Beta-expansion and continued fraction expansion

Bing Li, Jun Wu

Department of Mathematics, Wuhan University, Wuhan, Hubei 430072, PR China
Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

Received 19 November 2006
Available online 7 August 2007
Submitted by Goong Chen

Abstract

For any real number \( \beta > 1 \), let \( \epsilon(1, \beta) = (\epsilon_1(1), \epsilon_2(1), \ldots, \epsilon_n(1), \ldots) \) be the infinite \( \beta \)-expansion of 1. Define \( l_n = \sup\{k \geq 0: \epsilon_{n+j}(1) = 0 \text{ for all } 1 \leq j \leq k\} \). Let \( x \in [0, 1) \) be an irrational number. We denote by \( k_n(x) \) the exact number of partial quotients in the continued fraction expansion of \( x \) given by the first \( n \) digits in the \( \beta \)-expansion of \( x \). If \( \{l_n, n \geq 1\} \) is bounded, we obtain that for all \( x \in [0, 1) \setminus \mathbb{Q} \),

\[
\lim\inf_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta^* (x)}, \quad \lim\sup_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta_*(x)},
\]

where \( \beta^*(x), \beta_*(x) \) are the upper and lower Lévy constants, which generalize the result in [J. Wu, Continued fraction and decimal expansions of an irrational number, Adv. Math. 206 (2) (2006) 684–694]. Moreover, if \( \lim\sup_{n \to +\infty} \frac{l_n}{n} = 0 \), we also get the similar result except a small set.

© 2007 Elsevier Inc. All rights reserved.

Keywords: \( \beta \)-Expansion; Continued fraction expansion; Lévy constant

1. Introduction

Let \( \beta > 1 \) be a real number and denote by \( T_\beta \) the \( \beta \)-transformation on the unit interval \([0, 1)\) given by

\[ T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \). Then every \( x \in [0, 1) \) can be written as

\[
x = \sum_{n=1}^{+\infty} \epsilon_n(x) \beta^{-n}, \quad \text{where } T_\beta^0(x) = x \text{ and } \epsilon_n(x) = \lfloor \beta T_\beta^{n-1}(x) \rfloor \text{ for all } n \geq 1. \quad (1.1)
\]

We call the representation (1.1) the \( \beta \)-expansion in base \( \beta \) of \( x \) denoted by \((\epsilon_1(x), \epsilon_2(x), \ldots, \epsilon_n(x), \ldots)\) and \( \epsilon_n(x) \) the digit of \( x \), such an expansion was introduced by Rényi [10], more briefly, we will only write the \( \beta \)-expansion...
instead of the $\beta$-expansion in base $\beta$. Although the number 1 is not in the domain of $T_\beta$, we can still speak of the $\beta$-expansion of 1, denoted by $\epsilon(1, \beta) = (\epsilon_1(1), \epsilon_2(1), \ldots)$ where $\epsilon_n(1) = [\beta T_\beta^{n-1}(1)]$ with $T_\beta(1) = \beta - [\beta]$. 

**Remark 1.1.** If the $\beta$-expansion of 1 is finite, i.e., $\epsilon(1, \beta) = (\epsilon_1(1), \ldots, \epsilon_n(1), 0^\infty)$ with $\epsilon_n(1) \neq 0$, where $\omega^\infty$ denotes the sequence of all $\omega$, we call the purely periodic form $(\epsilon_1(1), \epsilon_2(1), \ldots, \epsilon_{n-1}(1), (\epsilon_n(1) - 1)^\infty$ the infinite $\beta$-expansion of 1.

Put $l_n = \sup\{k \geq 0 : \epsilon_{n+j}(1) = 0$ for all $1 \leq j \leq k\}$. Let

\[
A_0 = \left\{ \beta \in (1, +\infty) : \limsup_{n \to +\infty} l_n < +\infty, \text{ i.e., } \{l_n, n \geq 1\} \text{ is bounded} \right\},
\]

\[
A_1 = \left\{ \beta \in (1, +\infty) : \limsup_{n \to +\infty} \frac{l_n}{n} = 0 \right\},
\]

\[
A_2 = \left\{ \beta \in (1, +\infty) : \limsup_{n \to +\infty} \frac{l_n}{n} \neq 0 \right\}.
\]

Obviously, $A_0 \subset A_1$. From [11], the Lebesgue measure of the set $A_0$ is zero and its Hausdorff dimension (see [5]) is 1.

