \{s\}$
2. $\tilde{T} = (T \backslash \{t_1, t_2\}) \cup \{\tilde{t}\}$ (where $\tilde{t} \notin T$)
3. $\tilde{F} = F(t_1 \leftarrow \tilde{t}, t_2 \leftarrow \tilde{t}) \mid s$
4. $\forall s' \in \tilde{S}$: $\tilde{M}(s') = \begin{cases} M(s) & \text{if } s' \notin t_2^\bullet \\ M(s') + M(s) & \text{if } s' \in t_2^\bullet. \end{cases}$

The rule can be shown to be well defined by a similar argument to that used for the first rule.

We have the following result:

THEOREM 4.1. *R_1, R_2 are strongly sound with respect to* LBFC.

Proof. A stronger result follows easily from the definitions: R_1, R_2 are strongly sound with respect to the classes of Free Choice systems, live systems, and k-bounded systems independently. ▮

We denote by SR_1 and SR_2 the two structural rules corresponding to R_1 and R_2. There exists a strong connection between SR_1 and SR_2, which will be useful later.

PROPOSITION 4.2. $(N, \tilde{N}) \in SR_1$ *iff* $(N^{\mathrm{rd}}, \tilde{N}^{\mathrm{rd}}) \in SR_2$.

Proof. Immediate from the definitions. In particular, Fig. 5 (ignoring the marking) is obtained from Fig. 4 by interchanging places and transitions and reversing the arcs. ▮

4.2. *Linear Dependency Rules*

The third and fourth reduction rules consist of the removal of certain nodes. We deal with the removal of places first. The rôle of places in nets is to impose conditions on the occurrences of transitions. The fundamental property concerning a system Σ and the smaller system Σ' obtained after removing a place is that every occurrence sequence of Σ is also an occurrence sequence of Σ'. Moreover, it follows easily from the occurrence rule that the markings we obtain after letting the sequence occur in both Σ and Σ' coincide on the remaining places. To formalise these ideas some notations are necessary.

Let $N = (S, T, F)$ be a net with $|S| > 1$. We define the net $N^{-s} = (S \setminus \{s\}, T, F \mid s)$. The incidence matrices of N and N^{-s} are called C and C^{-s}, respectively. The row of C corresponding to a place s is denoted by $r(s)$. We then have

$$C = \begin{pmatrix} C^{-s} \\ r(s) \end{pmatrix}.$$

Given a marking M of N, M^{-s} denotes the marking of N^{-s} obtained by projecting M on $S \setminus \{s\}$.

The fundamental property described above can now be expressed as

$$M_1[\sigma\rangle M_2 \Rightarrow M_1^{-s}[\sigma\rangle M_2^{-s},$$

where the left part refers to N and the right part to N^{-s}. In particular, this property implies that the language of the bigger net is included in the language of the smaller net; formally, $L(N, M_0) \subseteq L(N^{-s}, M_0^{-s})$.

We are interested in places whose removal preserves some of the properties of the system.

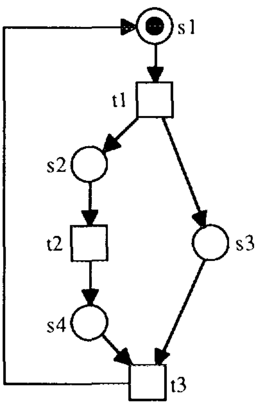

FIG. 6. s_3 is a linearly dependent place.

DEFINITION 4.3. Let C be the incidence matrix of a net $N = (S, T, F)$. A place $s \in S$ is *linearly dependent* iff $|S| > 1$ and $r(s)$ is a linear combination of the rows of C^{-s}; i.e. iff there exists a vector Λ such that $r(s) = \Lambda^{\mathrm{T}} \cdot C^{-s}$ (we require $|S| > 1$ because otherwise C^{-s} is not defined). The place s is *nonnegative linearly dependent* iff $\Lambda \geqslant 0$.

In Fig. 6, place s_3 is (nonnegative) linearly dependent, because $r(s_3) = r(s_2) + r(s_4)$.

Remark 4.4. A *self-loop* is a place s such that ${}^{\bullet}s = s^{\bullet}$. For a self-loop s, $r(s)$ is the null vector. According to the definition above, a self-loop of a net with at least two places is (nonnegative) linearly dependent with $\Lambda = 0$. However, if the self-loop is the only place of the net, then it is not a linearly dependent place.

The fundamental property of a linearly dependent place is that, at any reachable marking, the number of tokens it contains is a linear function of the number of tokens in the rest of the places:

PROPOSITION 4.5. *Let (N, M_0) be a system and s a linearly dependent place of N such that $r(s) = \Lambda^{\mathrm{T}} \cdot C^{-s}$. Then*

$$\forall M \in [M_0\rangle\colon M(s) = M_0(s) + \Lambda^{\mathrm{T}} \cdot (M^{-s} - M_0^{-s}).$$

Proof.

$$\begin{aligned} M(s) &= M_0(s) + r(s) \cdot \boldsymbol{\sigma} \text{ (where } M_0[\sigma\rangle M) && \{\text{state equation of } (N, M_0)\} \\ &= M_0(s) + \Lambda^{\mathrm{T}} \cdot C^{-s} \cdot \boldsymbol{\sigma} && \{r(s) = \Lambda^{\mathrm{T}} \cdot C^{-s}\} \\ &= M_0(s) + \Lambda^{\mathrm{T}} \cdot (M^{-s} - M_0^{-s}) && \{\text{state equation of } (N^{-s}, M_0^{-s})\}. \end{aligned}$$ ∎

This simple result has the following interesting consequence: if the initial number of tokens in a nonnegative linearly dependent place is large enough, then the place does not constrain the language of the net (places with this property are called *implicit* in [5]). This result can be found in [5]. However, for the sake of completeness, we include here a proof of our own.

PROPOSITION 4.6 [5]. *Let $N=(S, T, F)$ be a net, $s \in S$ a nonnegative linearly dependent place, and M_0 a marking of N^{-s}. Then there exists a marking M_1 of N such that:*

(1) $M_1^{-s} = M_0$

(2) $L(N, M_1) = L(N^{-s}, M_0)$.

Before proving this proposition, let us illustrate it by means of the example of Fig. 6. Consider the system obtained by removing place s_6 in Fig. 6. If the place s_6 is now added without tokens, the language of the new system is only a proper subset of the former language: the sequence t_3 cannot occur anymore. However, if s_6 is added with one token then the languages of the two systems are equal, as the reader can easily check.

Proof. Since s is nonnegative linearly dependent, there exists $\Lambda \geqslant 0$ such that $r(s) = \Lambda^{\mathrm{T}} \cdot C^{-s}$. Choose a marking M_1 of N given by

- $M_1^{-s} = M_0$ (hence M_1 satisfies (1))
- $M_1(s) = \Lambda^{\mathrm{T}} \cdot M_0 + 1$.

We show that $L(N, M_1) = L(N^{-s}, M_1^{-s})$. Since $M_1^{-s} = M_0$, we get $L(N^{-s}, M_1^{-s}) = L(N^{-s}, M_0)$.

(i) $L(N, M_1) \subseteq L(N^{-s}, M_1^{-s})$.

Follows from the fundamental property of linearly dependent places (Proposition 4.5).

(ii) $L(N^{-s}, M_1^{-s}) \subseteq L(N, M_1)$.

By induction on the length k of the occurrence sequences of $L(N^{-s}, M_1^{-s})$.

Base. $k=0$. Obvious.

Step. Assume that every occurrence sequence of $L(N^{-s}, M_1^{-s})$ of length k belongs to $L(N, M_1)$. Let σt be an arbitrary sequence of $L(N^{-s}, M_1^{-s})$ of length $k+1$. Since σ has length k, we have $\sigma \in L(N, M_1)$ by the induction hypothesis. Let M_2 be the marking reached by letting σ occur from M_1, i.e., $M_1[\sigma\rangle M_2$. Then:

$$M_2(s) = M_1(s) + \Lambda^{\mathrm{T}} \cdot (M_2^{-s} - M_1^{-s}) \qquad \{\text{Proposition 4.5}\}$$
$$= \Lambda^{\mathrm{T}} \cdot M_2^{-s} + 1 \qquad \{M_1(s) = \Lambda^{\mathrm{T}} \cdot M_1^{-s} + 1\}$$
$$> 0 \qquad \{\Lambda \geqslant 0\}.$$

By the fundamental property, $M_1^{-s}[\sigma\rangle\, M_2^{-s}$. Since $\sigma t \in L(N^{-s}, M_1^{-s})$, t is enabled at M_2^{-s}. Since $M_2(s) > 0$, t is enabled at M_2. Therefore $\sigma t \in L(N, M_1)$. ∎

Using Propositions 4.5 and 4.6, we can study the consequences of the removal of a nonnegative linearly dependent place on the liveness and boundedness of a net. The following proposition will help us to show that these consequences are particularly interesting for Free Choice systems.

