Spectral Expansion of a Non-Self-Adjoint Differential Operator on the Whole Axis

Gülen Başcanbaz-Tunca

E-mail: tunca@science.ankara.edu.tr

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In this article, we consider an operator $L$ defined by the differential expression

$$L(y) = -y'' + q(x)y, \quad x \in \mathbb{R} = (-\infty, \infty)$$

in $L_2(\mathbb{R})$, where $q$ is a complex-valued function. Under the condition

$$\sup_{-\infty < x < \infty} \left| \exp\left(\epsilon \sqrt{|x|}\right)|q(x)|\right| < \infty, \quad \epsilon > 0,$$

we have proved a spectral expansion of $L$ in terms of the principal functions, taking into account the spectral singularities. We have also investigated the convergence of the spectral expansion of $L$.

1. INTRODUCTION

Let us consider an operator $L_0$ defined by the differential equation

$$-y'' + q(x)y = \mu y, \quad x \in \mathbb{R}^+ = [0, \infty)$$

and the boundary condition $y'(0) - h y(0) = 0$, in $L_2(\mathbb{R}^+)$, where $q$ is a complex-valued function and $h \in \mathbb{C}$. The spectral analysis of $L_0$ has been investigated by Naimark [15]. In this article, he has proved that some of the poles of the resolvent’s kernel of $L$ are not the eigenvalues of the operator. He has also shown that those poles (which are called spectral singularities by Schwartz [17]) are on the continuous spectrum. Moreover, he has shown that the spectral singularities play an important role in the discussion of the spectral analysis of $L_0$, and if the condition

$$\int_0^\infty |q(x)| \exp(\epsilon x) \, dx < \infty, \quad \epsilon > 0$$

then...
holds, then the eigenvalues and the spectral singularities are of finite number, and each of them is of finite multiplicity.

Let \( E \) denote the set of all entire functions of exponential type which are integrable over the real axis, and let \( E' \) denote the dual of \( E \). Now we define

\[
\varphi(f_i, \lambda) = \int_0^\infty f_i(x) \varphi(x, \lambda) \, dx, \quad i = 1, 2,
\]

for any finite \( f_1, f_2 \in L_2(\mathbb{R}^+) \), where \( \varphi(x, \lambda) \) is the solution of \(-y'' + q(x)y = \lambda^2 y\), subject to the initial conditions \( \varphi(0, \lambda) = 1 \) and \( \varphi_x(0, \lambda) = h \). In [13] Marchenko has proved that

\[
\varphi(f_1, \lambda) \varphi(f_2, \lambda) \in E,
\]

and there exists a functional \( T \in E' \) such that

\[
\int_0^\infty f_1(x)f_2(x) \, dx = T[\varphi(f_1, \lambda) \varphi(f_2, \lambda)].
\]

This relation is a generalization of the well-known Parseval equality for the singular self-adjoint differential operators and is called a Marchenko–Parseval equality, where \( T \) is the generalized spectral function for the boundary value problem \(-y'' + q(x)y = \lambda^2 y\), \( \varphi(0, \lambda) = 1 \), \( \varphi_x(0, \lambda) = h \).

These results of Naimark and Marchenko have been extended to the case where the potential function is \( q(x) \) with \( q(x) = \int (\lambda + 1)x^{-2} + p(x) \), where \( p \) is summable on every finite interval of \((0, \infty)\), and the three-dimensional Schrödinger equations by Gasymow [6, 7] and Lyance [11].

The Laurent expansion of the resolvent of the abstract non-self-adjoint operators in the neighborhood of the spectral singularities has been investigated by Gasymov and Maksudov [8] and Maksudov and Al-lakhverdiev [12].

The spectral analysis of some class of dissipative operators with spectral singularities has been studied by Pavlov [16], using the theory of functional model [14] and scattering theory [10].

Let us consider an operator \( L \) generated in \( L_2(\mathbb{R}) \) by the equation

\[
-y'' + q(x)y = \mu y, \quad x \in \mathbb{R},
\]

where \( q \) is a complex-valued function and \( \mu \) is a spectral parameter.

The results of Naimark [15] have been generalized to the operator \( L \) by Blashak [5]; he has proved that the operator \( L \) has a finite number of eigenvalues and spectral singularities if

\[
\int_{-\infty}^{\infty} |q(x)| \exp(\epsilon|x|) \, dx < \infty, \quad \epsilon > 0,
\]

(1.2)
holds. Now we consider the quadratic pencil of Schrödinger operators $L(\lambda)$, generated in $L_2(\mathbb{R}^+)$ by the equation

$$-y'' + \left[q(x) + 2\lambda u(x) - \lambda^2\right]y = 0, \quad x \in \mathbb{R}^+,$$

and the boundary condition $y(0) = 0$, where $u, q$ are complex-valued functions, and $u$ is absolutely continuous in each finite subinterval of $\mathbb{R}^+$. If $u \equiv 0$, then the operator $L(\lambda)$ reduces to the operator $L_0$.

Discrete spectrum, principal functions, and eigenfunction expansion of the quadratic pencil of Schrödinger operators have been investigated in [2–4]. In [1] it has been proved that the operator $L$ has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity under the condition

$$\sup_{-\infty < x < \infty} \{\exp(\epsilon \sqrt{|q(x)|})\} < \infty, \quad \epsilon > 0,$$

which is weaker than (1.2). Moreover, the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of $L$ have been obtained.

In this paper, which is a continuation of [1], we investigated the spectral expansion of $L$ with respect to the principal functions, using a contour integral method and the regularization of divergent integrals, using summability factors.