Let $x \in [0, 1)$ be an irrational number and $[a_1(x), a_2(x), \ldots]$ be its regular continued fraction expansion. It is well known that such an expansion is induced by the continued fraction transformation $\tau : [0, 1) \to [0, 1)$ given by

\[
\tau(x) = \begin{cases} 
\frac{1}{x} - \lfloor \frac{1}{x} \rfloor & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

For any $n \geq 1$, we denote by $p_n(x), q_n(x) := [a_1(x), a_2(x), \ldots, a_n(x)]$ the $n$th convergent of $x$ (see [6]). With the conventions $p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1$, we have

\[
p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x), \quad n \geq 0,
\]

\[
q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x), \quad n \geq 0.
\]

It is easy to see $q_n(x) \geq q_{n-1}(x) + q_{n-2}(x) \geq 2q_{n-2}(x)$, then a successive application of this inequality gives

\[
q_n(x) \geq 2^{\frac{n-k-1}{2}}q_k(x) \quad \text{for all } 0 \leq k \leq n.
\]

Put $I(a_1(x), \ldots, a_n(x)) = \{\omega \in [0, 1) : a_k(\omega) = a_k(x), 1 \leq k \leq n\}$ and call it the $n$th fundamental interval in the continued fraction expansion of $x$. We have

\[
\left| I(a_1(x), \ldots, a_n(x)) \right| = \frac{1}{q_n(x)(q_n(x) + q_{n-1}(x))}
\]

and

\[
\frac{1}{2q_n^2(x)} \leq \left| I(a_1(x), \ldots, a_n(x)) \right| \leq \frac{1}{q_n^2(x)}.
\]

Then it follows that

\[
\liminf_{n \to +\infty} \frac{-\log \left| I(a_1(x), \ldots, a_n(x)) \right|}{n} = 2 \liminf_{n \to +\infty} \frac{\log q_n(x)}{n},
\]

\[
\limsup_{n \to +\infty} \frac{-\log \left| I(a_1(x), \ldots, a_n(x)) \right|}{n} = 2 \limsup_{n \to +\infty} \frac{\log q_n(x)}{n}.
\]

We call $\beta_n(x) = \liminf_{n \to +\infty} \frac{\log q_n(x)}{n}, \beta^*(x) = \limsup_{n \to +\infty} \frac{\log q_n(x)}{n}$ the lower and upper Lévy constant of $x$ respectively. If $\beta_n(x) = \beta^*(x)$, we call the common value $\beta(x)$ the Lévy constant of $x$. In this paper, “almost all” always in the sense of the Lebesgue measure on $[0, 1)$. It is well known [7] that
Theorem 1.2 (P. Lévy). For almost all $x \in [0, 1)$,

$$\beta(x) = \frac{\pi^2}{12 \log 2} = 1.1865691104\ldots.$$ 

In 1964, G. Lochs [8] firstly compared the decimal (i.e., $\beta = 10$) and the continued fraction expansions, and he obtained the following surprising result.

Let $k_n(x) = \sup \{m \geq 0 : J(\varepsilon_1(x), \ldots, \varepsilon_n(x)) \subset I(a_1(x), \ldots, a_m(x))\}$, that is, the exact number of partial quotients in the continued fraction expansion of $x$ given by the first $n$ digits in the $\beta$-expansion of $x$. When there is no confusion we will put $k_n$ instead of $k_n(x)$.

Theorem 1.3 (G. Lochs). For almost all $x \in [0, 1)$,

$$\lim_{n \to +\infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log 10}{\pi^2} = 0.97027014\ldots.$$ 

In [4], K. Dajani and A. Fieldsteel generalized the above theorem by comparing any two expansions with some special conditions: let $S$ and $T$ be number theoretic fibered maps (see also [13]) on $[0, 1)$ with invariant probability measure $\mu_1$ and $\mu_2$, respectively, each boundedly equivalent to Lebesgue measure and with generating partitions $P$ and $Q$ respectively. Denote by $P_n$ and $Q_n$ the interval partitions of $[0, 1)$ into cylinders of order $n$, and $P_n(x)$ the $n$th cylinder that containing $x$ (similarly for $Q_n(x)$). Put

$$m(x, n) = \sup \{m \geq 0 : P_n(x) \subset Q_m(x)\}.$$ 

Suppose that $H_{\mu_1}(P)$ is finite and $h_{\mu_1}(S)$ is positive, where $H_{\mu_1}(P)$ denotes the entropy of the partition $P$ with respect to $\mu_1$ and $h_{\mu_1}(S)$ the entropy of the map $S$ (the same to $H_{\mu_2}(Q)$ and $h_{\mu_2}(T)$). K. Dajani and A. Fieldsteel got the following theorem by the Shannon–McMillan–Breiman theorem (see [3]).

Theorem 1.4 (K. Dajani and A. Fieldsteel). Let $S$ and $T$ be given as above. Then for almost all $x \in [0, 1)$,

$$\lim_{n \to +\infty} \frac{m(x, n)}{n} = \frac{h_{\lambda}(S)}{h_{\lambda}(T)},$$

where $\lambda$ denotes the Lebesgue measure on $[0, 1)$.