PROPOSITION 4.7 [8]. *Liveness monotonicity.*
If $(N, M_0) \in \mathrm{LBFC}$ *and* $M_0' \geqslant M_0$ *then* $(N, M_0') \in \mathrm{LBFC}$.

THEOREM 4.8. *Let* $N = (S, T, F)$ *be a net and* $s \in S$ *a nonnegative linearly dependent place such that* N^{-s} *is connected and contains some place and some transition. We have:*

(a) *N is structurally bounded iff N^{-s} is structurally bounded*

(b) *If N^{-s} is structurally live, then N is structurally live*

(c) *If N is Free Choice, then N^{-s} is structurally live iff N is structurally live.*

Proof. (a⇐): Let M_0 be an arbitrary marking of N. We show that (N, M_0) is bounded.

Let $M_0[\sigma\rangle\, M$. By the fundamental property, $M_0^{-s}[\sigma\rangle\, M^{-s}$. Since N^{-s} is structurally bounded, (N^{-s}, M_0^{-s}) is k-bounded for some k. We then have

$$\forall s' \in S \backslash \{s\}: M(s') = M^{-s}(s') \leqslant k.$$

That is, all places of N in $S \backslash \{s\}$ are k-bounded. It remains to show that s is also bounded:

$$M(s) = M_0(s) + \Lambda^{\mathrm{T}} \cdot (M^{-s} - M_0^{-s}) \qquad \{\text{Proposition 4.5}\}$$
$$\leqslant M_0(s) + \Lambda^{\mathrm{T}} \cdot (K - M_0^{-s}) \qquad \{(N^{-s}, M_0^{-s})$$
$$\text{(where } K = (k, k, ..., k)) \qquad \text{is } k \text{ bounded and } \Lambda \geqslant 0\}$$
$$\leqslant M_0(s) + \Lambda^{\mathrm{T}} \cdot K \qquad \{\Lambda \geqslant 0\}.$$

So $M(s)$ is bounded by $M_0(s) + \Lambda^{\mathrm{T}} \cdot K$.

(a⇒): Let M_0 be an arbitrary marking of N^{-s}. By Proposition 4.6, there exists a marking M_1 of N such that $M_1^{-s} = M_0$ and $L(N, M_1) = L(N^{-s}, M_0)$. We have:

- (N, M_1) is bounded, because N is structurally bounded
- $M \in [M_1\rangle$ iff $M^{-s} \in [M_0\rangle$, because of the language equivalence and the fundamental property.

Hence, (N^{-s}, M_0) is also bounded. Since M_0 was chosen arbitrarily, N^{-s} is structurally bounded.

(b) By the definition of structural liveness, there exists a marking M_0 of N^{-s} such that (N^{-s}, M_0) is live. By Proposition 4.5, there exists a marking M_1 of N such that $L(N^{-s}, M_0) = L(N, M_1)$. Hence, (N, M_1) is live and N is structurally live.

(c⇒) Particular case of (b).

(c⇐) Since N is structurally live, there exists a marking M of N such that (N, M) is live. We show that (N^{-s}, M^{-s}) is live.

By Proposition 4.6 applied to (N^{-s}, M^{-s}), there exists a marking M_1 of N such that $M_1^{-s} = M^{-s}$ and $L(N, M_1) = L(N^{-s}, M^{-s})$. Consider two cases:

Case 1. $M_1 < M$. Then we have:

$$\begin{aligned} L(N^{-s}, M^{-s}) &= L(N, M_1) && \{\text{Proposition 4.6}\} \\ &\subseteq L(N, M) && \{M_1 < M\} \\ &\subseteq L(N^{-s}, M^{-s}) && \{\text{fundamental property of linearly dependent places}\}. \end{aligned}$$

So $L(N^{-s}, M^{-s}) = L(N, M)$ and, since (N, M) is live, (N^{-s}, M^{-s}) is live.

Case 2. $M_1 \geqslant M$. Since N is Free Choice, Proposition 4.7 can be applied to conclude that (N, M_1) is also live. Since $L(N, M_1) = L(N^{-s}, M^{-s})$, the system (N^{-s}, M^{-s}) is live. ▮

Theorem 4.8 leads to the following reduction rule:

Rule 3. Let $\Sigma = (N, M_0)$ be a system. $(\Sigma, \tilde{\Sigma}) \in R_3$, where $\tilde{\Sigma} = (\tilde{N}, \tilde{M}_0)$, iff:

Conditions on Σ.

1. N is Free Choice
2. Every nonempty siphon of N is marked at M_0

3. N contains a nonnegative linearly dependent place s
4. N^{-s} is connected and contains some place and some transition.

Changes in Σ to produce $\tilde{\Sigma}$.

1. $(\tilde{N}, \tilde{M}_0) = (N^{-s}, M_0^{-s})$.

The rule is well defined because of Condition 4 (this is the reason for the inclusion of this condition).

THEOREM 4.9. *R_3 is strongly sound with respect to the class* LBFC.

Proof. Let $((N, M_0), (N^{-s}, M_0^{-s})) \in R_3$. We show the following:

(a) $N \in$ WFFC iff $N^{-s} \in$ WFFC

$N \in$ WFFC $\Leftrightarrow$ N is Free Choice, structurally live and structurally bounded	{Theorem 2.5}
$\Leftrightarrow$ N^{-s} is Free Choice, structurally live and structurally bounded	{Theorem 4.8, parts (a) and (c)}
$\Leftrightarrow$ $N^{-s} \in$ WFFC	{Theorem 2.5}.

(b) Every nonempty siphon of N is marked at M_0 iff every nonempty siphon of N^{-s} is marked at M_0^{-s}.

($\Rightarrow$) By the definition of siphon, every siphon of N^{-s} is also a siphon of N.

($\Leftarrow$) By Condition 2 of application of the rule.

By (a), (b), and Theorem 2.7, we have that $(N, M_0) \in$ LBFC iff $(N^{-s}, M_0^{-s}) \in$ LBFC. So R_3 is strongly sound with respect to LBFC. ∎

Two limitations of R_3 should be pointed out:

1. A rule is local if, in order to decide if its conditions of application hold, only the neighbourhood of the intended point of application has to be examined, and the changes affect only this part of the system. Local rules are clearly preferable to non-local ones. R_1 and R_2 are examples of local rules. R_3, however, is non-local, because in order to find the linear combination showing that s can be removed it can be necessary to examine the whole net N.

2. R_3 is not sound with respect to the class of live and k-bounded Free Choice systems. Let (N, M_0) be the system of Fig. 7. It is easy to see that $((N, M_0), (N^{-s}, M_0^{-s})) \in R_3$, and that (N, M_0) is 2-bounded. However, (N^{-s}, M_0^{-s}) is 1-bounded.

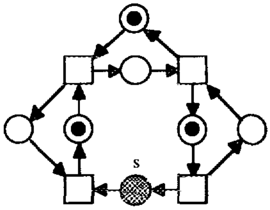

FIG. 7. R_3 does not preserve k-boundedness.

We now consider the removal of nonnegative linearly dependent transitions. *A transition t is nonnegative linearly dependent iff the net has at* least two transitions and the column corresponding to t in the incidence matrix (denoted by $c(t)$) is a linear combination of other columns with nonnegative coefficients.

We define the following rule, where, given $N=(S, T, F)$ with $|T|>1$, N^{-t} denotes the net $(S, T\backslash\{t\}, F \mid t)$.

Rule 4. Let $\Sigma=(N, M_0)$ be a system. $(\Sigma, \tilde{\Sigma}) \in R_4$, where $\tilde{\Sigma}=(\tilde{N}, \tilde{M}_0)$, iff:

Conditions on Σ.

1. N is a Free Choice net
2. N contains a nonnegative linearly dependent transition t
3. N^{-t} is connected and contains some place and some transition.

Changes on Σ to obtain $\tilde{\Sigma}$.

1. $(\tilde{N}, \tilde{M}_0)=(N^{-t}, M_0)$.

The rule is well defined because of Condition 3. In order to prove the strong soundness of this rule with respect to the class LBFC, we use the following relationship between the structural rules SR_3 and SR_4:

PROPOSITION 4.10. $(N, \tilde{N}) \in SR_4$ iff $(N^{\mathrm{rd}}, \tilde{N}^{\mathrm{rd}}) \in SR_3$.

Proof. It is easy to see that, if C and C^{rd} are the incidence matrices of N and N^{rd}, respectively, then $C^{\mathrm{rd}}=C^{\mathrm{T}}$ (the transpose of C). This implies that the vector $r(s)$ is a linearly dependent row of C iff $r(s)^{\mathrm{T}}$ is a linearly dependent column of C^{rd}. Hence, s is a (nonnegative) linearly dependent place of N iff it is a (nonnegative) linearly dependent transition of N^{rd}. ∎

THEOREM 4.11. *R_4 is strongly sound with respect to* LBFC.