2. SPECIAL SOLUTIONS

Let us suppose that $q$ satisfies the condition

$$\int_{-\infty}^{\infty} (1 + |x|) |q(x)| \, dx < \infty. \quad (2.1)$$

Under the condition (2.1), Eq. (1.1) has the solutions for $\mu = \lambda^2, \lambda \in \mathbb{C}$,

$$e^+(x, \lambda) = e^{i\lambda x} + \int_{x}^{\infty} K^+(x, t) e^{i\lambda t} \, dt \quad (2.2)$$

$$e^-(x, \lambda) = e^{-i\lambda x} + \int_{-\infty}^{x} K^-(x, t) e^{-i\lambda t} \, dt \quad (2.3)$$

for $\lambda \in \mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \Im \lambda \geq 0\}$, and the kernels $K^\pm(x, t)$ satisfy

$$|K^\pm(x, t)| \leq C\sigma^\pm \left(\frac{x + t}{2}\right), \quad (2.4)$$
where
\[
\sigma^+(x) = \int_{x}^{\infty} |q(t)| \, dt, \quad \sigma^-(x) = \int_{-\infty}^{x} |q(t)| \, dt \tag{2.5}
\]
and \( C > 0 \) is a constant.

Therefore \( e^+(x, \lambda) \) and \( e^-(x, \lambda) \) are analytic with respect to \( \lambda \) in \( \mathbb{C}_+ := \{ \lambda : \lambda \in \mathbb{C}, \text{Im} \lambda > 0 \} \), continuous on the real axis, and satisfy
\[
e^+(x, \lambda) = e^{i\lambda x} + o(1), \quad \lambda \in \mathbb{C}_+, |\lambda| \to \infty, \tag{2.6}
\]
\[
e^-(x, \lambda) = e^{-i\lambda x} + o(1), \quad \lambda \in \mathbb{C}_+, |\lambda| \to \infty. \tag{2.7}
\]

The above results have been given in [1].

3. THE SPECTRUM OF \( L \)

Let us define
\[
\alpha(\lambda) := W(e^+(x, \lambda), e^-(x, \lambda)).
\]

It is clear that (see [1, 5])
\[
\sigma_\Omega(L) = \{ \mu : \mu = \lambda^2, \lambda \in \mathbb{C}_+, \alpha(\lambda) = 0 \},
\]
\[
\sigma_\text{s}(L) = \{ \mu : \mu = \lambda^2, \lambda \in \mathbb{R}, \alpha(\lambda) = 0 \},
\]
\[
\sigma_\varepsilon(L) = [0, \infty),
\]
\[
\rho(L) = \{ \mu : \mu \in \lambda^2, \lambda \in \mathbb{C}_+, \alpha(\lambda) \neq 0 \},
\]

where \( \sigma_\Omega(L), \sigma_\text{s}(L), \sigma_\varepsilon(L) \), and \( \rho(L) \) denote the eigenvalues, the spectral singularities, the continuous spectrum, and the resolvent set of \( L \), respectively. Let
\[
R(x, t; \lambda^2) = \frac{1}{\alpha(\lambda)} \begin{cases} e^+(x, \lambda)e^-(t, \lambda); & -\infty < t < x \\ e^-(x, \lambda)e^+(t, \lambda); & x \leq t < \infty \end{cases} \tag{3.1}
\]
be the Green function of \( L \) for \( \lambda \in \mathbb{C}_+ \).

It is known from [1] that under the condition (1.3) \( L \) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Let \( \lambda_1, \ldots, \lambda_n \) denote the zeros of \( \alpha \) in \( \mathbb{C}_+ \) (i.e., \( \lambda_1^2, \ldots, \lambda_n^2 \) are the eigenvalues of \( L \)) with multiplicities \( m_1, \ldots, m_n \), respectively. Similarly, let \( \lambda_{n+1}, \ldots, \lambda_k \) be the zeros of \( \alpha \) on the real axis (i.e., \( \lambda_{n+1}^2, \ldots, \lambda_k^2 \) are the
spectral singularities of $L$ with multiplicities $m_{\ell+1}, \ldots, m_k$, respectively. We need the Hilbert spaces

$$H(a, b; \pm m) = \left\{ f : \int_a^b (1 + |x|)^{\pm 2m} |f(x)|^2 \, dx < \infty \right\},$$

$$m = 0, 1, 2, \ldots$$

Here $(a, b)$ may be $(-\infty, 0), (0, \infty), (-\infty, \infty)$. We have previously obtained [1]

$$U_n(x, \mu_j) = \left\{ \sum_{i=0}^n \binom{n}{i} a_{n-i}(\lambda_j) \left( \frac{\partial}{\partial \lambda} \right)^i (x, \lambda) \right\}_{\lambda = \lambda_j}; \quad -\infty < x < 0$$

$$\left( \frac{\partial}{\partial \lambda} \right)^n e^{+}(x, \lambda) \big|_{\lambda = \lambda_j}; \quad 0 \leq x < \infty$$

$$U_n(x, \mu_j) \in L^2(\mathbb{R}), \quad n = 0, 1, \ldots, m_j - 1, j = 1, \ldots, \ell.$$  \hspace{1cm} (3.2)

Here $U_0(x, \mu_j)$ is an eigenfunction and $U_l(x, \mu_j), \ldots, U_{m_j-1}(x, \mu_j)$ are the associated functions of $U_0(x, \mu_j)$,

$$U_n(x, \mu_j) \in H(-\infty, \infty; -(m_0 + 1)),$$

$$n = 0, 1, \ldots, m_j - 1, j = \ell + 1, \ldots, k.$$  \hspace{1cm} (3.3)

Here $U_n(x, \mu_j)$ are the principal functions corresponding to the spectral singularities and

$$m_0 = \max\{m_{\ell+1}, \ldots, m_k\}. \hspace{1cm} (3.4)$$

We can easily obtain that the resolvent operator of $L$ is

$$R_\mu(L) f(x) = \int_{-\infty}^{\infty} R(x, t; \mu) f(t) \, dt$$

for $\mu \in \rho(L)$, where $R(x, t; \mu)$ was given in (3.1).