Applying Theorem 1.4 to the $\beta$-transformation and the continued fraction transformation, we have

Corollary 1.5. For almost all $x \in [0, 1)$,

$$\lim_{n \to +\infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta}{\pi^2}.$$ 

G. Lochs’s theorem follows by taking $\beta = 10$. In [14], the second author improved the result of G. Lochs to all irrationals and proved that

Theorem 1.6. For any irrational $x \in [0, 1)$,

$$\liminf_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log 10}{2\beta^*(x)}, \quad \limsup_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log 10}{2\beta_*(x)}.$$ 

In this paper, we generalize Theorem 1.6 from decimal expansion to general $\beta$-expansion and obtain

Theorem 1.7. Let $\beta \in A_0$. We have for any irrational $x \in [0, 1)$,

$$\liminf_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta^*(x)}, \quad \limsup_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta_*(x)}.$$ 

In particular, if $\beta$ is a Pisot number, the results also hold.
Theorem 2.2. Let \( \beta \in A_1 \). Then for all irrational \( x \notin [0, 1): \beta_\times(x) = +\infty, \beta_\sigma(x) < +\infty \),
\[
\liminf_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta_\times(x)}, \quad \limsup_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta_\sigma(x)}.
\]

Remark 1.9. We conjecture Theorem 1.8 can hold for all irrational \( x \in [0, 1) \) and the results (1.5) will be not true for some irrationals if \( \beta \in A_2 \). But at present time, we cannot prove or disprove them.

2. Preliminary

In this section we will give some properties of the \( \beta \)-expansion.

Definition 2.1. An \( n \)-block \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) is called admissible in base \( \beta \) if there exists \( x \in [0, 1) \) such that \( \varepsilon_k(x) = \varepsilon_k \) for all \( 1 \leq k \leq n \). An infinite sequence \((\varepsilon_1, \varepsilon_2, \ldots, \) \( \varepsilon_n, \ldots, \) \) is admissible in base \( \beta \) if \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)\) is admissible in base \( \beta \) for all \( k \geq 1 \).

We denote by \( \mathcal{D}_\beta \) the set of admissible sequences obtained from the \( \beta \)-expansion of all real number in \([0, 1)\). The digits of the \( \beta \)-expansion \( \varepsilon_n \) belong to the set \( A = [0, 1, \ldots, \beta - 1] \) if \( \beta \) is an integer or the set \( A = [0, 1, \ldots, \lfloor \beta \rfloor] \) if \( \beta \) is not an integer. Let \( \mathcal{W} = A^\mathbb{N} \) be the symbolic space defined on \( A \) with \( \sigma \)-one-sided shift on \( \mathcal{W} \) and \( \leq_{\text{lex}} \) the lexicographical ordering on \( \mathcal{W} \), that is, \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots) <_{\text{lex}} (\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n, \ldots)\) means that there exists \( k \geq 1 \) such that \( \varepsilon_j = \varepsilon'_j \) for all \( 1 \leq j < k \) and \( \varepsilon_k < \varepsilon'_k \).

We know that not all sequences in \( \mathcal{W} \) belong to \( \mathcal{D}_\beta \) when \( \beta \) is not an integer. W. Parry characterized the set \( \mathcal{D}_\beta \) in [9].

Theorem 2.2 (W. Parry). Let \( \beta > 1 \) be a real number and \( \varepsilon(1, \beta) \) be the \( \beta \)-expansion of 1. We denote by \( \omega \) an infinite sequence of positive integer.

1. If \( \varepsilon(1, \beta) \) is infinite, \( \omega \in \mathcal{D}_\beta \) if and only if

\[
\sigma^k(\omega) <_{\text{lex}} \varepsilon(1, \beta) \quad \text{for all } k \geq 0.
\]

2. If \( \varepsilon(1, \beta) \) is finite, i.e., \( \varepsilon(1, \beta) = (\varepsilon_1(1), \ldots, \varepsilon_n(1), 0^\infty) \) with \( \varepsilon_n(1) \neq 0 \), \( \omega \in \mathcal{D}_\beta \) if and only if

\[
\sigma^k(\omega) <_{\text{lex}} \varepsilon^*(1, \beta) \quad \text{for all } k \geq 0,
\]

where \( \varepsilon^*(1, \beta) = (\varepsilon_1(1), \varepsilon_2(1), \ldots, \varepsilon_{n-1}(1), (\varepsilon_n(1) - 1)^\infty) \) is a purely periodic sequence.