Proof. Let $((N, M_0), (N^{-t}, M_0)) \in R_4$. We show the following:

(a) $N \in \text{WFFC}$ iff $N^{-t} \in \text{WFFC}$

$$((N, M_0), (N^{-t}, M_0^{-t})) \in R_4 \Rightarrow (N, N^{-t}) \in SR_4$$

{definition of structural rule}

$$\Rightarrow (N^{\text{rd}}, (N^{-t})^{\text{rd}}) \in SR_3$$

{Proposition 4.10}

$$\Rightarrow N^{\text{rd}} \in \text{WFFC} \Leftrightarrow (N^{-t})^{\text{rd}} \in \text{WFFC}$$

{strong soundness of R_3 with respect to LBFC}

$$\Rightarrow N \in \text{WFFC} \Leftrightarrow N^{-t} \in \text{WFFC}$$

{Theorem 2.6}.

(b) If $(N, M_0) \in \text{LBFC}$, then $(N^{-t}, M_0) \in \text{LBFC}$.

Since $N \in \text{WFFC}$, we have $N^{-t} \in \text{WFFC}$. By Remark 2.4, N^{-t} is strongly connected. Moreover, it is structurally bounded by Theorem 2.5. By Theorem 2.8, it suffices to show that (N^{-t}, M_0) is deadlock-free.

Since both N ad N^{-t} are strongly connected nets, there exists a transition t' of N, $t' \neq t$, such that ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$, where the dot notation refers to N. Since N is Free Choice, we have ${}^\bullet t = {}^\bullet t'$; it follows that t is enabled at a marking iff t' is also enabled at it.

Let now M be an arbitrary reachable marking of (N^{-t}, M_0). Clearly, M is a reachable marking of (N, M_0) as well. Since (N, M_0) is live, some transition of N is enabled at M. Moreover, since t is enabled at M iff t' is enabled at M, the marking M enables in particular some transition different from t. So M enables this transition in N^{-t} too. Since M was arbitrarily chosen, (N^{-t}, M_0) is deadlock-free.

(c) If $(N^{-t}, M_0) \in \text{LBFC}$, then $(N, M_0) \in \text{LBFC}$.

By (a) and Theorem 2.7, it suffices to prove that every nonempty siphon of N is marked at M_0.

Let R be a nonempty siphon of N. By the definition of a siphon, R is a nonempty siphon of N^{-t}. Since $(N^{-t}, M_0) \in \text{LBFC}$, R is marked at M_0 by Theorem 2.7.

(a), (b) and (c) imply that (N, M_0) satisfies conditions (i) and (ii) of Theorem 2.7 iff (N^{-t}, M_0^{-t}) satisfies them. Hence, $(N, M_0) \in \text{LBFC}$ iff $(N^{-t}, M_0^{-t}) \in \text{LBFC}$, which proves the strong soundness of R_4. ∎

5. The Kits $\{R_1, R_3, R_4\}$ and $\{R_2, R_3, R_4\}$

In the next two sections we study the kits $\{R_1, R_3, R_4\}$ and $\{R_2, R_3, R_4\}$ acting on the following class of atomic systems.

Definition 5.1. *Atomic systems.*
A net isomorphic to $(\{s\}, \{t\}, \{(s, t), (t, s)\})$ is called an *atomic net.* A system (N, M_0) is *atomic* iff N is atomic and $M_0 > 0$.

There exist therefore infinitely many atomic systems, which differ in the number of tokens put in the only place of the—up to isomorphism—unique atomic net. Since, according to our convention, net systems must have at least one place and one transition, the atomic systems are the live and bounded systems with a minimal number of nodes. It is also easy to see that none of our rules is applicable to an atomic system: atomic systems do not satisfy Condition 1 of R_1 and R_2, and do not contain linearly dependent places nor transitions—according to our definition, the net must have for that at least two places or two transitions, respectively.

Once the class of atomic systems has been fixed, the classes $\mathscr{R}(R_1, R_3, R_4)$ and $\mathscr{R}(R_2, R_3, R_4)$ are well defined. Our goal is to prove that

$$\mathscr{R}(R_1, R_3, R_4) = \text{LBFC} = \mathscr{R}(R_2, R_3, R_4).$$

Since all atomic systems are in LBFC and all the rules are strongly sound, we have $\mathscr{R}(R_1, R_3, R_4) \subseteq \text{LBFC}$ and $\mathscr{R}(R_2, R_3, R_4) \subseteq \text{LBFC}$ by Proposition 3.1. In the next section we will prove the converse of these two inclusions.

Before that, we present a reduction sequence of a system in $\mathscr{R}(R_1, R_3, R_4)$. We have chosen the system that is used as the main example in Hack's master thesis [13], one of the first works in which Free Choice systems were studied. It has been slightly simplified in order to reduce the number of reduction steps. The system is shown in Fig. 8(a).

1. Apply R_3 to remove s_{11}:

$$r(s_{11}) = r(s_7) + r(s_{10}).$$

2. Apply R_4 to remove t_5:

$$c(t_5) = c(t_1) + c(t_2) + c(t_6) + c(t_7) + c(t_8) + c(t_9) + c(t_{10}).$$

3. Apply R_1 to fuse s_{10} and s_2 into $\tilde{s}_0$ (Fig. 8(b)).
4. Apply R_4 to remove t_3 and t_{10}:

$$c(t_3) = c(t_{10}) = 0.$$

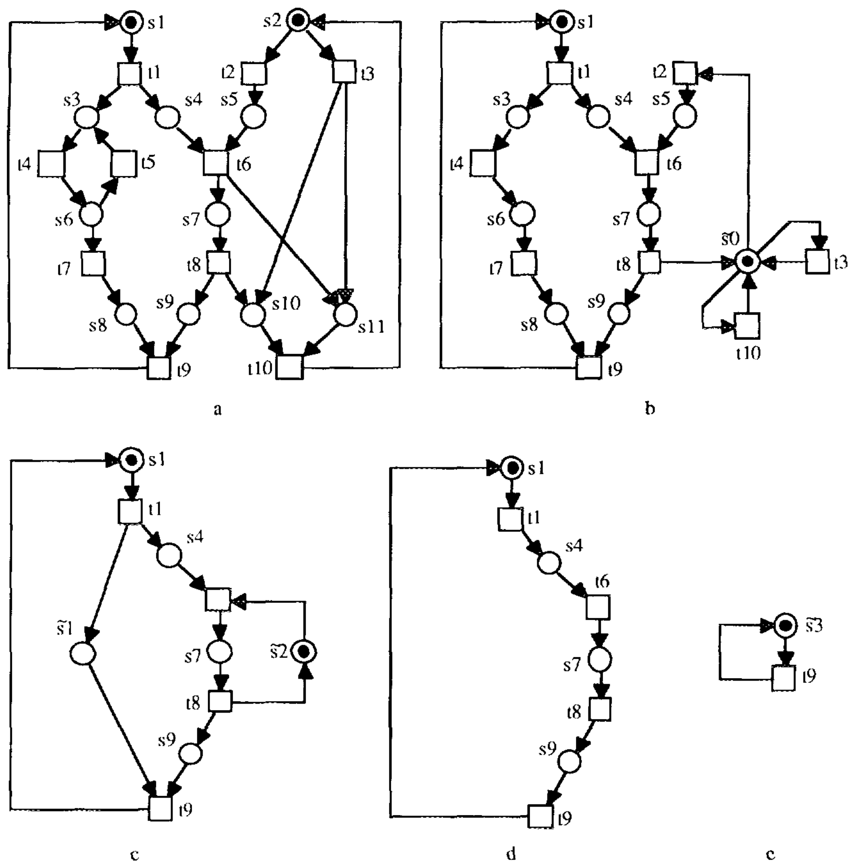

FIG. 8. Reduction of an LBFC system.

5. Apply R_1 twice to fuse s_3, s_6, and s_8 into $\tilde{s}_1$.
6. Apply R_1 to fuse $\tilde{s}_0$ and s_5 into $\tilde{s}_2$ (Fig. 8(c)).
7. Apply R_3 twice to remove $\tilde{s}_1$ and $\tilde{s}_2$ (Fig. 8(d)):

$$r(\tilde{s}_1) = r(s_4) + r(s_7) + r(s_9)$$

$$r(\tilde{s}_2) = r(s_9) + r(s_1) + r(s_4).$$

8. Apply R_1 three times to fuse s_1, s_4, s_7, and s_9 into $\tilde{s}_3$ (Fig. 8(e)).

Since we claim that $\mathscr{R}(R_1, R_3, R_4) = \mathscr{R}(R_2, R_3, R_4)$, there should be a reduction sequence of this system using $\{R_2, R_3, R_4\}$. This is in fact the case. The two first steps are as before. After them, we may go on as follows:

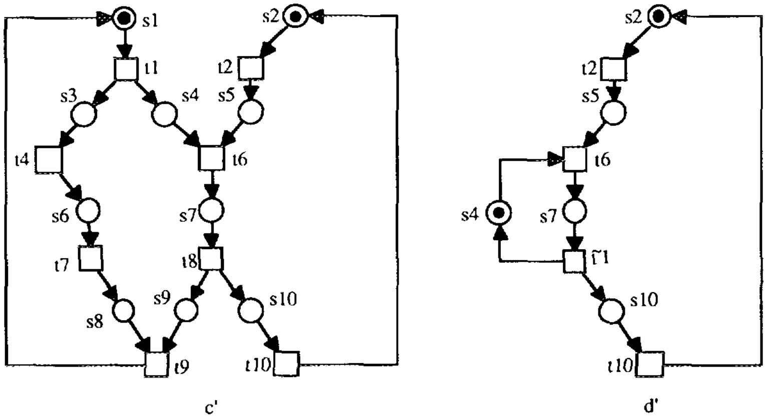

FIG. 9. An alternative reduction of the system of Fig. 8.