### 4. SPECTRAL EXPANSION

Let $D$ denote the set of infinitely differentiable functions in $\mathbb{R}$ with compact support. Evidently,

$$\psi(x) = \int_{-\infty}^{\infty} R(x, t; \mu) \left[ -\psi''(t) + q(t) \psi(t) - \mu \psi(t) \right] dt$$
for each \( \psi \in D \). Therefore,

\[
\frac{\psi(x)}{\mu} = \frac{1}{\mu} \int_{-\infty}^{\infty} R(x,t; \mu) \left[ -\psi''(t) + q(t)\psi(t) \right] dt \\
- \int_{-\infty}^{\infty} R(x,t; \mu) \phi(t) dt.
\] (4.1)

Let \( \gamma_r \) denote the disc centered at the origin having radius \( r \), and let \( \partial \gamma_r \) be the boundary of \( \gamma_r \). \( r \) will be chosen such that all eigenvalues and spectral singularities of \( L \) are in \( \gamma_r \). Let \( P_{r,\eta} \) denote the part of \( \gamma_r \) consisting of the points \( \mu \) satisfying

\[
|\text{Im} \mu| \leq \eta, \quad \text{Re} \mu > 0, \\
\gamma_{r,\eta} = \gamma_r - P_{r,\eta}, \quad \gamma_r = \gamma_{r,\eta} \cup P_{r,\eta}.
\]

So we easily see that

\[
\partial \gamma_r = \partial \gamma_{r,\eta} \cup \partial P_{r,\eta}. \tag{4.2}
\]

From (4.1) we obtain

\[
\psi(x) = \frac{1}{2\pi i} \int_{\partial \gamma_r} \left\{ \frac{1}{\mu} \int_{-\infty}^{\infty} R(x,t; \mu) \left[ -\psi''(t) + q(t)\psi(t) \right] dt \\
- \int_{-\infty}^{\infty} R(x,t; \mu) \phi(t) dt \right\} d\mu. \tag{4.3}
\]

Using (2.6), (2.7), and (3.1), we see that the first term of the right-hand side of (4.3) vanishes as \( r \to \infty \). Then considering (4.2) we get

\[
\psi(x) = -\lim_{r \to \infty, \eta \to 0} \frac{1}{2\pi i} \int_{\partial \gamma_{r,\eta}} R_{\mu}(L) \psi(x) d\mu \\
- \lim_{r \to \infty, \eta \to 0} \frac{1}{2\pi i} \int_{\partial P_{r,\eta}} R_{\mu}(L) \psi(x) d\mu. \tag{4.4}
\]
We easily obtain that the first integral in (4.4) gives

\[ \int_{\gamma R, \gamma} R_{\mu}(L) \, d\mu = \sum_{j=1}^{\ell} \text{Res}_{\mu = \mu_j} R_{\mu}(L) \psi(x). \quad (4.5) \]

Here

\[ R_{\mu}(L) \psi(x) = \int_{-\infty}^{\infty} R(x, t; \mu) \psi(t) \, dt. \]

Now we want to reduce (4.4) to a spectral expansion. We know from [9] that if \( \mu_j, j = 1, \ldots, \ell \) are poles of the resolvent, then the principal part of the resolvent for \( \mu_j \) can be expressed in the form

\[
\frac{U_0 V_0}{(\mu - \mu_j)m_{j-1}} + \frac{U_1 V_1 + U_1 V_0}{(\mu - \mu_j)m_{j-2}} + \ldots + \frac{U_m V_{m-1} + U_{m-1} V_0}{\mu - \mu_j},
\]

where \( U_0, U_1, \ldots, U_{m-1} \) are the eigenfunction and associated functions of \( L \) corresponding to the eigenvalue \( \mu_j \) with order \( m_j \), and \( V_0, V_1, \ldots, V_{m-1} \) are the eigenfunction and associated functions of the operator \( L^* \) adjoint to \( L \), corresponding to the eigenvalue \( \bar{\mu}_j, j = 1, \ldots, \ell \), which is uniquely determined for the given \( U_0, U_1, \ldots, U_{m-1} \). And \( U_0 V_0 \) denotes the operator defined by

\[ B\psi = (\psi, V_0)U_0. \]

Moreover, for every function \( \psi \in L_2(\mathbb{R}), \)

\[ (\psi, V_m), \quad m = 1, 2, \ldots, m_j - 1, j = 1, \ldots, \ell \]

is defined because of (3.2). Considering (4.6) in (4.5), we obtain the first integral in (4.4) as follows:

\[
\int_{\gamma R, \gamma} R_{\mu}(L) \psi(x) \, d\mu = \sum_{j=1}^{\ell} \sum_{\nu=0}^{m_j-1} U_\nu(x, \mu) V_{m_j-1-\nu}(\psi, \mu) \bigg|_{\mu = \mu_j}. \quad (4.7)
\]

**Assumption.** Let us suppose that the operator \( L \) has no spectral singularities.

Under the assumption, to reduce (4.4) to a spectral expansion, we first obtain the second integral in (4.4), by using the classical method in [15], as
follows:

\[
\lim_{r \to \infty, \eta \to 0} \frac{1}{2\pi i} \int_{\partial P_{r,\eta}} R_\mu(L) \psi(x) \, d\mu
\]

\[
= \frac{1}{2\pi i} \int_0^\infty \{ R_{\mu+i\eta}(L) \psi(x) - R_{\mu-i\eta}(L) \psi(x) \} \, d\mu. \quad (4.8)
\]

Now we must evaluate the right-hand side of (4.8). Since

\[
W\{e^+(x, \lambda), e^+(x, -\lambda)\} = -2i\lambda, \quad \text{Im} \lambda = 0, x \to \infty
\]

and

\[
W\{e^-(x, \lambda), e^-(x, -\lambda)\} = 2i\lambda, \quad \text{Im} \lambda = 0, x \to -\infty,
\]

then we can write \( e^-(t, \lambda) \) and \( e^+(x, -\lambda) \) as

\[
e^-(t, \lambda) = \frac{2i\lambda}{\alpha(-\lambda)} e^+(t, -\lambda) - \frac{W\{e^-(t, \lambda), e^+(t, -\lambda)\}}{\alpha(-\lambda)} e^-(t, -\lambda),
\]

\[
e^+(x, -\lambda) = \frac{W\{e^+(x, -\lambda), e^-(x, \lambda)\}}{\alpha(\lambda)} e^+(x, \lambda) - \frac{2i\lambda}{\alpha(\lambda)} e^-(x, \lambda). \quad (4.10)
\]