When \( \varepsilon(1, \beta) \) is finite, i.e., \( T^\mathbb{N}_\beta(1) = 0 \) for some \( n \), we call \( \beta \) a simple \( \beta \)-number. The set of simple \( \beta \)-numbers is everywhere dense in \([1, +\infty)\) from W. Parry [9]. He also defined \( \beta \) to be a \( \beta \)-number if \( \varepsilon(1, \beta) \) is ultimately periodic, i.e., the orbit \( \{T^\mathbb{N}_\beta(1)\} \) is finite. K. Schmidt [12] proved that every Pisot number is a \( \beta \)-number, which was proved independently by A. Bertrand [1,2]. From the definition of \( A_0 \), we know that all \( \beta \)-numbers (of course Pisot numbers) belong to \( A_0 \).

In the case that the \( \beta \)-expansion of 1 is finite, we will always write \( \varepsilon(1, \beta) \) instead of \( \varepsilon^*(1, \beta) \) in this paper.

Remark 2.3. An \( n \)-block \((\varepsilon_1, \ldots, \varepsilon_n)\) is admissible in base \( \beta \) if and only if \((\varepsilon_1, \ldots, \varepsilon_n, 0^\infty) \in \mathcal{D}_\beta \).

Given an admissible \( n \)-block \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\) in base \( \beta \), we define
\[
J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \{ x \in [0, 1): \varepsilon_k(x) = \varepsilon_k, \ 1 \leq k \leq n \}
\]
and call \( J(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x)) \) the \( n \)th fundamental interval in the \( \beta \)-expansion of \( x \).

Definition 2.4. We call \( J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) a full interval of rank \( n \) in base \( \beta \) if \( T^\mathbb{N}_\beta(J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)) = [0, 1) \) and nonfull otherwise.
Remark 2.5. If $J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is full in base $\beta$, we get $|J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)| = \beta^{-n}$ where $| \cdot |$ is the Lebesgue measure on $[0, 1)$. In particular, all the fundamental intervals are full if $\beta$ is an integer and otherwise if $\beta$ is not an integer. However, we always have $|J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)| \leq \beta^{-n}$.

In the proof of Theorems 1.7 and 1.8, the Lebesgue measure of the $n$th fundamental interval in the $\beta$-expansion of $x$ is very important. We firstly give the following lemma at first.

**Proposition 2.6.** $\beta \in A_0$ if and only if there exists a constant $C$ such that for all $x \in [0, 1)$ and $n \geq 1$,

$$ C \frac{1}{\beta^n} \leq \left| J(\varepsilon_1(x), \ldots, \varepsilon_n(x)) \right| \leq \frac{1}{\beta^n}. $$

**Proposition 2.7.** Let $\beta > 1$ be a real number and $\limsup_{n \to +\infty} \frac{\log n}{n} = c$. Then for all $x \in [0, 1)$,

$$ \liminf_{n \to +\infty} \frac{-\log |J(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta $$

and

$$ \limsup_{n \to +\infty} \frac{-\log |J(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x))|}{n} \leq (c + 1) \log \beta. $$

Applying Proposition 2.7, we can get the following corollary.

**Corollary 2.8.** $\beta \in A_1$ if and only if for all $x \in [0, 1)$,

$$ \lim_{n \to +\infty} \frac{-\log |J(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta. $$

Before we prove Prepositions 2.6, 2.7 and Corollary 2.8, we give the following lemma at first.

**Lemma 2.9.** Let $\beta > 1$ be a real number and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be an admissible $n$-block in base $\beta$.

(i) If $\varepsilon_n \neq 0$, we have $J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0)$ is a full interval of rank $n$ in base $\beta$.

(ii) Denote $M_n = \max_{1 \leq k \leq n} |l_k|$, then for any $m > M_n$, $J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0^{\infty})$ is a full interval in base $\beta$.

**Proof.** (i) It need only to prove $T^n_\beta (J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0)) \supset [0, 1)$.

In fact, for any $x \in [0, 1)$, we denote by $(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_m(x), \ldots)$ the $\beta$-expansion of $x$. Let $\omega = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0, \varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_m(x), \ldots)$.

Since $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is admissible, then $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0^{\infty}) \in D_\beta$ by Remark 2.3. Note that $\varepsilon_n \neq 0$, by Theorem 2.2 we have for all $0 \leq k < n$,

$$ \sigma_k(\omega) \prec_{\text{lex}} \sigma_k(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, \varepsilon_n, 0^{\infty}) \prec_{\text{lex}} \varepsilon(1, \beta); $$

for all $k \geq n$, $\sigma_k(\omega) = \sigma^{k-n}(\varepsilon_1(x), \varepsilon_2(x), \ldots) \prec_{\text{lex}} \varepsilon(1, \beta)$, therefore $\omega \in D_\beta$.

Then there exists $y \in [0, 1)$ such that the sequence of its $\beta$-expansion is just $\omega$, so $T^n_\beta y = x$. Note that $y \in J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0)$, then $x \in T^n_\beta (J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0))$. Therefore $J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0)$ is full in base $\beta$.