3. Apply R_4 to remove t_3 (Fig. 9(c')):

$$c(t_3) = c(t_1) + c(t_2) + c(t_4) + c(t_6) + c(t_7) + c(t_8) + c(t_9).$$

4. Apply R_2 twice to fuse t_1, t_4, and t_7 into $\tilde{t}_0$.
5. Apply R_3 to remove s_8:

$$r(s_8) = r(s_4) + r(s_7) + r(s_9).$$

6. Apply R_2 twice to fuse t_8, t_9, and $\tilde{t}_0$ into $\tilde{t}_1$ (Fig. 9(d')).
7. Apply R_3 to remove s_4:

$$r(s_4) = r(s_{10}) + r(s_2) + r(s_5).$$

8. Apply R_2 three times to fuse t_2, t_6, $\tilde{t}_1$, and t_{10} into one transition.

6. Completeness of the Kits

We give in this section a direct proof of the completeness of $\{R_2, R_3, R_4\}$ with respect to the class LBFC. Let us first show that, once this is achieved, the completeness of $\{R_1, R_3, R_4\}$ follows.

PROPOSITION 6.1. *If* $\{R_2, R_3, R_4\}$ *is complete for* LBFC *then so is* $\{R_1, R_3, R_4\}$.

Proof. Let $\Sigma = (N, M_0) \in$ LBFC. We have to show that there exists $\Sigma' \in \mathcal{A}$ such that Σ can be reduced to Σ' using $\{R_1, R_3, R_4\}$. We first show that N can be reduced to an atomic net using $\{SR_1, SR_3, SR_4\}$.

$N \in \text{WFFC}$

$\Leftrightarrow N^{rd} \in \text{WFFC}$	{Theorem 2.6}
$\Leftrightarrow (N^{rd}, N_0) \in (SR_2 \cup SR_3 \cup SR_4)^*$ for some atomic net N_0	{completeness of $\{R_2, R_3, R_4\}$, and definition of structural rule}
$\Leftrightarrow (N, N_0^{rd}) \in (SR_1 \cup SR_4 \cup SR_3)^*$	{Propositions 4.2 and 4.10}.

Since N_0 is atomic iff N_0^{rd} is atomic, the result follows.

Let $(SR_{i_1}, ..., SR_{i_n})$ be the sequence of structural rules reducing N to an atomic net. We show that the sequence $(R_{i_1}, ..., R_{i_n})$ reduces Σ to an atomic system. We only have to prove that this sequence is applicable to N.

Assume this is not the case. Then, there exists an index i_j, $1 \leqslant j \leqslant n$, such that $(R_{i_1}, ..., R_{i_{j-1}})$ is applicable to N, but not $(R_{i_1}, ..., R_{i_j})$. Let (N_j, M_j) be the system obtained after the application of $(R_{i_1}, ..., R_{i_{j-1}})$. We have that SR_{i_j} is applicable to N_j, but R_{i_j} is not applicable to (N_j, M_j). A simple inspection of the conditions of application shows that $i_j = 3$, and M_j does not mark all the nonempty siphons of N_j. But then, by Theorem 2.7, (N_j, M_j) is not live, which contradicts the strong soundness of $R_1, ..., R_4$. ∎

Our task is to show that for every $\Sigma \in \text{LBFC}$ there exists $\Sigma' \in \mathscr{A}$ such that Σ can be reduced to Σ' using $\{R_2, R_3, R_4\}$. A first important observation is that, in order to prove this, it suffices to show that the following statement (A) holds:

(A) If Σ is a non-atomic LBFC system, then some rule in $\{R_2, R_3, R_4\}$ can be applied to it.

LEMMA 6.2. *Let* $\Sigma \in \text{LBFC}$. *If* (A) *holds, then* Σ *can be completely reduced using* $\{R_2, R_3, R_4\}$.

Proof. Assume that (A) holds. Consider the function $f\colon \text{LBFC} \to \mathbb{N}^2$ given by

$$f((S, T, F, M_0)) = (|S|, |T|)$$

and the partial order $\preccurlyeq$ on $\mathbb{N}^2$ given by

$$(x_1, x_2) \preccurlyeq (x_3, x_4) \quad \text{iff} \quad x_1 \leqslant x_2 \text{ and } x_3 \leqslant x_4.$$

Due to our definition of a net system (which requires the existence of at least one place and one transition), the range of f is bounded from below by $(1, 1)$. Moreover, this minimum is reached by all and only atomic systems. Finally, f is monotonically decreasing with respect to $(R_2 \cup R_3 \cup R_4)$, because for all the rules the target system has always less places and/or transitions than the source system.

Therefore, any maximal sequence of reductions starting with Σ terminates in an atomic system. ▮

Furthermore, we can easily prove (A) for LBFC systems with exactly one transition. Assume Σ is non-atomic and has exactly one transition. Then, due to our definition of a net system, it has more than one place. The rows corresponding to these places in the incidence matrix are identical (null vectors). Any of them is then a nonnegative linear combination of the others, and hence R_3 is applicable.

It remains to prove (A) for LBFC systems with more than one transition (and hence non-atomic). Let (A′) be the restriction of (A) to these systems.

The concept of shower subnet is central to the proof of (A′). We define it next.

Given a subnet $N' = (S', T', F')$ of N, $t \in T'$ is a *way-in* transition to N iff there exists $s \in {}^\bullet t \backslash S'$.

Definition 6.3. Let $N = (S, T, F)$ be a net and $\hat{N} = (\hat{S}, \hat{T}, \hat{F}) \subseteq N$ a T-graph with $|\hat{T}| > 1$. $\hat{N}$ is a *shower* subnet of N iff:

(i) ${}^\bullet\hat{S} \cup \hat{S}^\bullet = \hat{T}$ (where the dot notation refers to the net N),

(ii) $\hat{N}$ has exactly one way-in transition, and

(iii) for every $x \in \hat{S} \cup \hat{T}$, there exists a path in $\hat{N}$ from the way-in transition of $\hat{N}$ to x.

Figure 10 shows a shower subnet with a certain marking, and explains the reason for the name. In showers, water gets in through one single pipe and gets out concurrently through many small holes. The behaviour of shower subnets is similar: tokens "get into" the subnet through one single way-in transition (t_{in} in the figure), and "leave" it concurrently through possibly many way-out transitions (t^1_{out}, t^2_{out}, t^3_{out} in the figure).

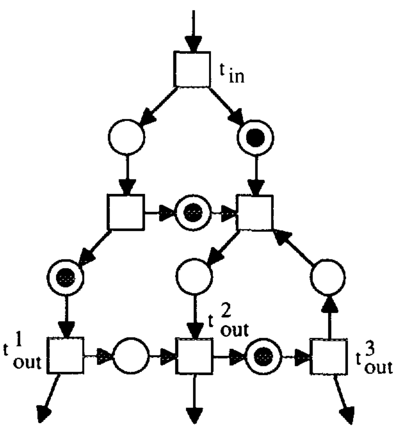

Fig. 10. A shower subnet.

The proof of (A′) has a hierarchical structure. (A′) is implied by the conjunction of the two following statements, which will also be derived from the conjunction of others.

Let (N, M_0) be an LBFC system, where $N=(S, T, F)$ and $|T|>1$:

(A.1) If (N, M_0) contains a shower subnet, then R_2 or R_3 is applicable.

(A.2) If (N, M_0) contains no shower subnets, then R_4 is applicable.

6.1. LBFC *Systems with Shower Subnets*

PROPOSITION 6.4. *Statement* (A.1).

Let $(N, M_0) \in$ LBFC, *where* $N=(S, T, F)$ *and* $|T|>1$, *and* $\hat{N}$ *a shower subnet of* N. *Then* R_2 *or* R_3 *is applicable to* (N, M_0).

Proof. The proof is by induction on $|\hat{T}|$, the set of transitions of $\hat{N}$.

Base. $|\hat{T}|=2$. Then $\hat{T}=\{t_{\text{in}}, t\}$, where t_{in} is the unique way-in transition of $\hat{N}$. By Condition (iii) of the definition of a shower subnet, there exists a path (t_{in}, s, t) in $\hat{N}$.

If ${}^{\bullet}t=\{s\}$ (the dot notation always refers to N), then it is easy to see that the place s satisfies all the conditions of application of the rule R_2: Condition 1 holds because $\hat{N}$ is a T-graph and $t \neq t_{\text{in}}$; Condition 2 holds because N is strongly connected (Remark 2.4); Condition 3 holds because ${}^{\bullet}t=\{s\}$.