Substituting (4.9) and (4.10) in the right-hand side of (4.8), we obtain

\[
\frac{1}{2\pi i} \int_{-\infty}^\infty \{ R_{\mu+i\eta}(L) \psi(x) - R_{\mu-i\eta}(L) \psi(x) \} \, d\mu
\]

\[
= \int_{-\infty}^\infty \frac{\sqrt{\mu}}{\pi \alpha(\sqrt{\mu}) \alpha(-\sqrt{\mu})} \{ e^+(x, \sqrt{\mu}) e^+(\psi, \sqrt{\mu})
\]

\[
+ e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu}) \} \, d\mu, \quad (4.11)
\]

where

\[
e^\pm(\psi, -\sqrt{\mu}) := \int_{-\infty}^\infty e^\pm(t, -\sqrt{\mu}) \psi(t) \, dt
\]

and

\[
\{ e^+(x, \sqrt{\mu}), e^-(x, \sqrt{\mu}) \}, \quad \{ e^+(x, -\sqrt{\mu}), e^-(x, -\sqrt{\mu}) \}, \quad \mu > 0,
\]
are the principal functions of the continuous spectrum of $L$ and $L^*$, respectively. Therefore,
\[
\left[ e^+(x, \sqrt{\mu}) \right]^* = e^+(x, \sqrt{-\mu}), \quad \left[ e^-(x, \sqrt{\mu}) \right]^* = e^-(x, -\sqrt{\mu}),
\]
\[
\alpha^* = (\sqrt{\mu}) = \alpha(-\sqrt{\mu}).
\]

The integral in (4.11) converges in the norm of $L_2(\mathbb{R})$. Taking (4.7) and (4.11) into account, (4.4) takes the form
\[
\psi(x) = \sum_{j=1}^{\nu - 1} \sum_{\nu = 0}^{m_j - 1} U_j(x, \mu) V_{m_j - 1 - \nu}^*(\psi, \mu) \bigg|_{\mu = \mu_j} \nonumber
\]
\[
+ \int_{-\infty}^{\infty} \frac{\sqrt{\mu}}{\pi \alpha(\sqrt{\mu}) \alpha(-\sqrt{\mu})} \left( e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu}) + e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu}) \right) d\mu. \quad (4.12)
\]

Since $\bar{D} = L_2(-\infty, \infty)$, we have the following:

**Remark.** For every $f \in L_2(\mathbb{R})$ the spectral expansion formula (4.12) is valid under the assumption, and the integral converges in the norm of $L_2(\mathbb{R})$.

Now we want to reduce (4.4) to a spectral expansion under the condition (1.3), which guarantees the finiteness of eigenvalues and spectral singularities with finite multiplicities. It is clear that if $\alpha(\lambda) = 0$ then $\alpha(-\lambda) = 0$.

Let $\Gamma_+$ be the contour that isolates the positive numbers $\mu_j = \lambda_j^2$, $\alpha(\lambda_j) = 0$, $\lambda_j > 0$, by semicircles with centers at $\mu_j$ having the same radius $\delta_0$ in the upper half-plane; similarly, let $\Gamma_-$ be the corresponding contour for the positive numbers $\mu_j = \lambda_j^2$, $\alpha(-\lambda_j) = 0$, $\lambda_j > 0$, in the lower half-plane, where $j = 1, \ldots, k$. The radius $\delta_0$ will be chosen such that two neighborhoods have no common points (see Fig. 1).

As easily seen from Fig. 1, we obtain
\[
\lim_{r \to \infty, \eta \to -\infty} \frac{1}{2\pi i} \int_{\partial P_{\nu, \eta}} R_\mu(L) \psi(x) \, d\mu
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma_-} R_\mu(L) \psi(x) \, d\mu - \frac{1}{2\pi i} \int_{\Gamma_+} R_\mu(L) \psi(x) \, d\mu.
\]
Hence, taking (4.7) into account, (4.4) will be as follows:

$$\psi(x) = \sum_{j=1}^{m_j-1} \sum_{\nu=0}^{r_{j-1}} U_{\nu}(x, \mu)V_{m_j-1-\nu}(\psi, \mu)|_{\mu=\mu_j}$$

$$+ \frac{1}{2\pi i} \left\{ \int_{\Gamma_+} R_{\mu}(L) \psi(x) \, d\mu - \int_{\Gamma_-} R_{\mu}(L) \psi(x) \, d\mu \right\}. \quad (4.13)$$

**Lemma 4.1.** There is a number $C > 0$ such that, for every finite function $\psi \in L_2(\mathbb{R})$,

$$\int_0^\infty |e^{\pm}(\psi, -\sqrt{\mu})|^2 \, d\mu \leq C \int_{-\infty}^\infty |\psi(x)|^2 \, dx. \quad (4.14)$$

**Proof.** Since

$$e^{\pm}(\psi, -\sqrt{\mu}) = \int_{-\infty}^\infty e^{\pm}(x, -\sqrt{\mu}) \psi(x) \, dx,$$

using (2.2), we get

$$e^{\pm}(\psi, -\sqrt{\mu}) = \int_{-\infty}^\infty \left\{ e^{-i\sqrt{\mu}x} + \int_x^\infty K^+(x, t) e^{-i\sqrt{\mu}t} \, dt \right\} \psi(x) \, dx$$

$$+ \int_{-\infty}^\infty \int_x^\infty \psi(x) K^+(x, t) e^{-i\sqrt{\mu}t} \, dt \, dx.$$