(ii) It is sufficient to prove that $(\varepsilon_1, \ldots, \varepsilon_n, 0^{\infty})$ is admissible by (i).

Denote $u = (\varepsilon_1, \ldots, \varepsilon_n, 0^{\infty})$. In fact, for $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is admissible, then $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0^{\infty}) \in D_\beta$.

Since $m > M_n = \max_{1 \leq k \leq n} |l_k|$, then $\sigma_k(u) \prec_{\text{lex}} \varepsilon(1, \beta)$ for all $0 \leq k < n$. Note that $\varepsilon^m_1(1) = [\beta] \geq 1$, so for all $k \geq n$, $\sigma_k(u) \prec_{\text{lex}} \varepsilon(1, \beta)$. Therefore $u \in D_\beta$ by Theorem 2.2, then $(\varepsilon_1, \ldots, \varepsilon_n, 0^{\infty})$ is admissible by Remark 2.3.  \(\square\)
Proof of Proposition 2.6. On the one hand, for $\beta \in A_0$, there exists $M > 0$ such that $l_n \leq M$ for all $n \geq 1$.

For all $x \in [0, 1)$ and $n \geq 1$, $J(\varepsilon_1(x), \ldots, \varepsilon_n(x), 0, \ldots, 0)$ is full and its length is $\beta^{-(n+M+1)}$ by Lemma 2.9. Thus we can take $C = \beta^{-(M+1)}$.

On the other hand, when $\beta \notin A_0$, there exists a subsequence $\{l_{n_k}\}$ of $\{l_n\}$ such that $l_{n_k} \to +\infty$ as $k \to +\infty$. We suppose for all $x \in [0, 1)$ and $n \geq 1$, there exists a constant $C$ such that

$$C \frac{1}{\beta^n} \leq |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| \leq \frac{1}{\beta^n}. \tag{2.1}$$

We choose $k$ large enough such that $\beta^{-\ln k} < C$.

Note that $J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1)) = J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1), 0, \ldots, 0)$ from Theorem 2.2 and $J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1), 0, \ldots, 0)$ is full for $\varepsilon_{n_k+1}(1) \neq 0$, so

$$\frac{1}{\beta^{n_k+l_{n_k}+1}} \leq |J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1))| \leq \frac{1}{\beta} < C \frac{1}{\beta^{n_k}}.$$  

Choose $x \in J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1))$, from the representation (1.1) we know

$$J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x)) = J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1)).$$

Then $|J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x))| < C \beta^{-n_k}$, which contradicts to (2.1).

Therefore the results hold. \(\square\)

Proof of Proposition 2.7. Case 1. The $\beta$-expansion of $x$ is finite, i.e., $x = \frac{e_1(x)}{\beta} + \frac{e_2(x)}{\beta^2} + \cdots + \frac{e_{n_0}(x)}{\beta^{n_0}}$ with $e_{n_0}(x) \neq 0$.

Let $N = \max_{1 \leq n \leq n_0} \{l_n\} + 1$, by Lemma 2.9(ii), we have $J(\varepsilon_1(x), \ldots, \varepsilon_n(x))$ is a full interval for all $n \geq n_0 + N$ and $|J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| = \beta^{-n}$. Thus

$$\lim_{n \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta.$$

Case 2. The $\beta$-expansion of $x$ is infinite, we denote by $(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x), \ldots)$ the $\beta$-expansion of $x$. Choose the subsequence $\{n_k\}$ of $\{n\}$ such that $\varepsilon_{n_k+1}(x) \neq 0$.

Then $J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x), 0)$ is full by Lemma 2.9(i) and $|J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x), 0)| = \beta^{-(n_k+1)}$. Note that $J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x), 0) \subset J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x))$, we have

$$\frac{1}{\beta^{n_k+1}} \leq |J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x))| \leq \frac{1}{\beta^{n_k}}.$$  

Thus $\lim_{k \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_{n_k}(x))|}{n_k} = \log \beta$.

On the other hand, $|J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| \leq \beta^{-n}$ for all $n \geq 1$, i.e.,

$$\lim_{n \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} \geq \log \beta.$$  

Therefore $\lim_{n \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta$.

In the following, we will consider the upper limit. By $\limsup_{n \to +\infty} \frac{l_n}{n} = c$, we have for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $l_n < (c+\varepsilon)n$. Let $M_N = \max\{l_1, l_2, \ldots, l_N, [(c+\varepsilon)n]\}$.