If ${}^{\bullet}t \neq \{s\}$, then there exists a place $s' \in {}^{\bullet}t$, $s' \neq s$. Since $\hat{N}$ is a T-graph, s' has exactly one input transition, which is either t_{in} or t. In the first case, we have $r(s)=r(s')$; in the second, $r(s')=0$. In both cases, s' is a nonnegative linearly dependent place of N, and it is easy to see that the rule R_3 is applicable.

Step. $|\hat{T}|>2$. By the definition of a shower subnet, there exist a transition t of $\hat{N}$, $t \neq t_{\text{in}}$, and a path (t_{in}, s, t). If ${}^{\bullet}t=\{s\}$, then the rule R_2 is applicable, as shown in the base case. If ${}^{\bullet}t \neq \{s\}$, then there exists a place $s' \in {}^{\bullet}t$, $s' \neq s$. Since $\hat{N}$ is a shower subnet, some path of $\hat{N}$ leads from t_{in} to s'. Consider two cases:

- Some path of $\hat{N}$ leading from t_{in} to s' does not contain s. Then, $r(s)$ is the sum of the rows of the incidence matrix corresponding to the places contained in this path. Moreover, N^{-s} is connected. Since M_0 marks all nonempty siphons of N because (N, M_0) belongs to LBFC (Theorem 2.7), R_3 is applicable to (N, M_0).

- Every path of $\hat{N}$ leading from t_{in} to s' contains s (and therefore t). Let $N'=(S', T', F')$ be the subnet of $\hat{N}$ generated by all the nodes contained in the elementary paths of $\hat{N}$ leading from t to s'. In particular, t_{in} is not a transition of N', because every path from t to s' containing t_{in} must contain t twice.

We prove that N' is a shower subnet of N.

— ${}^\bullet S' \cup S'^\bullet = T'$. Follows easily from ${}^\bullet \hat{S} \cup \hat{S}^\bullet = \hat{T}$ and the definition of N'.

— N' has exactly one way-in transition. t is a way-in transition of N'. We show that it is the only one. Let $t' \in T'$ be a way-in transition of N'. We prove $t' = t$.

t' is a transition of $\hat{N}$, and moreover $t' \neq t_{\text{in}}$, because t_{in} is not a transition of N'. Since $\hat{N}$ is a shower subnet, all the input places of t' are contained in $\hat{N}$. Since t' is a way-in transition of N', some input place r of t' is furthermore not contained in N'.

There exists a path from t_{in} to r in $\hat{N}$ because $\hat{N}$ is a shower subnet. This path can be extended to a path from t_{in} to t' because r is an input place of t'. It can moreover be extended to a path π from t_{in} to s' by the definition of N'. Since every path from t_{in} to s' contains t, π contains t. So every element of π, with the exception of t_{in} and s, belongs to N'. Since r is not a place of N', we have $r = s$. This implies $t' = t$.

— For every $x \in S' \cup T'$, there exists a path $(t, ..., x)$ in N'. Obvious from the definition of N'.

Since N' does not contain the transition t_{in}, we have $|T'| < |\hat{T}|$. By the induction hypothesis, R_2 or R_3 is applicable to (N, M_0). ∎

Using this proposition, the following result can be easily proved.

PROPOSITION 6.5. *Let (N, M_0) be a live and bounded T-system. (N, M_0) can be completely reduced using $\{R_2, R_3\}$.*

Proof. Let $N = (S, T, F)$. By an analogous argument to the one used to prove Lemma 6.2, we conclude that it suffices to prove that some rule is applicable to (N, M_0). Moreover, we only need consider the case $|T| > 1$ (if $|T| = 1$, then R_3 can be used to remove all but one place, yielding an atomic system).

Choose $t \in T$. By Remark 2.4, N is strongly connected. Consider the net obtained by removing from N all the input places of t, together with their input and output arcs. It is easy to see that this net is a shower subnet of N, with t as way-in transition. By Proposition 6.4, R_2 or R_3 is applicable. ∎

6.2. LBFC *Systems without Shower Subnets*

We prove statement (A.2): if an LBFC system having more than one transition contains no shower subnets, then R_4 is applicable. This statement is implied by the conjunction of the following two statements. Let (N, M_0) be a LBFC system, where $N = (S, T, F)$ and $|T| > 1$:

(A.2.1) If (N, M_0) has no shower subnets, then there exists $t \in T$ such that N^{-t} is strongly connected.

(A.2.2) If there exists $t \in T$ such that N^{-t} is strongly connected, then $((N, M_0), (N^{-t}, M_0)) \in R_4$ (i.e., R_4 is applicable to (N, M_0)).

We deal with (A.2.2) first. The proof is based on the next proposition.

PROPOSITION 6.6. *Let $N = (S, T, F)$ be a net with $|T| > 1$ and let $t \in T$. If N and N^{-t} are structurally live and structurally bounded nets, then t is a nonnegative linearly dependent transition.*

Proof. Let C and C^{-t} be the incidence matrices of N and N^{-t} respectively. By Theorem 2.2, N and N^{-t} are consistent. Hence, there exist vectors $X_1 > 0$ and $X_2 > 0$ (where the dimension of X_1 is 1 more than the dimension of X_2) such that

$$C \cdot X_1 = 0 \tag{1}$$

$$C^{-t} \cdot X_2 = 0 \tag{2}$$

Assume w.l.o.g. that $c(t)$ is the last column of C. Then, X_1 can be written in the form $[X_1' \mid X_1(t)]$, where X_1' and X_2 have the same dimension, and (1) can be written as

$$C^{-t} \cdot X_1' + X_1(t)\, c(t) = 0 \tag{3}$$

Take

$$X = \frac{1}{X_1(t)} (kX_2 - X_1'),$$

where k is positive and large enough to make $X > 0$. We then have:

$$\begin{aligned} C^{-t} \cdot X &= -\frac{1}{X_1(t)} C^{-t} \cdot X_1' && \{\text{definition of } X\text{, Eq. (2)}\} \\ &= c(t) && \{\text{Eq. (3)}\}. \end{aligned}$$

So, since $X > 0$, the transition t is nonnegative linearly dependent. ∎

THEOREM 6.7. *Statement* (A.2.2).
Let $(N, M_0) \in$ LBFC, where $N = (S, T, F)$ and $|T| > 1$. Let $t \in T$. If N^{-t} is strongly connected, then $((N, M_0), (N^{-t}, M_0)) \in R_4$.

Proof. By Theorem 2.7, every nonempty siphon of N is marked at M_0. Therefore, it suffices to show that t is a nonnegative linearly dependent transition. By Proposition 6.6, it suffices to prove that N and N^{-t} are structurally live and structurally bounded nets. Since $(N, M_0) \in$ LBFC, we have $N \in$ WFFC. By Theorem 2.5, N is structurally live and structurally bounded.

We now prove that $(N^{-t}, M_0) \in$ LBFC. We collect some preliminary facts:

(a) $L(N^{-t}, M_0) \subseteq L(N, M_0)$. Follows from the definition of the occurrence rule.

(b) (N^{-t}, M_0) is bounded. Follows easily from (a) and the boundedness of (N, M_0).

(c) There exists $t' \in T$, $t' \neq t$, such that t is enabled iff t' is enabled. By the strong connectedness of N^{-t}, there exists $t' \in T$, $t' \neq t$ such that ${}^{\bullet}t' \cap {}^{\bullet}t \neq \emptyset$. Since N is Free Choice, ${}^{\bullet}t' = {}^{\bullet}t$. Hence, t' is enabled iff t is enabled.

(d) (N^{-t}, M_0) is deadlock-free. Assume there exists an occurrence sequence σ in (N^{-t}, M_0) such that $M_0[\sigma\rangle M$ and no transition of $T \setminus \{t\}$ is enabled at M. By (a), this sequence can also occur in (N, M_0), leading to the same marking. By (c), t is not enabled at M as well. Hence, no transition in T is enabled at M, and (N, M_0) is not deadlock-free. This contradicts the liveness of (N, M_0).

(N^{-t}, M_0) is bounded by (b), strongly connected by hypothesis, and deadlock-free by (c). By Theorem 2.8, (N^{-t}, M_0) is live. So $(N^{-t}, M_0) \in$ LBFC, which implies $N \in$ WFFC. By Theorem 2.5, N is structurally live and structurally bounded. ∎

We now prove (A.2.1): If (N, M_0) has no shower subnets, then there exists a transition t such that N^{-t} is strongly connected.

This part is based on the notion of private subnet, which is introduced now.

A set $\mathcal{T}$ of T-components of a net N is a *cover* iff every node of N is contained in some element of $\mathcal{T}$. $\mathcal{T}$ is *minimal* iff no proper subset of $\mathcal{T}$ is itself a cover. Every T-component of a minimal cover $\mathcal{T}$ has at least one *own node*: a node that does not belong to any other T-component of the cover. To prove it, just notice that a T-component without own nodes can be removed from $\mathcal{T}$, and the remaining T-components are still a cover, against the minimality of $\mathcal{T}$. Private subnets are certain subnets of a T-component containing only own nodes.