Changing the order of integration, we obtain

$$e^{\pm}(\psi, -\sqrt{\mu}) = \int_{-\infty}^\infty \{ (I + K^+) \psi(t) \} e^{-i\sqrt{\mu}t} \, dt, \quad (4.15)$$

in which the operator $I$ is the unit operator, and $K^+$ is the operator defined by

$$K^+ \psi(t) = \int_{-\infty}^t K^+(x, t) \psi(x) \, dx.$$

We know from [1] that under the condition (1.3)

$$|K^+(x, t)| \leq C \exp \left\{ -\frac{\epsilon}{2} \left( \sqrt{\frac{|x+t|}{2}} \right) \right\}. $$

Hence $K^+$ is a compact operator in $L_2(\mathbb{R})$. Thus $(I + K^+)$ is continuous and one-to-one on $L_2(\mathbb{R})$. Using Parseval’s equality for (4.15), we obtain (4.14) for $e^{\pm}(\psi, -\sqrt{\mu})$. The estimate for $e^{-}(\psi, -\sqrt{\mu})$ can be proved similarly.
By Lemma 4.1, for each function $\psi \in L_2(\mathbb{R})$ the limits

$$e^{\pm}(\psi, -\sqrt{\mu}) = \lim_{N \to \infty} \int_{-N}^{N} e^{\pm}(x, -\sqrt{\mu})\psi(x) \, dx$$  \hspace{1cm} (4.16)

exist in the sense of convergence in the mean square on the real axis. Since $\mathcal{D} = L_2(\mathbb{R})$, (4.14) may be extended onto $L_2(\mathbb{R})$, where $e^{\pm}(\psi, -\sqrt{\mu})$ must be understood in the sense of (4.16). We shall need a generalization of these estimates.

**Lemma 4.2.** If

$$\int_{-\infty}^{\infty} \left| (1 + |x|)^{\nu} \psi(x) \right|^2 \, dx < \infty
$$

then the functions $e^{\pm}(\psi, -\sqrt{\mu})$ have a derivative of order $(\nu - 1)$ which is absolutely continuous on every finite interval of the half-axis $\mu > 0$.

There exists a number $C_{\nu}$ such that

$$\int_{0}^{\infty} \left| \frac{d}{d\mu} \left( e^{\pm}(\psi, -\sqrt{\mu}) \right) \right|^2 \, d\mu \leq C_{\nu} \int_{-\infty}^{\infty} \left| (1 + |x|)^{\nu} \psi(x) \right|^2 \, dx.$$  \hspace{1cm} (4.17)

The proof is similar to that of Lemma 4.1.

To transform (4.4) into the spectral expansion of $L$, it is natural to transform formula (4.13) so that the integration contour shall become the positive half-axis $\mu \geq 0$. Since the spectral singularities of $L$ are the squares of the real zeros of $\alpha(\lambda)$, then the integrals over the positive real axis are divergent in the norm of $L_2(\mathbb{R})$. Now we will investigate the convergence of these integrals in a norm weaker than the norm of $L_2(\mathbb{R})$. For this, we will use the technique of the regularization of divergent integrals. So we will define the following summability factors:

$$\Phi_{\mu} = \begin{cases} 
\frac{(\mu - \mu_j)^n}{n!} ; & |\mu - \mu_j| < \delta, j = 1, \ldots, k \\
0 ; & |\mu - \mu_j| \geq \delta
\end{cases} \hspace{1cm} (4.18)$$

where $\delta > 0$ is a sufficiently small number such that the $\delta$-neighborhoods of $\mu_j$ are distinct. Furthermore, for an arbitrary function $f(\mu)$ which is
differentiable often at the points \( \mu_{\ell+1}, \ldots, \mu_k \) we put

\[
\Phi(f(\mu)) = f(\mu) - \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \left( \frac{d}{d\mu} \right)^n f(\mu)|_{\mu=\mu_j} \Phi_{j\mu}(\mu). \quad (4.19)
\]

By (4.19) the points \( \mu_{\ell+1}, \ldots, \mu_k \) are roots of orders of at least \( m_{\ell+1}, \ldots, m_k \) for the function \( \Phi(f(\mu)) \). In the neighborhood of a spectral singularity or, what amounts to the same thing, a generalized eigenvalue (as Gasymov and Maksudov called it in [8]), we can write the resolvent in the generalized Laurent expansion (which means the function does not need to be analytic at the neighborhood of the generalized eigenvalue, but it has a derivative of each order at these points), in terms of a generalized eigenfunction and adjoint functions corresponding to the generalized eigenvalue \( \mu_j, j = \ell + 1, \ldots, k \) as we would for an ordinary singularity. Therefore, we can write the resolvent as follows. When \( \psi \in H_{(m_\ell + 1)},{ } \]

\[
R_\mu(L) \psi(x) = \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \sum_{p=0}^{n} U_p(x, \mu)V_n(\psi, \mu)|_{\mu=\mu_j} \int_{\gamma_j} \frac{\Phi_{j\mu}(\mu)}{(\mu - \mu_j)^{m_j}} d\mu \\
+ \frac{\Phi(R_1(x, \psi, \mu))}{\alpha(\sqrt{\mu})}, \quad (4.20)
\]

where

\[
R_1(x, \psi, \mu) := \int_{-\infty}^{\infty} (R_1(x, t, \mu)) \psi(t) dt
\]

and

\[
R_1(x, t, \mu) := \begin{cases} 
  e^+(x, \sqrt{\mu})e^-(t, \sqrt{\mu}); & -\infty < t < x \\
  e^-(x, \sqrt{\mu})e^+(t, \sqrt{\mu}); & x \leq t < \infty,
\end{cases}
\]

the kernel of the resolvent is

\[
R(x, t; \mu) = \frac{1}{\alpha(\sqrt{\mu})} R_1(x, t; \mu).
\]

Furthermore, by (2.20), \( U_\ell(x, \mu) \) is a generalized eigenfunction; \( U_j(x, \mu), \ldots, U_{m_j - 1}(x, \mu) \) are associated functions of generalized eigenvalue \( \mu_j \); and \( V_\ell(\psi, \mu), \ldots, V_{m_j - 1}(\psi, \mu) \) are \( \psi \)-Fourier transformations of the generalized eigenfunction and associated functions of the operator \( L^\ast \) adjoint to \( L \), corresponding to the generalized eigenvalue \( \mu_j, j = \ell + 1, \ldots, k \). That is,
for $\psi \in H_{(m_{n+1})}$,
\[
(\psi, V_k), \quad k = 0, \ldots, m_j - 1, j = \ell + 1, \ldots, k,
\]
is defined because of (3.3).