By Lemma 2.9(ii), we know $J(\varepsilon_1(x), \ldots, \varepsilon_n(x), 0, \ldots, 0)$ is full, then

$$|J(\varepsilon_1(x), \ldots, \varepsilon_n(x), 0, \ldots, 0)| \leq \frac{1}{\beta^{n+M_N+1}}.$$
Note that $J(\varepsilon_1(x), \ldots, \varepsilon_n(x), 0, \ldots, 0) \subset J(\varepsilon_1(x), \ldots, \varepsilon_n(x))$, so
\[
|J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| \geq \frac{1}{\beta^{n+M_n+1}}.
\]
Therefore $\lim\sup_{n \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} \leq (1 + c) \log \beta$ for the arbitrary of $\varepsilon$. $\square$

**Proof of Corollary 2.8.** We need only to prove that there exists $x \in [0, 1)$ such that
\[
\lim\sup_{n \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} > \log \beta
\]
when $\beta \notin A_1$.

In fact, we suppose that $\lim\sup_{n \to +\infty} -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta$ for all $x \in [0, 1)$, then for any $0 < \varepsilon < \frac{c}{2}$, there exists $N \in \mathbb{N}$ such that for all $n > N$,
\[
\left( \frac{1}{\beta^n} \right)^{1+\varepsilon} \leq |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| \leq \left( \frac{1}{\beta^n} \right)^{1-\varepsilon}.
\]
(2.2)

Note that $\lim\sup_{n \to +\infty} \frac{\beta^n}{n} = c > 0$, we can choose $k$ large enough such that $n_k > N$ and $l_{n_k} > (c - \varepsilon)n_k$ where the sequence $(n_k)$ is a subsequence of $(n)$. Similar in the proof Proposition 2.6, note that $c > 2\varepsilon$, we can get
\[
\frac{1}{\beta^{n_k+l_{n_k}}+1} \leq |J(\varepsilon_1(1), \ldots, \varepsilon_{n_k}(1))| \leq \frac{1}{\beta^{n_k+l_{n_k}}} \left( \frac{1}{\beta^{n_k}} \right)^{1+\varepsilon},
\]
which contradicts to (2.2). Thus we get our results. $\square$

3. Proofs of theorems

Before we prove Theorems 1.6 and 1.7, we give some lemmas firstly.

**Lemma 3.1.** Let $x \in [0, 1)$ be an irrational number. Then for all $n \geq 1$,
\[
\frac{1}{6q_{k_n+2}(x)} \leq |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| \leq \frac{1}{q_{k_n}^2(x)}.
\]
(3.1)

**Proof.** Recall $k_n(x) = \sup\{m \geq 0: J(\varepsilon_1(x), \ldots, \varepsilon_n(x)) \subset I(a_1(x), \ldots, a_m(x))\}$, we have $J(\varepsilon_1(x), \ldots, \varepsilon_n(x)) \subset I(a_1(x), \ldots, a_{k_n}(x))$ and
\[
|J(\varepsilon_1(x), \ldots, \varepsilon_n(x))| \leq |I(a_1(x), \ldots, a_{k_n}(x))| \leq \frac{1}{q_{k_n}^2(x)}.
\]

Note that $J(\varepsilon_1(x), \ldots, \varepsilon_n(x)) \not\subset I(a_1(x), \ldots, a_{k_n+1}(x))$, we know at least one endpoint of $J(\varepsilon_1(x), \ldots, \varepsilon_n(x))$ does not belong to $I(a_1(x), \ldots, a_{k_n+1}(x))$. Without loss of generality, we assume the left endpoint of $J(\varepsilon_1(x), \ldots, \varepsilon_n(x))$ does not belong to $I(a_1(x), \ldots, a_{k_n+1}(x))$, i.e., the left endpoint of $I(a_1(x), \ldots, a_{k_n+1}(x))$ belongs to $J(\varepsilon_1(x), \ldots, \varepsilon_n(x))$.

(i) If $k_n$ is odd, we know $I(a_1(x), \ldots, a_{k_n+1}(x))$ is decomposed into a countable fundamental intervals of order $k_n + 2$ and these intervals $I(a_1(x), \ldots, a_{k_n+1}(x), 1)$, $I(a_1(x), \ldots, a_{k_n+1}(x), 2)$, ... run from right to left. Since $x \in J(\varepsilon_1(x), \ldots, \varepsilon_n(x)) \cap I(a_1(x), \ldots, a_{k_n+1}(x), a_{k_n+2}(x))$, we have
\[
I(a_1(x), \ldots, a_{k_n+1}(x), a_{k_n+2}(x) + 1) \subset J(\varepsilon_1(x), \ldots, \varepsilon_n(x)).
\]

By (1.3) and (1.4), we know
\[
|I(a_1(x), \ldots, a_{k_n+1}(x), a_{k_n+2}(x) + 1)| > \frac{1}{6q_{k_n+2}(x)} \geq \frac{1}{6q_{k_n+2}(x)}.
\]
so

\[ |J(\epsilon_1(x), \ldots, \epsilon_n(x))| \geq \frac{1}{6q_{kn+3}^2(x)}. \]

(ii) If \( kn \) is even, we consider \( I(a_1(x), \ldots, a_{kn+3}(x)) \). \( I(a_1(x), \ldots, a_{kn+2}(x), 1) \) is decomposed into \( I(a_1(x), \ldots, a_{kn+2}(x), 2) \) and these intervals run from right to left. For \( x \in J(\epsilon_1(x), \ldots, \epsilon_n(x)) \cap I(a_1(x), \ldots, a_{kn+3}(x)) \), thus \( I(a_1(x), \ldots, a_{kn+3}(x) + 1) \subset J(\epsilon_1(x), \ldots, \epsilon_n(x)) \).