DEFINITION 6.8. Let $\mathcal{T}$ a minimal cover of a net N, and $N_1 = (S_1, T_1, F_1)$ an element of $\mathcal{T}$. $N' = (S', T', F') \subseteq N_1$ is a *private subnet* of N_1 iff the following conditions hold:

(i) N' is nonempty and connected

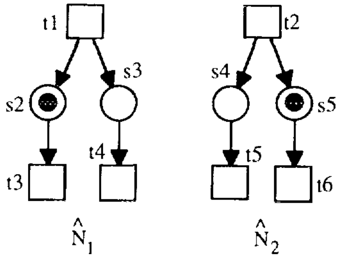

FIG. 11. Private subnets of the net of Fig. 1.

(ii) $S' \cap (S \backslash S_1) = T' \cap (T \backslash T_1) = \emptyset$

(iii) There exists no net N'' satisfying $N' \subset N'' \subseteq N_1$, (i), and (ii).

The T-components N_1, N_2 of the minimal cover shown in Fig. 3 have one single private subnet each, namely the subnets $\hat{N}_1$ and $\hat{N}_2$ shown in Fig. 11.

Given $\hat{N} = (\hat{S}, \hat{T}, \hat{F}) \subseteq N = (S, T, F)$, we denote $N \backslash \hat{N} = (S \backslash \hat{S},\ T \backslash \hat{T},\ F | (\hat{S} \cup \hat{T}))$.

Let (N, M_0) be an LBFC system with at least two transitions and containing no shower subnet. The conjunction of the following statements (A.2.1.1) and (A.2.1.2) implies (A.2.1):

(A.2.1.1) There exists a T-component of N containing a private subnet $\hat{N}$ such that $N \backslash \hat{N}$ is strongly connected.

(A.2.1.2) $\hat{N}$ is composed by one isolated transition.

PROPOSITION 6.9. *Statement* (A.2.1.1).

Let (N, M_0) *be an* LBFC *system with at least two transitions and containing no shower subnet. There exists a T-component* N_1 *of* N *such that for every private subnet* $\hat{N}$ *of* N_1, $N \backslash \hat{N}$ *is strongly connected.*

Proof. By Theorem 2.3, there is a cover $\mathscr{T}$ of N. If $|\mathscr{T}| = 1$, then N is a T-graph and, as shown in the proof of Proposition 6.5, N contains a shower subnet, against our hypothesis. Hence, $|\mathscr{T}| \geqslant 2$. We construct the (non-directed) graph $G = (V, A)$ as follows:

$$V = \mathscr{C}$$

$$(N_i, N_j) \in A \qquad \text{iff} \quad N_i \cap N_j \neq \emptyset.$$

Because T-components are strongly connected, it is immediate to see that G is connected iff the net N is strongly connected.

In our particular case N is strongly connected by Remark 2.4. So G is connected. There exists a node of G such that, when we remove it, the remaining graph G' is still connected, and non-empty. This graph G' corresponds to the net N' covered by $\mathscr{T}\backslash\{N_1\}$ for some T-component N_1 of $\mathscr{T}$. Since G' is connected, N' is strongly connected.

Let $\hat{N}$ be an arbitrary private subnet of N_1. We show that $N\backslash\hat{N}$ is strongly connected.

Let x, y be two arbitrary nodes of $N\backslash\hat{N}$. Since N is strongly connected, there is a path $\pi = u_1 \cdots u_k$ of N such that $x = u_1$ and $y = u_k$. We find a path of $N\backslash\hat{N}$ also leading from x to y.

Let u_{i+1} and u_{j-1} be the first and last elements of π that belong to $\hat{N}$ (they may be the same node). By the maximality property of private subnets (Definition 6.8(iii)), $u_i, u_j \in N'$.

Since N' is strongly connected, there is a path $u_i v_1 \cdots v_l u_j$ of N' leading from u_i to u_j. Since N' is a subnet of $N\backslash\hat{N}$, this path is also a path of $N\backslash\hat{N}$. Hence, the path

$$x \cdots u_i v_1 \cdots v_l u_j \cdots y$$

is a path of $N\backslash\hat{N}$. ∎

PROPOSITION 6.10. *Statement* (A.2.1.2).

Let $\Sigma = (N, M_0)$ be an LBFC *system with at least two transitions and containing no shower subnet. Let N_1 be a T-component of N and $\hat{N} = (\hat{S}, \hat{T}, \hat{F})$ a private subnet of N_1 such that $N\backslash\hat{N}$ is strongly connected. Then $|\hat{T}| = 1$ and $\hat{S} = \varnothing$.*

Proof. Assume that $\hat{N}$ has more than one way-in transition. Using that $\hat{N}$ is a connected T-graph, it is not difficult to see that there exist two way-in transitions t_1, t_2 with the following property. There exist two elementary paths $\pi_1 = (t_1, ..., t)$ and $\pi_2 = (t_2, ..., t)$ in $\hat{N}$ such that the only node contained in both paths is t. Moreover, due to the strong connectedness of $\tilde{N}$, there exists an elementary path $\pi_3 = (s_1, ..., s_2)$ in $\tilde{N}$ of minimal length with $s_1 \in {}^\bullet t_1$, $s_2 \in {}^\bullet t_2$. This setting is graphically described in Fig. 12.

Let S', T' be the set of places and transitions contained in these paths, respectively. Consider the mapping $J: S' \to \mathbb{Z}$ described in Fig. 13.[2]

Let, for a marking M:

$$J(M) = \sum_{s \in S'} J(s)\, M(s).$$

[2] Roughly speaking, we move clockwise along the triangle, and assign consecutive integers to the places. When moving in the opposite direction to the arcs of F, these integers are given a minus sign.

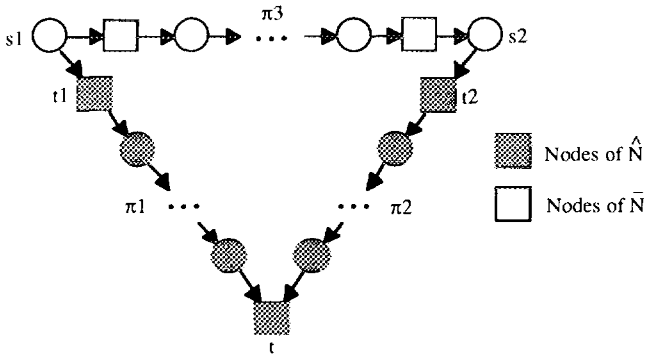

FIG. 12. The setting of the proof of Proposition 6.10.

We now show that for every $M \in [M_0\rangle$ there is a marking $M' \in [M\rangle$ such that $J(M) < J(M')$. Note that, if we are able to prove this, we are done, because this fact contradicts the boundedness of N.

Consider two cases:

(i) There is a transition $t' \in s'^{\bullet}$ enabled at M, where $s' \in S'$. Since N is Free Choice, all transitions in $s'^{\bullet}$ are enabled. Select a transition $t' \in s'^{\bullet}$ as follows:

- If $s' = s_1$, $s' \neq s_2$, then let t' be the successor of s' in π_3.
- If $s' = s_2$, then let t' be t_2.
- If $s_1 \neq s' \neq s_2$, then let t' be the successor of s' in its respective path.

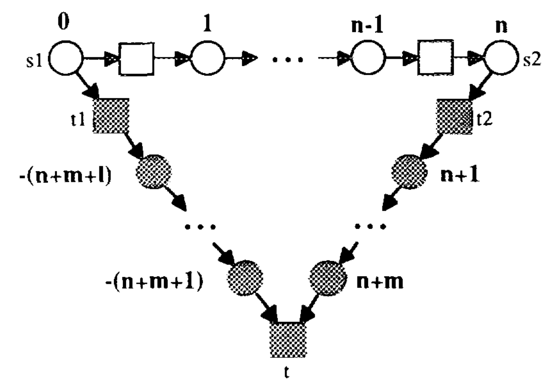

FIG. 13. Mapping considered in the proof of Proposition 6.10.

We show that $M[t'\rangle M'$ implies $J(M) < J(M')$. Due to our choice of J, and since t_1 is never selected, it suffices to show that:

(a) No place in π_1 has more than one input transition in $T' \backslash \{t_1\}$.

(b) No place of π_3 or π_2 has more than one output transition in $T' \backslash \{t_1\}$.

(a) and the part of (b) concerning π_2 hold because $\hat{N}$ is a T-graph. It remains to show (b) for π_3.

Let s' be a place of π_3 and let t' be its selected transition. Assume there exists $\tilde{t} \in s'^{\bullet} \cap (T' \backslash \{t_1\})$, $\tilde{t} \neq t'$. We show that this leads to a contradiction. There are three possible cases:

Case 1. $\tilde{t} = t_2$. We have $s' \neq s_2$, because otherwise $\tilde{t} = t'$. Then t_2 has more than one place of π_3 in its preset, contradicting the minimality of π_3.