Let us define
\[
\Lambda_j = (\mu_j - \delta, \mu_j + \delta), \quad j = \ell + 1, \ldots, k,
\]
\[
\Lambda_0 = R \setminus \bigcup_{j=\ell+1}^{k} \Lambda_j.
\]

$\gamma_j^\pm$ are the semicircles, centered at $\mu_j$, lying on $\Gamma^\pm$.

Integrating (4.20) over $\Gamma_+$ and $\Gamma_-$, we get
\[
\frac{1}{2\pi i} \int_{\Gamma_+} R_\mu(L) \psi(x) \, d\mu
\]
\[
= \frac{1}{2\pi i} \sum_{j=\ell+1}^{k} \left\{ \sum_{n=0}^{m_j} \sum_{p=0}^{n} U_p(x, \mu)V_n(\psi, \mu)\big|_{a=\mu_j} \int_{\gamma_j^+} \frac{\Phi_j(\mu)}{(\mu - \mu_j)^{m_j}} \, d\mu \right. \\
+ \int_{\gamma_j^+} \frac{\Phi(R_1(x, \psi, \mu))}{\alpha(\sqrt{\mu})} \, d\mu \right\}
\]
\[
+ \frac{1}{2\pi i} \int_{\Lambda_0} \frac{\Phi(R_1(x, \psi, \mu))}{\alpha(\sqrt{\mu})} \, d\mu
\]
\[
(4.21)
\]
\[
\frac{1}{2\pi i} \int_{\Gamma_-} R_\mu(L) \psi(x) \, d\mu
\]
\[
= \frac{1}{2\pi i} \sum_{j=\ell+1}^{k} \left\{ \sum_{n=0}^{m_j} \sum_{p=0}^{n} U_p(x, \mu)V_n(\psi, \mu)\big|_{a=\mu_j} \int_{\gamma_j^-} \frac{\Phi_j(\mu)}{(\mu - \mu_j)^{m_j}} \, d\mu \right. \\
+ \int_{\gamma_j^-} \frac{\Phi(R_1(x, \psi, -\mu))}{\alpha(-\sqrt{\mu})} \, d\mu \right\}
\]
\[
+ \frac{1}{2\pi i} \int_{\Lambda_0} \frac{\Phi(R_1(x, \psi, -\mu))}{\alpha(-\sqrt{\mu})} \, d\mu.
\]

Here $\Phi(R_1(x, \psi, \mu))/\alpha(\sqrt{\mu})\Phi(R_1(x, \psi, -\mu))/\alpha(-\sqrt{\mu})$ is analytic in $\text{Im}\, \lambda > 0$ ($\text{Im}\, \lambda < 0$) and has a derivative of each order on a real axis.
Subtracting (4.22) from (4.21), we obtain
\[
\frac{1}{2\pi i} \left\{ \int_{\Gamma} R_\mu(L) \psi(x) \, d\mu - \int_{\Gamma-} R_\mu(L) \psi(x) \, d\mu \right\}
\]
\[
= \frac{1}{2\pi i} \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \sum_{p=0}^{n} U_p(x, \mu) V_n(\psi, \mu) |_{\mu=\mu_j} \int_{\gamma_j} \frac{\Phi_j(\mu)}{(\mu - \mu_j)^{m_j}} \, d\mu
\]
\[
- \frac{1}{2\pi i} \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \sum_{p=0}^{n} U_p(x, \mu) V_n(\psi, \mu) |_{\mu=\mu_j} \int_{\gamma_j} \frac{\Phi_j(\mu)}{(\mu - \mu_j)^{m_j}} \, d\mu
\]
\[
+ \frac{1}{2\pi i} \int_0^{\infty} \left\{ \Phi(R_1(x, \psi, \mu)) - \Phi(R_1(x, \psi, -\mu)) \right\} \frac{1}{\alpha(\sqrt{\mu})} \, d\mu. \tag{4.23}
\]

Taking (4.9), (4.10), and (4.11) into account, (4.23) then will be
\[
\frac{1}{2\pi i} \left\{ \int_{\Gamma} R_\mu(L) \psi(x) \, d\mu - \int_{\Gamma-} R_\mu(L) \psi(x) \, d\mu \right\}
\]
\[
= \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \sum_{p=0}^{n} U_p(x, \mu) V_n(\psi, \mu) |_{\mu=\mu_j, \alpha_j}
\]
\[
+ \frac{1}{\pi} \int_0^{\infty} \frac{\sqrt{\mu}}{\alpha(\sqrt{\mu})} \frac{1}{\alpha(-\sqrt{\mu})} \Phi(e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu})
\]
\[
+ e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu}) \right\} \, d\mu, \tag{4.24}
\]
where
\[
\alpha_j := \begin{cases} 
\frac{1}{2\pi i} \int_{\gamma_j} \frac{\Phi_j(\mu)}{(\mu - \mu_j)^{m_j}} \, d\mu, \\
\frac{1}{2\pi i} \int_{\gamma_j} \frac{\Phi_j(\mu)}{(\mu - \mu_j)^{m_j}} \, d\mu.
\end{cases}
\tag{4.25}
\]

Let us consider the operators \(\tau_1\) and \(\tau_2\) given by
\[
\tau_1 \psi(x) = \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \sum_{p=0}^{n} U_p(x, \mu) V_n(\psi, \mu) |_{\mu=\mu_j, \alpha_j}, \tag{4.26}
\]
where $\alpha_{j_n}$ is defined by (4.25) and

$$
\tau_2(\psi) = \frac{1}{\pi} \int_0^\infty \sqrt{\frac{\mu}{\alpha(\sqrt{\mu})}} \Phi\{e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu}) + e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu})\} \, d\mu. \quad (4.27)
$$

So, from (4.26) and (4.27) the right-hand side of (4.24) is $\tau_1(x) + \tau_2(x)$. Since $\psi \in H_{(m_0+1)}$, we can apply $\Phi$, defined by (4.19), to $e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu}) + e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu})$.