Note that

\[ |I(a_1(x), \ldots, a_{kn+3}(x) + 1)| \geq \frac{1}{6q_{kn+3}^2(x)}, \]

so we get the left inequality of (3.1). Therefore, we have

\[ \frac{1}{6q_{kn+3}^2(x)} \leq |J(\epsilon_1(x), \ldots, \epsilon_n(x))| \leq \frac{1}{q_{kn}^2(x)}. \]

Lemma 3.2. Let \( x \in [0, 1) \) be an irrational number. Suppose one of the following conditions is satisfied:

(i) \( \beta \in A_0 \);
(ii) \( \beta \in A_1 \) and \( \lim inf_{n \to +\infty} \frac{k_n(x)}{n} \neq 0. \)

Then \( \lim_{n \to +\infty} \frac{k_{n+1}(x)}{k_n(x)} = 1. \)

**Proof.** Note that the inequalities (3.1), we know

\[ \frac{|J(\epsilon_1(x), \ldots, \epsilon_n(x))|}{|J(\epsilon_1(x), \ldots, \epsilon_{n+1}(x))|} \geq \frac{q_{kn+1}^2(x)}{6q_{kn+3}^2(x)}. \]

(3.2)

If the condition (i) is satisfied, the left of (3.2) is less than \( C^{-1}\beta \) by Proposition 2.6. By (1.2), we have \( q_{kn+1}(x) \geq 2^{k_{n+1}-(kn+3)\frac{1}{2}}q_{kn+3}(x), \) then

\[ \frac{2^{k_{n+1}}}{6} \leq \frac{q_{kn+1}^2(x)}{6q_{kn+3}^2(x)} \leq \frac{\beta}{C}, \]

i.e.,

\[ k_{n+1} - k_n \leq 4 + \frac{\log 6\beta}{\log 2}. \]

This gives evidently \( \lim_{n \to +\infty} \frac{k_{n+1}(x)}{k_n(x)} = 1 \) for \( k_n(x) \to +\infty \) as \( n \to +\infty. \)

If the condition (ii) is satisfied, by Corollary 2.8, we have for all \( x \in [0, 1), \)

\[ \lim_{n \to +\infty} -\frac{\log |J(\epsilon_1(x), \ldots, \epsilon_n(x))|}{n} = \log \beta, \]

thus for any \( \epsilon > 0, \) there exists \( N \in \mathbb{N}, \) such that for all \( n > N, \)

\[ \left( \frac{1}{\beta^n} \right)^{1+\epsilon} < |J(\epsilon_1(x), \ldots, \epsilon_n(x))| \leq \frac{1}{\beta^n}, \]

then the left of (3.2) is less than \( \beta^{1+\epsilon}\beta^n. \) Since \( q_{kn+1} \geq 2^{\frac{kn+1-(kn+3)\frac{1}{2}}{2}}q_{kn+3}, \) we have

\[ \frac{2^{k_{n+1}-k_n-4}}{6} \leq \frac{q_{kn+1}^2}{6q_{kn+3}^2} \leq \beta^{1+\epsilon}\beta^n, \]
\[ k_{n+1} - k_n \leq 4 + \frac{\log 6}{\log 2} + (1 + \varepsilon + \varepsilon n) \log \beta, \]

note that \( 0 \leq k_1(x) \leq k_2(x) \leq k_3(x) \leq \cdots \), then \( \lim_{n \to +\infty} \frac{k_{n+1} - k_n}{k_n(x)} = 0 \) by the arbitrary of \( \varepsilon \), thus \( \lim_{n \to +\infty} \frac{k_{n+1}}{k_n(x)} = 1 \) for \( \lim_{n \to +\infty} \frac{k_n(x)}{n} \neq 0 \).