Case 2. $\tilde{t}$ is a transition of π_1 or π_2, different from t_2. Then $\tilde{t}$ is the only output transition of its predecessor in the path because $\hat{N}$ is a T-graph; however, s' has more than one output transition. This contradicts the Free Choice property.

Case 3. $\tilde{t}$ is a transition of π_3. If $\pi_3 = (s_1, ..., s', t', ..., \tilde{t}, ..., s_2)$, then $(s_1, ..., s', \tilde{t}, ..., s_2)$ is a shorter path, contradicting the minimality of π_3.

If $\pi_3 = (s_1, ..., \tilde{t}, ..., s', t', ..., s_2)$ then, since N is Free Choice, there is an arc from the predecessor $\tilde{s}$ of $\tilde{t}$ to t', and the path $(s_1, ..., \tilde{s}, t', ..., s_2)$ contradicts the minimality of π_3.

Since in all cases we reach a contradiction, we have $\tilde{t} = t'$.

(ii) No transition of $S'^{\bullet}$ is enabled at M. Due to the liveness of (N, M_0), there is an occurrence sequence σ of minimal length with $M[\sigma\rangle M''$ such that a transition $t' \in S'^{\bullet}$ is enabled at M''.

Let now $M[\sigma\rangle M''[t'\rangle M'$. Since π_1 and π_2 are paths of a T-graph, no transition occurring in σ changes the marking of the places of π_1 and π_2. Since no transition in $S'^{\bullet}$ occurs in σ, we have

- $\forall s \in \pi_1, \pi_2 : M''(s) = M(s)$
- $\forall s \in \pi_3 : M''(s) \geqslant M(s)$.

Due to our choice of J, we have $J(M) \geqslant J(M'')$ and $J(M'') < J(M')$. Hence $J(M) < J(M')$, which completes the proof.

So $\hat{N}$ has one single transition t. There exists a place $s' \in {}^{\bullet}t$ of $N \backslash \hat{N}$. Since $N \backslash \hat{N}$ is strongly connected, s' has some output transition in $N \backslash \hat{N}$. Since N is Free Choice, ${}^{\bullet}t = \{s\}$. Therefore, t has no input place in $\hat{N}$. Since every place of $\hat{N}$ must have some output transition in $\hat{N}$, the net $\hat{N}$ contains no places. ∎

7. The Complexity of Deciding Membership in the Class LBFC

A sound and complete kit of reduction rules for LBFC provides an algorithm for testing membership: a system is in LBFC iff it can be completely reduced by the kit. We show in this section that this algorithm is polynomial on the size of a reasonable encoding of the system, which compares favorably with the *NP*-completeness of deciding if a Free Choice system is not live [15].

PROPOSITION 7.1. *Let* (N, M_0) *be a Free Choice system. The following problems can be solved in polynomial time in the size of (a reasonable encoding of)* (N, M_0):

(a) *For* $i = 1, ..., 4$: *applicability of* R_i *to* (N, M_0)

(b) *Membership of* (N, M_0) *in* LBFC.

Proof. Let $N = (S, T, F)$.

(a) The conditions of application of R_1 and R_2 can be easily checked in polynomial time. R_4 is applicable when N contains a nonnegative linearly dependent transition, i.e., when for a transition t the system $C^{-t} \cdot X = c(t)$ of linear inequalities has a nonnegative solution X. We have to solve $|T|$ systems in the worst case to check this condition. Solving one of these systems is a polynomial problem on the size of the net (see, for instance, [9]). R_3 is applicable iff:

(i) Every nonempty siphon of N is marked at M_0,

(ii) N contains a nonnegative linearly dependent place s, and

(iii) N^{-s} is connected and contains some place and some transition.

Condition (i) can be checked in polynomial time using the following (polynomial) greedy algorithm, a slight modification of an algorithm of [21]. The algorithm returns the maximal siphon of N unmarked at M_0. If the algorithm yields the empty set, then every nonempty siphon of N is marked at M_0.

begin
  $R := \{s \in S \mid M_0(s) = 0\}$;
  while $\exists s \in R: {}^\bullet s \not\subseteq R^\bullet$ **do**
    $R := R \setminus \{s\}$
  endwhile
end

Checking condition (ii) reduces to the problem of deciding if N contains a place s such that for some $\Lambda \geqslant 0$, $\Lambda^{\mathrm{T}} \cdot C^{-s} = r(s)$. Finally, condition (iii) can be checked in polynomial time using standard algorithms.

(b) A reduction step consists of finding an applicable rule and then performing the corresponding transformation. By (a), a reduction step can be carried out in polynomial time because, once it is known that a certain rule is applicable, the corresponding transformation can be performed in polynomial time as well, as the reader can check by simple inspection.

Since the application of any of the rules reduces at least by 1 the number of nodes of the net, $(N, M_0) \in \mathrm{LBFC}$ iff it has been reduced to an atomic system after at most $|S| - 1 + |T| - 1$ reduction steps. Hence, the membership problem is polynomial. ∎

Checking if the net contains a non-negative linearly dependent place or transition is the most expensive computation required to decide the applicability of a rule. Therefore, the actual degree of the polynomial that bounds the time complexity of the reduction algorithm depends on the algorithm used to solve equations in the nonnegative orthant. Note also that the simplex algorithm, although of exponential complexity, behaves in practice better than the polynomial linear programming algorithms.

We finish the section by describing informally some possible improvements in the algorithm.

1. It is easy to see that, for $i = 1, ..., 4$, and for all $((N, M_0), (\tilde{N}, \tilde{M}_0)) \in R_i$, if every nonempty siphon of N is marked at M_0, then every nonempty siphon of $\tilde{N}$ is marked at $\tilde{M}_0$ (independently of whether $(N, M_0) \in \mathrm{LBFC}$ or not). Hence, it is not necessary to check this condition *every* time we try to apply R_4, but only the *first* time.

2. Whenever N contains a shower subnet (and all nonempty siphons of N are marked), this shower subnet can be reduced to a transition by means of a sequence of applications of R_2 and R_4. It is possible to introduce a "macro" that performs this reduction in one single step—this "macro" can be seen as a dual version of the *macroplace* reduction technique of [20]. Since shower subnets can be easily identified using graph algorithms (see [20]), the introduction of the "macro" improves the performance of the reduction procedure. The details are left to the reader.

8. Synthesis Rules

Reduction rules can be used "backwards" as *synthesis rules*, in order to generate a complex system starting from an atomic one. If R is a reduction rule, then $S = R^{-1}$ is a synthesis rule.

A system is *synthesised* by a kit $\{S_1, ..., S_a\}$ of synthesis rules iff it is reduced by $\{S_1^{-1}, ..., S_a^{-1}\}$. We denote by $\mathscr{S}(S_1, ..., S_a)$ the class of systems synthesised by the kit, defined as $\mathscr{S}(S_1, ..., S_a) = \mathscr{R}(S_1^{-1}, ..., S_a^{-1})$. The concepts of (strongly) sound and complete kit are defined as for reduction rules.

The formulation of the inverses of the reduction rules is quite straightforward. There is, however, an interesting point. When dealing with synthesis rules it is possible to exploit the fact that the atomic systems are known to be in LBFC. While a reduction rule can only be useful if it is strongly sound with respect to a certain class of systems, a synthesis rule need only be sound: by applying sound rules, since the initial seed is in LBFC, we stay within LBFC.

The kit formed by the inverses of a strongly sound and complete kit of reduction rules is also strongly sound and complete. Since a sound and complete kit suffices, we can try to weaken some rules, which can have the advantage that the conditions of application are easier to check. This can in fact be done with R_3 and R_4. We consider here the case of R_4 only, that of R_3 is analogous.

Checking the conditions of application of R_4 requires to solve a system of linear inequalities in the nonnegative orthant. Although, due to the polynomiality of linear programming, this is a polynomial problem, it is still time consuming. The following proposition allows us to do better.

Given a net N, N^{+t} denotes a net containing a transition t such that $(N^{+t})^{-t} = N$.

PROPOSITION 8.1. *Let N be a structurally live and structurally bounded net, and t a linearly dependent transition of a net N^{+t}. Then t is also a non-negative linearly dependent transition.*

Proof. Let C be the incidence matrix of N. Since t is linearly dependent, there exists a vector Λ such that $C \cdot \Lambda = c(t)$. By Theorem 2.2, N is consistent. Therefore, there exists $X > 0$ such that $C \cdot X = 0$. Take k such that $\Lambda' = \Lambda + kX > 0$. We have

$$C \cdot \Lambda' = c(t) + kC \cdot X = c(t).$$

Hence, t is a non-negative linearly dependent transition. ∎

We can now define the following synthesis rule. Note that we no longer have conditions on Σ but on $\tilde{\Sigma}$:

Rule 5. Let $\Sigma = (N, M_0)$ be a system. $(\Sigma, \tilde{\Sigma}) \in S_4$, where $\tilde{\Sigma} = (\tilde{N}, \tilde{M}_0)$, iff:

Changes in Σ to produce $\tilde{\Sigma}$.