**Lemma 4.3.** For each $\psi \in H_{(m_0+1)}$ there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$
\|\tau_1 \psi\|_{-(m_0+1)} \leq C_1 \|\psi\|_{-(m_0+1)} \quad (4.28)
$$

$$
\|\tau_2 \psi\|_{-(m_0+1)} \leq C_2 \|\psi\|_{-(m_0+1)} \quad (4.29)
$$

hold, where $m_0$ is defined by (3.4), and $\|\cdot\|_{\pm n}$ denote the norms of $H_{\pm n}$.

**Proof.** From (4.18) we get the absolute convergence of $\alpha_{j_n}$. Using (3.3), (4.14), and (4.17), we obtain that $\tau_n(x)$ is continuous from $H_{m_0}$ into $H_{-m_0}$ or from $H_{+(m_0+1)}$ into $H_{-(m_0+1)}$. Therefore there exists a constant $C_1 > 0$ such that (4.28) holds. Now using the integral form of the remainder in the generalized Taylor expansion, we have

$$
\Phi\{e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu}) + e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu})\} =

\begin{cases}
\frac{1}{(m_j - 1)!} \int_0^\mu (\mu - \xi)^{m_j - 1} \left\{ e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu}) + e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu}) \right\} \, d\xi, \\
\mu \in \Lambda_0 \\
\times \left\{ \frac{\partial}{\partial \xi} \right\}^{m_j} \left[ e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu}) + e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu}) \right] d\xi, \\
\mu \in \Lambda_j, \ j = \ell + 1, \ldots, k.
\end{cases}
$$

(4.30)
If we use the notations
\[
\tau_{\ell}^{j}(x) = \frac{1}{\pi} \int_{\Lambda_j} \frac{\sqrt{\mu}}{\alpha(\sqrt{\mu})\alpha(-\sqrt{\mu})} \Phi\{e^{+}(x, \sqrt{\mu})e^{+}(\psi, -\sqrt{\mu}) + e^{-}(x, \sqrt{\mu})e^{-}(\psi, -\sqrt{\mu})\} \, d\mu, \quad j = 0, \ell' + 1, \ldots, k,
\]
then from (4.30) we have
\[
\tau_2 = \tau_2^0 + \tau_2^{\ell+1} + \cdots + \tau_2^k. \tag{4.31}
\]
First we will prove the continuity of \(\tau_{\ell}^{j} , j = \ell' + 1, \ldots, k\), from \(H^{(m_j+1)}\) into \(H^{-(m_j+1)}\). It is trivial from (4.30) that
\[
\tau_{\ell}^{j}(x) = \frac{1}{\pi} \int_{\Lambda_j} \int_{\mu_j}^{\mu_j+\delta} \frac{(\lambda - \xi)^{m_j-1}\sqrt{\mu}}{\alpha(\sqrt{\mu})\alpha(-\sqrt{\mu})} \left\{ \frac{d}{d\xi} \right\} \left\{ e^{+}(x, \sqrt{\mu})e^{+}(\psi, -\sqrt{\mu}) + e^{-}(x, \sqrt{\mu})e^{-}(\psi, -\sqrt{\mu}) \right\} \, d\xi \, d\mu. \tag{4.32}
\]
Changing the order of integration, we get
\[
\tau_{\ell}^{j}(x) = \frac{1}{2\pi i(m_j - 1)!} \times \left\{ \int_{\mu_j}^{\mu_j+\delta} \frac{d}{d\xi} \int_{\mu_j}^{\mu_j+\delta} \left( \frac{d}{d\xi} \right)^{m_j} \left\{ e^{+}(x, \sqrt{\xi})e^{+}(\psi, -\sqrt{\xi}) + e^{-}(x, \sqrt{\xi})e^{-}(\psi, -\sqrt{\xi}) \right\} \frac{(\mu - \xi)^{m_j-1}\sqrt{\mu}}{\alpha(\sqrt{\mu})\alpha(-\sqrt{\mu})} \, d\mu \\
- \int_{\mu_j-\delta}^{\mu_j} \frac{d}{d\xi} \int_{\mu_j-\delta}^{\mu_j} \left( \frac{d}{d\xi} \right)^{m_j} \left\{ e^{+}(x, \sqrt{\xi})e^{+}(\psi, -\sqrt{\xi}) + e^{-}(x, \sqrt{\xi})e^{-}(\psi, -\sqrt{\xi}) \right\} \frac{(\mu - \xi)^{m_j-1}\sqrt{\mu}}{\alpha(\sqrt{\mu})\alpha(-\sqrt{\mu})} \, d\mu \right\}.
\]
Observing that
\[
\alpha(\sqrt{\mu})\alpha(\sqrt{\mu}) = (\mu - \mu_j)^{m_j}a_j(\mu),
\]
where \( a_j \) is holomorphic in a neighborhood of the point \( \mu_j \) and \( a_j(\mu_j) \neq 0 \) and that
\[
\int_{\xi}^{\mu_j + \delta} \left( \frac{\mu - \xi}{\alpha(\sqrt{\mu}) \alpha(-\sqrt{\mu})} \right)^{m_j-1} d\mu \\
\leq \int_{\xi}^{\mu_j + \delta} \left( \frac{d\mu}{\mu - \mu_j} \right) \leq M_j^{(1)}(\xi) \int_{\xi}^{\mu_j + \delta} \left( \frac{d\mu}{\mu - \mu_j} \right)
\]
\[
= M_j^{(1)}(\xi) \left[ \ln(\mu - \mu_j) \right]_{\xi}^{\mu_j + \delta} \\
= M_j^{(1)}(\xi) \left[ \ln(\delta) - \ln(\xi - \mu_j) \right], \quad \text{if } \xi > \mu_j, \quad (4.33)
\]
similarly,
\[
\int_{\mu_j - \delta}^{\xi} \left( \frac{\mu - \xi}{\alpha(\sqrt{\mu}) \alpha(-\sqrt{\mu})} \right)^{m_j-1} d\mu \\
\leq \int_{\mu_j - \delta}^{\xi} \left( \frac{d\mu}{\mu - \mu_j} \right) \leq M_j^{(2)}(\xi) \ln(\mu - \mu_j) \left[_{\mu_j - \delta}^{\xi} \right] \\
= M_j^{(2)}(\xi) \left[ \ln(\mu - \mu_j) \right]_{\mu_j - \delta}^{\xi} \\
= M_j^{(2)}(\xi) \left[ \ln(\mu_j - \mu_j) - \ln(\delta) \right], \quad \text{if } \xi < \mu_j, \quad (4.34)
\]
where
\[
M_j^{(1)}(\xi) = \max_{\mu \in [\xi, \mu_j + \delta]} \left| \frac{1}{a_j(\mu)} \right|, \quad M_j^{(2)}(\xi) = \max_{\mu \in [\mu_j - \delta, \xi]} \left| \frac{1}{a_j(\mu)} \right|.
\]
(4.33) and (4.34) show that \( \tau_j^k, j = 1, \ldots, k, \) are integral operators with kernels having logarithmic singularities. Equation (4.32) can be written as
\[
\tau_j^k \psi(x) = \int_{A_j} \sum_{k=0}^{m_j} b_{k,j}(x, \xi) \left( \frac{d}{d\xi} \right)^k \left( e^+(\psi, -\sqrt{\xi}) + e^-(\psi, -\sqrt{\xi}) \right) d\xi.
\]
If we define
\[
B_{kj} := \int_0^\infty \int_{A_j} \left| \frac{b_{k,j}(x, \xi)}{(1 + |x|)^{m_n+1}} \right|^2 d\xi dx,
\]
then \( B_{kj} < \infty \) by (3.3), (4.33), and (4.34).
Since
\[
\left\| \tau_2^j \psi(x) \right\|^2_{-(m_0 + 1)} \leq \sum_{k=0}^{m_j} B_k \int_{\lambda_j} \left( \frac{d}{d\xi} \right)^k \left( e^+ (\psi, \sqrt{-\xi}) + e^- (\psi, \sqrt{-\xi}) \right) d\xi
\]
holds, considering Lemma 4.1 and 4.2, we get
\[
\left\| \tau_2^j \psi \right\|_{-(m_0 + 1)} \leq C_j \left\| \psi \right\|_{m_0} \leq C_j \left\| \psi \right\|_{(m_0 + 1)}, \quad j = \ell + 1, \ldots, k, \quad (4.35)
\]
where \( C_j > 0 \) are constants.