**Lemma 3.3.** Let \( x \in [0, 1) \) be an irrational number and \( \lim_{n \to +\infty} \frac{k_{n+1}}{k_n(x)} = 1 \). Then for any fixed \( m \geq 0 \),

\[
\liminf_{n \to +\infty} \frac{\log q_{k_n(x)+m}(x)}{k_n(x) + m} = \beta^*(x), \quad \limsup_{n \to +\infty} \frac{\log q_{k_n(x)+m}(x)}{k_n(x) + m} = \beta^*(x). 
\]

**Proof.** For any \( i \geq 1 \), there exists \( n \) such that \( k_n + m \leq i \leq k_{n+1} + m \), so

\[
\frac{k_n(x) + m}{k_{n+1}(x) + m} \leq \frac{\log q_i(x)}{i} \leq \frac{k_{n+1}(x) + m}{k_n(x) + m} \cdot \log q_{k_{n+1}(x)+m}(x). \tag{3.3}
\]

Note that \( \lim_{n \to +\infty} \frac{k_{n+1}}{k_n(x)} = 1 \), from the left of (3.3) we have

\[
\liminf_{n \to +\infty} \frac{\log q_{k_n(x)+m}(x)}{k_n(x) + m} \leq \liminf_{i \to +\infty} \frac{\log q_i(x)}{i} = \beta^*(x).
\]

Thus \( \lim_{n \to +\infty} \frac{\log q_{k_n(x)+m}(x)}{k_n(x) + m} = \beta^*(x) \) for the \( \{k_n(x) + m\} \) is the subsequence of \( \{n\} \).

Similarly, from the right of (3.3) we can get \( \limsup_{n \to +\infty} \frac{\log q_{k_n(x)+m}(x)}{k_n(x) + m} = \beta^*(x) \).

**Lemma 3.4.** Let \( x \in [0, 1) \) be an irrational number satisfying

\[
\lim_{n \to +\infty} \frac{k_{n+1}}{k_n(x)} = 1 \text{ and } \lim_{n \to +\infty} \frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta.
\]

Then

\[
\liminf_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta^*(x)}, \quad \limsup_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta^*(x)}.
\]

**Proof.** By (3.1), we have

\[
2 \frac{\log q_{k_n(x)}}{n} \leq -\frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} \leq \frac{\log 6 + 2 \log q_{k_{n+3}(x)}}{n}. \tag{3.4}
\]

Note that \( \lim_{n \to +\infty} \frac{\log |J(\varepsilon_1(x), \ldots, \varepsilon_n(x))|}{n} = \log \beta \), from the left of the (3.4), we get

\[
2 \liminf_{n \to +\infty} \frac{k_n(x)}{n} \limsup_{n \to +\infty} \frac{\log q_{k_n(x)}(x)}{k_n(x)} \leq \log \beta. \tag{3.5}
\]

From the right of the (3.4), we have

\[
2 \liminf_{n \to +\infty} \frac{k_n(x) + 3}{n} \limsup_{n \to +\infty} \frac{\log q_{k_n(x)+3}(x)}{k_n(x) + 3} \geq \log \beta. \tag{3.6}
\]

By Lemma 3.3, we know \( \limsup_{n \to +\infty} \frac{\log q_{k_n(x)}(x)}{k_n(x)} = \limsup_{n \to +\infty} \frac{\log q_{k_n(x)+3}(x)}{k_n(x)+3} = \beta^*(x) \), then from (3.5) and (3.6), we have

\[
\liminf_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta^*(x)}. \tag{3.7}
\]

In the similar way, we can get

\[
\limsup_{n \to +\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{2\beta^*(x)}. \quad \Box
\]
Remark 3.5. Let $\beta \in A_1$. Then $\lim \inf_{n \to \infty} \frac{k_n(x)}{n} = 0$ if and only if $\beta^*(x) = +\infty$.

Proof. ($\Rightarrow$) If $\lim \inf_{n \to +\infty} \frac{k_n(x)}{n} = 0$, we have $\lim \sup_{n \to +\infty} \frac{\log q_n(x) + k_n(x)}{k_n(x)} = +\infty$ by (3.6). So $\beta^*(x) = +\infty$.

($\Leftarrow$) If $\lim \inf_{n \to +\infty} \frac{k_n(x)}{n} \neq 0$, we have $\lim \inf_{n \to +\infty} k_n(x) = \log \beta_2 \beta^*(x)$ by Lemmas 3.2 and 3.4. Then $\beta^*(x) \neq +\infty$.

Proof of Theorem 1.7. By Lemmas 3.2 and 3.4, we have our conclusion.

Proof of Theorem 1.8. If $\beta^*(x) < +\infty$, i.e., $\lim \inf_{n \to +\infty} \frac{k_n(x)}{n} \neq 0$, we can get results by Lemmas 3.2 and 3.4. If $\beta^*(x) = +\infty$ and $\beta_n(x) = +\infty$, we have $\lim \inf_{n \to +\infty} \frac{k_n(x)}{n} = \lim \sup_{n \to +\infty} \frac{k_n(x)}{n} = 0$ from (3.4), this can give the results.

References