1. $(\tilde{N}, \tilde{M}_0) = (N^{+t}, M_0)$.

Conditions on $\tilde{\Sigma}$:

1. $\tilde{N}$ is Free Choice
2. t is a linearly dependent transition of N^{+t}.

In order to check if t is a linearly dependent transition, it suffices to solve an ordinary system of linear equations (using, for instance, Gauss elimination). This is easier than solving a system in the nonnegative orthant. We now show that S_4 is the inverse of R_4 within LBFC.

PROPOSITION 8.2. *Let $\Sigma \in$ LBFC. Then $(\Sigma, \tilde{\Sigma}) \in S_4$ iff $(\tilde{\Sigma}, \Sigma) \in R_4$.*

Proof. ($\Rightarrow$): Since $\Sigma \in$ LBFC and t is linearly dependent, t is nonnegative linearly dependent by Proposition 8.1. It is easy to see that $\tilde{N} = N^{+t}$ satisfies the condition of application of R_4, and that the result of applying it is Σ.

($\Leftarrow$): Follows easily from the definitions. ∎

Using this property, we can replace in the kit of inverses the inverse of R_4 by the rule S_4. The new kit is still sound and complete, and the conditions of application easier to check. It is left to the interested reader to show that the inverse of R_3 can be weakened in a similar way.

9. CONCLUSIONS

We have introduced two complete kits of reduction rules for the class LBFC, taking as atomic systems those whose underlying net contains one place and one transition. We have also shown that the reduction algorithm runs in polynomial time on the size of the system. The algorithm can be reversed to yield a synthesis algorithm.

Three papers [12, 6, 16] contain results closely related to ours:

In [12], Genrich and Thiagarajan study Bipolar Schemes, a model very similar to Petri nets. They provide a complete kit of eight synthesis rules for the class of "well behaved" Bipolar Schemes, with atomic systems very similar to ours. Every well behaved Bipolar Scheme can be translated into an equivalent (in a strong sense) live and 1-bounded Free Choice system, but the converse does not hold. The kit contains non-local rules.

Thiagarajan has conjectured that well behaved Bipolar Schemes are equivalent to live and 1-bounded Free Choice systems without frozen tokens. Absence of frozen tokens can be interpreted as a particular kind of fairness.

Desel [6] provides a complete kit of four rules for this class of Free Choice systems, with all live and 1-bounded S- and T-systems as atomic systems. All the rules are local.

Kovalyov studies in [16] LBFC systems in which all S-components contain a certain transition of the net. He provides a complete reduction kit of three local rules, with the empty system as atomic system.

Five parameters can be considered in order to relate these results to each other:

1. *The structural conditions imposed on the systems*—the weaker the better: the Free Choice property in [12, 6] and this paper, the Free Choice property plus an extra condition in [16].

2. *The atomic systems*—the simpler the better: very simple ones in [12, 16] and this paper, more complicated in [6].

3. *The simplicity of the rules, their number and local character*: simple local rules in [6, 16] while [12] and this paper contain non-local rules.

4. *The complexity of the reduction procedure*: this point is not considered in [12] nor in [16]. The procedure is polynomial in [6][3] and in this paper.

5. *The behavioural properties preserved by the rules*: it is difficult to compare different results, because the properties of interest depend on the application. In favour of this paper we can say that liveness and boundedness are two of the most studied properties in net theory [19, 18]. However, it is sometimes more interesting to preserve liveness and 1-boundedness.

There exists so far no (strongly) sound and complete kit of rules for the class of live and 1-bounded Free Choice systems. Our kit is sound and complete for a larger class, namely LBFC, while the kits of [6, 12, 16] are sound and complete for smaller classes. Obtaining such a kit is a very interesting topic for further research.

Annex

A net is a triple (S, T, F) such that $S \cap T = \emptyset$ and $F \subseteq (S \times T) \cup (T \times S)$. Since a net can be viewed as a directed graph, terminology can be transferred (for instance, strong or weak connectedness). We assume that nets are connected.

The *pre-set* ${}^\bullet x$ of $x \in (S \cup T)$ is defined as the set $\{y \in (S \cup T) \mid (y, x) \in F\}$, and the *post-set* $x^\bullet$ of $x \in (S \cup T)$ is defined as $\{y \in (S \cup T) \mid (x, y) \in F\}$. The notation is extended to sets $X \subseteq (S \cup T)$ by ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$, and similarly for $X^\bullet$.

A net N is an *S-graph* iff $\forall t \in T$: $|{}^\bullet t| = |t^\bullet| = 1$. N is a *T-graph* iff $\forall s \in S$: $|{}^\bullet s| = |s^\bullet| = 1$. N is a *Free Choice* net iff $\forall s \in S$, $\forall t \in s^\bullet$: $s^\bullet = \{t\} \vee {}^\bullet t = \{s\}$.

[3] This result is not contained in [6]; it was privately communicated by the author.

A net $N' = (S', T, F')$ is a *subnet* of $N = (S, T, F)$ (denoted by $N' \subseteq N$) iff $S' \subseteq S$, $T' \subseteq T$, and $F' = F \cap ((S \times T) \cup (T \times S))$. $N' \subseteq N$ is a *T-component* of N iff it is a strongly connected T-graph and ${}^{\bullet}T' = S' = T'^{\bullet}$. N' is an *S-component* of N iff it is a strongly connected S-graph and ${}^{\bullet}S' = T' = S'^{\bullet}$.

A *path* of N is a sequence $(x_1, ..., x_r)$ of elements of $S \cup T$ such that $\forall i$, $1 \leqslant i \leqslant (r-1)$: $(x_i, x_{i+1}) \in F$. A path is *elementary* iff the elements of the sequence are distinct.

A *siphon* of N is a subset of places $R \subseteq S$ such that ${}^{\bullet}R \subseteq R^{\bullet}$.

Let $N = (S, T, F)$ be a net with $S = \{s_1, ..., s_n\}$ and $T = \{t_1, ..., t_m\}$. The matrix $C = \|c_{ij}\|$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant M)$ such that

$$c_{ij} = \begin{cases} -1 & \text{if } (s_i, t_j) \in F \backslash F^{-1} \\ +1 & \text{if } (t_j, s_i) \in F \backslash F^{-1} \\ 0 & \text{otherwise} \end{cases}$$

is called the *incidence matrix of N*.

N is *conservative* iff there exists a vector $Y > 0$ (i.e., every component of Y is positive) such that $Y^{\mathrm{T}} \cdot C = 0$. Analogously, N is *consistent* iff there exists a vector $X > 0$ such that $C \cdot X = 0$.

A *marking* of N is a function $M: S \to \mathbb{N}$. A *marked net* or *system* is a pair (N, M_0), where $N = (S, T, F)$ is a net such that S and T are non-empty, and M_0 is an (initial) marking of N.

A marking M *enables* a transition $t \in T$ iff $\forall s \in {}^{\bullet}t$: $M(s) \geqslant 1$. An enabled transition can *occur*, yielding a new marking M', denoted by $M[t\rangle M'$. M' is defined by the following rule: $M'(s) = M(s) - 1$ for $s \in {}^{\bullet}t \backslash t^{\bullet}$, $M'(s) = M(s) + 1$ for $s \in t^{\bullet} \backslash {}^{\bullet}t$, and $M'(s) = M(s)$ otherwise.

An *occurrence sequence* is a sequence

$$\sigma = M_0[t_1\rangle M_1[t_2\rangle M_2 \cdots M_n.$$

We say that σ starts with M_0 and leads to M_n. Sometimes we omit the intervening markings since they are determined by M_0 and the sequence of transitions. We also say that M enables σ iff there are intermediate markings such that σ is an occurrence sequence starting with M. The set of all occurrence sequences enabled by M_0 (without the intervening markings) is the *language* of N, denoted by $L(N, M_0)$. The set $[M\rangle$ is defined as the set of all markings M' such that some occurrence sequence leads from M to M'.

The *Parikh vector* of an occurrence sequence σ, denoted by $\boldsymbol{\sigma}$, is the vector having $|T|$ components, and whose ith component is the number of appearances of t_i in σ.

Let (N, M_0) be a system and C the incidence matrix of N. The equation

$$M = M_0 + C \cdot X$$

is called the *state equation* of (N, M_0). This equation has the following property: if $M_0[\sigma\rangle M$, then $X=\boldsymbol{\sigma}$ satisfies the equation.

Let (N, M_0) be a system with $N=(S, T, F)$. A transition $t \in T$ is *live* iff for every $M \in [M_0\rangle$, there exists $M' \in [M\rangle$ such that M' enables t. A place $s \in S$ is k-bounded iff all markings $M \in [M_0\rangle$ satisfy $M(s) \leqslant k$. A place is *bounded* iff it is k-bounded for some number k.

(N, M_0) is live iff all its transitions are live. (N, M_0) is bounded iff all its places are bounded.

A net N is *structurally bounded* iff it is bounded for every marking. N is *structurally live* iff there exists a marking that makes it live.

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