Now we consider the operator \( \tau_2^0 \) defined by
\[
\tau_2^0 \psi(x) = \frac{1}{2\pi i} \int_0^{\infty} \frac{\chi_0(\mu)\sqrt{\mu}}{\alpha(\sqrt{\mu})\alpha(-\sqrt{\mu})} \times \left( e^+ (x, \sqrt{\mu}) e^+ (\psi, \sqrt{\mu}) + e^- (x, \sqrt{\mu}) e^- (\psi, -\sqrt{\mu}) \right) d\mu,
\]
where \( \chi_0 \) is the characteristic function of the interval. From (4.36), similar to the proof of Lemma 4.1, we obtain
\[
\int_{-\infty}^{\infty} \left\| \tau_2^0 \psi(x) \right\|^2 dx \leq C_0 \int_{-\infty}^{\infty} \left\| \psi(x) \right\|^2 dx,
\]
where \( C_0 \) is a constant. Since
\[
H_{(m_0 + 1)} \subsetneq L_2(\mathbb{R}) \subsetneq H_{-(m_0 + 1)}
\]
holds, we get
\[
\left\| \tau_2^0 \psi \right\|_{-(m_0 + 1)} \leq C_0 \left\| \psi \right\|_{(m_0 + 1)}, \quad (4.37)
\]
Theorem 4.4. Under the condition (1.3), the spectral expansion

\[ \psi(x) = \sum_{j=1}^{m_j-1} \sum_{l=0}^{m_j-1} U_l(x, \mu)V_{m_j-1-l}(\psi, \mu)|_{\mu=\mu_j} \]

\[ + \frac{1}{2\pi i} \int_0^\infty \sqrt{\mu} \alpha(\sqrt{\mu}) \frac{e^+(x, \sqrt{\mu}) e^+(\psi, -\sqrt{\mu})}{\Phi(x, \sqrt{\mu})} + \sqrt{\mu} \alpha(-\sqrt{\mu}) \frac{e^-(x, \sqrt{\mu}) e^-(\psi, -\sqrt{\mu})}{\Phi(x, \sqrt{\mu})} \, d\mu \]

\[ + \sum_{j=\ell+1}^{k} \sum_{n=0}^{m_j-1} \sum_{p=0}^{\ell} U_n(x, \mu)V_n(\psi, \mu)|_{\mu=\mu_j} \alpha_{j\alpha_j} \]

(4.38)

of \( L \) in terms of the principal functions holds for any \( \psi \in H_{(m_0,1)} \), and the integrals in (4.38) converge in the norm of \( H^{-2(m_0+1)} \), where \( \Phi \) and \( \alpha_{j\alpha_j} \) are defined by (4.19) and (4.25), respectively.

Proof. Using (4.13), (4.24), and (4.25), we obtain (4.38). The convergence of the integrals in the norm of \( H^{-2(m_0+1)} \) has been given in Lemma 4.3.

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References